Stochastic Time and Temporal Non-locality

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We introduce "stochasticity" and "non-locality" to the time variable as an attempt to contemplate "time". These concepts are normally considered as "space" concepts. However, we try to consider ways to export these concepts onto the time axis through simple dynamical models. For stochasticity in time, we introduce noise in the time variable but not in the "space" variable, which is opposite to the normal description of stochastic dynamics. Similarly with respect to temporal non-locality, we consider delayed and predictive dynamics, which involve two points separated on the time axis. With certain combinations of fluctuations and non-locality in time, we show that there appears to be a "resonance" effect. We discuss how this line of approach may further develop thinking on the nature of "time"

INTRODUCTION

The concept of "Time" has drawn a lot of attention from thinkers in virtually all disciplines [1]. One of the central issues is its relation to space. We ask "Are space and time the same? and Can we consider them on the same footing?" These questions gained more attention with the development of the theory of relativity, which has conceptually brought space and time closer to receiving equal treatment. On the other hand, issues such as "direction" or "arrows" of time [2], which have also been discussed and developed in the last century, appear to connotate the difference between space and time.

We propose an idea on the relation between space and time through the concepts of "stochasticity" and "non-locality". These are normally considered as "spatial" concepts. We present possible ways to transport them onto the time axis through concrete examples.

Let us first consider the treatment of noise or fluctuations in dealing with dynamical systems. Whether it is classical, quantum, or relativistic, time is commonly viewed as not having stochastic characteristics when we consider dynamical systems. In stochastic dynamical systems, we associate noise and fluctuations with only "space" variables, such as the position of a particle, but not with the time variable. In quantum mechanics, the concept of time fluctuation is embodied in the time-energy uncertainty principle. However, time is not considered as a dynamical quantum observable, and the time-energy uncertainty is not viewed as the same as the position-momentum uncertainty [3].

Difference also exists in our cognition of non-locality in space and time. Non-local effects in space are naturally incorporated in the physical theories. It has a close relation to the concepts of waves, fields, and so on. The issue of spatial non-locality is more intricate in quantum mechanics, constituting a backbone of such quantum effects as the Einstein-Podolsky-Rosen paradox. With respect to time, there have been investigations of memory effects in dynamical systems. However, less attention has been paid to the non-locality on the time axis. The characteristics associated with delay differential equations, which can be considered as one way to incorporate non-locality in time, are not yet fully understood [4–7].

Against this background, we would like to present models for treating stochasticity and non-locality on the time axis in classical dynamics [8]. We discuss delayed and predictive dynamics as examples of temporal non-locality. Then, for stochasticity, we propose a delayed dynamical model with fluctuating time, termed as stochastic time. We observe that this combination of temporal stochasticity and non-locality lead to characteristics similar to "stochastic resonance". Stochastic resonance has been studied in a variety of fields from the stance that "spatial" noise provides constructive or beneficial effects [9–12]. We will close with a discussion of the direction of further development of this approach toward contemplation of the nature of "time".

TEMPORAL NON-LOCALITY: DELAYED AND ANTICIPATIVE DYNAMICS

We would like to start by discussing temporal non-locality. Dynamical systems that depend on states at two separated time points on the time axis can be thought of as being examples of temporal non-locality. The differential equation with this property can be cast as

$$\frac{dx(t)}{dt} = F(\bar{x}(\bar{t}), x(t)). \tag{1}$$

Here, $x(t)$ is the dynamical variable at time point t, and $\bar{x}$ is a (possibly predicted) state of x at another time point $\bar{t}$. Thus, the dynamical function F contains the state of x at two points, t and $\bar{t}$.

If we set $\bar{t} = t - \tau$, with $\tau > 0$, it gives a delayed dynamical equation, or delay differential equation, with a delay of τ. This dynamical differential equation incorporates the effect from a time point in a fixed interval in the past and the current time point. The delayed dynamics play an important role in situations with a feedback delay, such as that in physiological-control systems. These have also been applied in studies such as those on ecological or population dynamics and economic activities [4–7].

However, when we set $\bar{t} = t + \eta$ with $\eta > 0$, a future temporal point is taken and anticipative dynamics are obtained. η is called an advance. These types of equations have been treated as forms of functional differential equations with advances in mathematical studies [13, 14]. However, they have been studied less than delayed dynamical models. In anticipative dynamics, we also need to define the state of the dynamical variable x at this future point in time. There are a variety of ways we can choose this anticipation, $\bar{x}(\bar{t})$. Here, $\bar{x}(\bar{t})$ is estimated as

$$\bar{x}(\bar{t} = t + \eta) = \eta \frac{dx(t)}{dt} + x(t). \tag{2}$$

This prediction is termed "fixed rate prediction". Namely, we estimate $\bar{x}(\bar{t})$ as the value that would be obtained if the current rate of change extends for a duration from the present point to a future point.

We investigated the properties of these delayed and anticipative dynamical models with computer simulations. To avoid ambiguity and for the sake of simplicity, time-discretized map dynamical models, which incorporate the above-mentioned general properties of the delayed and anticipative dynamical equations, will be discussed throughout the rest of this paper. The general form of the dynamical map is given as

$$x_{n+1} = (1 - \mu)x_n + f[\bar{x}(\bar{t_n})] \tag{3}$$

Here, μ is a parameter controlling the rate of change. For a delayed map with a delay τ, we define

$$\bar{x}(\bar{t_n}) = x_{n-\tau}, \tag{4}$$

while for an anticipative map with advance η, we have

$$\bar{x}(\bar{t_n}) = x_n + \eta(x_n - x_{n-1}). \tag{5}$$

We use the Mackey-Glass function as the dynamical function (Fig. 1A), i.e.,

$$f(x) = \frac{\beta x}{1 + x^s}, \tag{6}$$

where β and s are parameters. This function was first proposed for modeling the cell reproduction process and is known to induce chaotic behavior with a large delay [4].

Figure 1B shows examples of computer simulations of the delayed and anticipative cases. The parameters μ, β, and s were chosen so that with $\tau = \eta = 0$, i.e., without temporal non-locality, the dynamical variable $x(t)$ monotonically approaches a stable fixed point. However, the stability of the fixed point is lost as τ or η increases, giving rise to complex dynamics. Thus, non-locality in time induces a complex behavior in otherwise simple dynamical systems.

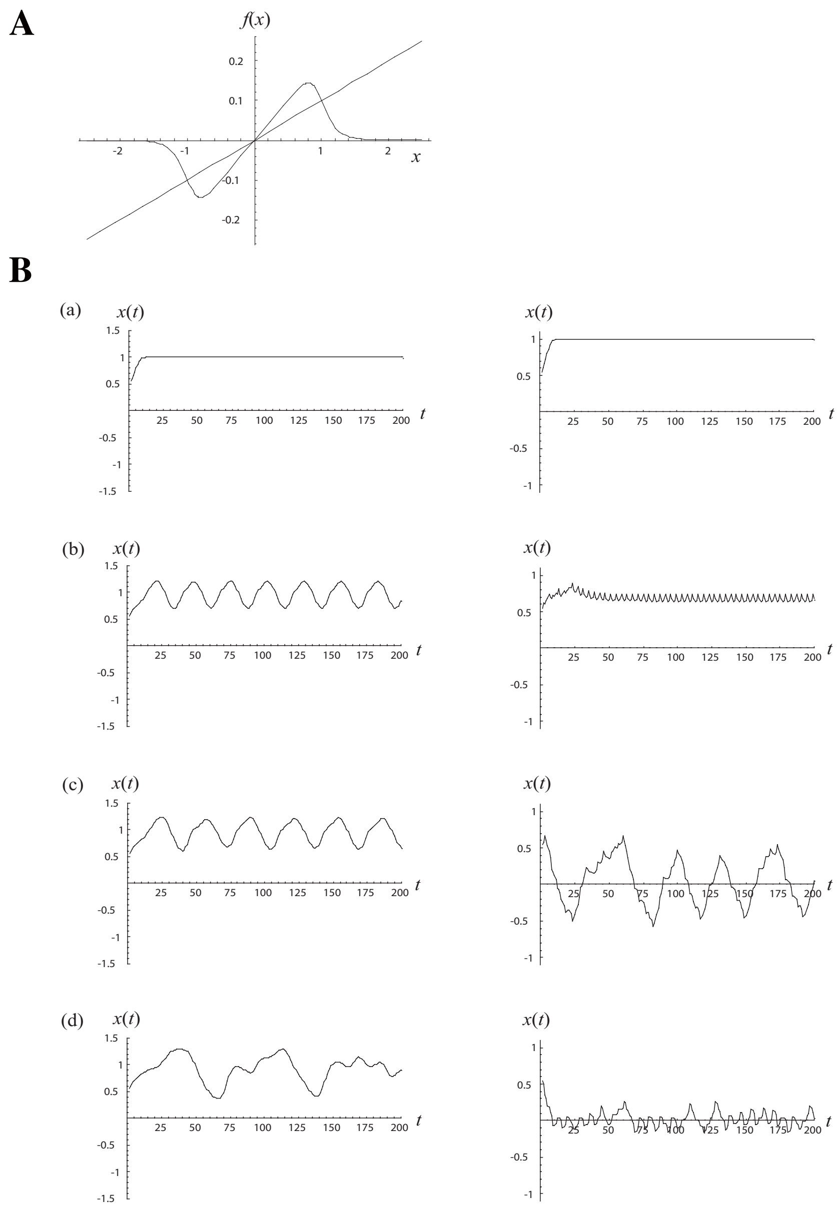

FIG. 1: **A**: Mackey-Glass function $f(x)$ with $\beta = 0.8$ and $n = 10$. Straight line has slope of $\alpha = 0.1$. **B**: Examples of delayed (left column) and predictive (right column) dynamics for Mackey-Glass map with $\mu = 0.5$, $\beta = 0.8$, and $n = 10$. For delayed dynamics, initial condition is fixed at $x_0 = 0.5$ for interval of $(-\tau, 0)$. For predictive dynamics, initial condition is $x_0 = 0.5$ and $x_1 = (1 - \mu)x_0 + f(x_0)$. Values of delay and advance $\tau = \eta$ are (a) 0, (b) 8, (c) 10, (d) 20.

It should be noted that spatial non-locality leads to intricate effects in quantum mechanics. When considered on the time axis, non-locality can lead to complex behavior even within classical dynamics. Linear stability analysis gives an estimate of the critical delay or advance at which the stability of the fixed point is lost. However, the nature of the dynamics beyond these critical levels of temporal non-locality is not yet clearly understood.

Also, in delayed dynamics, we need to decide on the initial function and delay. Analogously, in predictive dynamics, the prediction scheme and advance need to be specified. Common to delayed and predictive dynamical systems, a level of temporal non-locality and these boundary conditions affect the nature of the dynamics.

It has been shown that when delayed dynamics is combined with a spatial noise of a suitable "strength", a characteristic similar to stochastic resonance is obtained [15]. This phenomenon, called "delayed stochastic resonance," has a different mechanism in the sense that it does not require external oscillatory signals or forces, but instead it uses a delay as the source of the oscillation combined with noise. An analogous resonance phenomenon has recently been observed in anticipative dynamics with added spatial noise [16]. These observations suggest that non-locality on the time axis and fluctuation on the space axis suitably combined can lead to a non-standard type of resonance phenomena.

STOCHASTIC TIME

We will now turn out attention to stochasticity on the time axis, which we term "stochastic time". There are various ways of approaching this. As in non-locality, we think in the context of classical dynamical systems. We see that stochastic time combined with delayed dynamics leads again to phenomena similar to stochastic resonance.

The general differential equation for the class of delayed dynamics with stochastic time is

$$\frac{dx(\bar{t})}{d\bar{t}} = f(x(\bar{t}), x(\bar{t} - \tau)). \tag{7}$$

Here, x is the dynamical variable, and f is the "dynamical function" governing the dynamics, and τ is the delay. Its difference from the normal delayed dynamical equation is in "time" $\bar{t}$, which contains stochastic characteristics. We can define $\bar{t}$ in a variety of ways as well as the function, f. Again, to avoid ambiguity and for simplicity, we discuss the following dynamical map system, which incorporates the basic ideas of the general definition above.

$$\begin{aligned} x_{n_{k+1}} &= f(x_{n_k}, x_{n_k - \tau}), \\ n_{k+1} &= n_k + \xi_k \end{aligned} \tag{8}$$

Here, ξ_k is the stochastic variable, which takes either $+1$ or -1 with certain probabilities. n is an integral variable, which we interpret as "time". The dynamics progress by incrementing integer k. By stochasticity, n occasionally "moves back a unit" with the occurrence of $\xi = -1$. Let the probability of $\xi_k = -1$ be p for all k, and we set $n_0 = 0$. Then, with $p = 0$, this map reduces to a normal delayed map with $n_k = k$. We update the variable, x_n, with the larger k. Hence, x_n in the "past" could be "re-written" as n decreases with the probability, p.

We can qualitatively make an analogy of this model with a tele-typewriter or a tape recorder , whose recording head occasionally moves back on a tape with certain probabilities (Fig. 2A). The recording head writes on the tape the values of x at each step, and "time" is associated with positions on the tape. When there are no fluctuations in time ($p = 0$), the head only moves in one direction on the tape and it records values of x for normal delayed dynamics. With probability $0 < p$, it moves back a unit of time to overwrite the value of x. We ask how the recorded patterns of x on the tape are affected as we change p.

The dynamical function is called a "negative feedback function" (Fig. 2B) and the discretized map model that we discuss is given as

$$x_{n_{k+1}} = x_{n_k} + \epsilon(-\alpha x_{n_k} - \frac{2}{1 + e^{-\beta x_{n_k - \tau}}} + 1), \tag{9}$$

with α, β, and ϵ as parameters. With both α and β being positive and with no stochasticity in time, this map has an origin as a stable fixed point with no delay. By analyzing the linearization around a fixed point [6], the conditions for stability can be derived as

$$\alpha > \frac{\beta}{2} \tag{10}$$

or

$$\alpha < \frac{\beta}{2}, \quad \tau < \frac{\cos^{-1}(\frac{2\alpha}{\beta})}{\sqrt{(\frac{\beta}{2})^2 - \alpha^2}}. \tag{11}$$

We investigated $\alpha = 0.5$, $\beta = 6$, and $\epsilon = 0.01$ for this paper. The critical delay, τ_c, at which the fixed point loses its stability, is $\tau_c \sim 59$, according to the above stability analysis. The larger delay gives an oscillatory dynamical path. Computer simulations show that an interesting characteristic arises even when the delay is smaller than this critical delay. The tuned level of stochasticity in time creates a tendency in the system for oscillation. In other words, adjusting the value of p to control the temporal stochasticity induces an oscillatory dynamical path. There are examples of this in Fig. 2C. With increasing probability for the time flow to reverse, i.e., with p increasing, we can observe oscillation both in the sample dynamical path as well as in the corresponding power spectrum. However, when p reaches beyond an optimal value, oscillation begins to deteriorate. The change in the peak heights is plotted in Fig. 2D. Again, this phenomenon resembles stochastic resonance. An analytical understanding of the mechanism is yet to be explored for our model. Yet, this example indicates that a combination of stochasticity and non-locality in time may lead to entirely new phenomena.

DISCUSSION

We have presented a different point of view about time through examples to transfer the concepts of "stochasticity" and "non-locality" onto the time axis. These examples indicate that even within the boundary of classical dynamics, temporal stochasticity and non-locality could produce rather strange phenomena. If these models capture some aspects of reality, particularly with respect to temporal stochasticity, this resonance may be used as an experimental indication for probing fluctuations or stochasticity in time. As already mentioned, we have previously proposed "delayed stochastic resonance" [15], a resonance that occurs through the interplay of noise and a delay. It was theoretically extended [17], and recently, it was experimentally observed in a solid-state laser system with a feedback loop [18]. We leave it for the future to see if an analogous experimental test could be developed with respect to the stochasticity and non-locality of time.

By extending these lines of thinking and models, we have a representation of dynamical systems with fluctuations and non-locality on both the time and space axes. The analytical frameworks and tools for such descriptions need to be developed. For example, stochastic time could be mathematically treated by extending the theory of stochastic calculus [19].

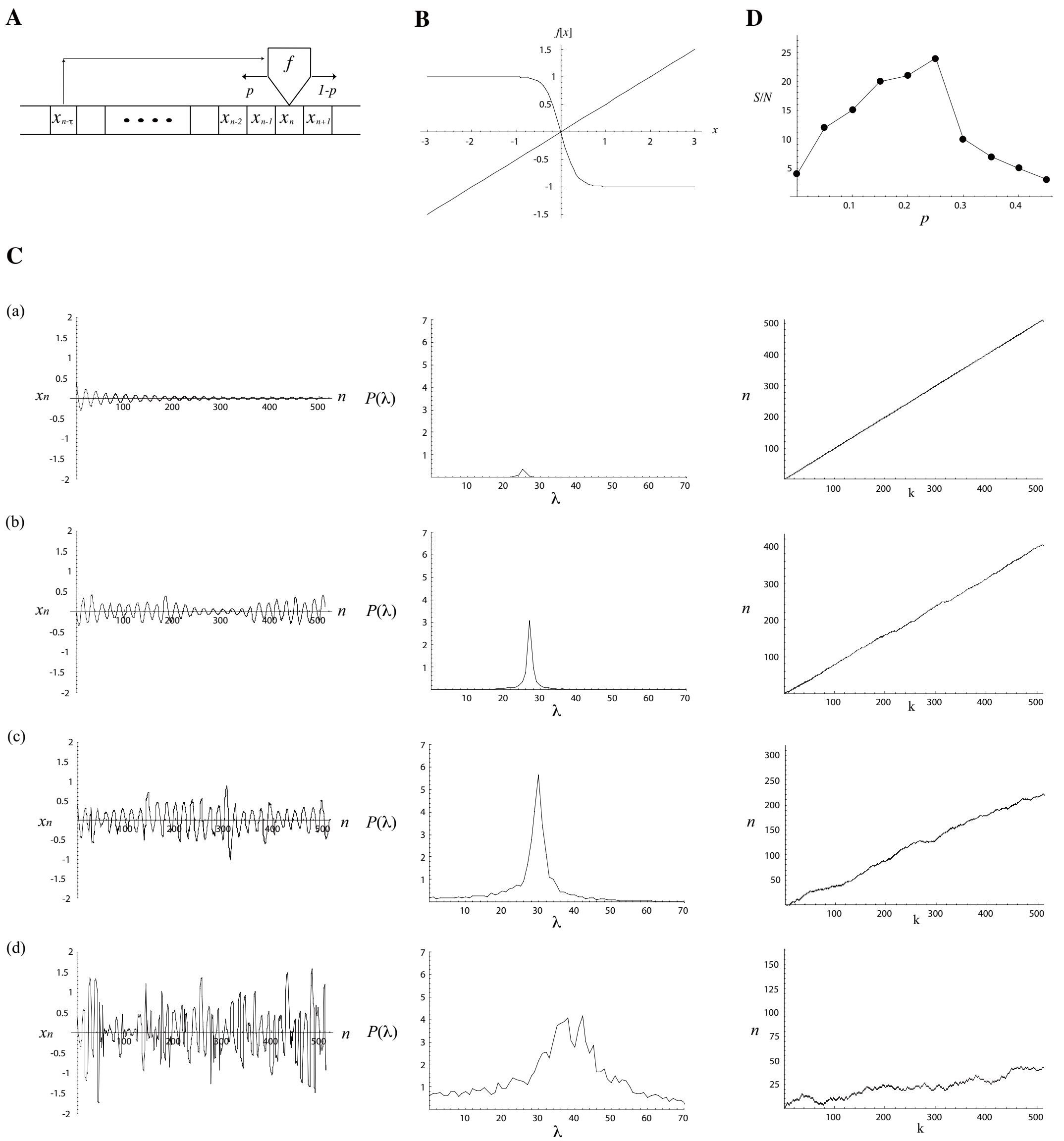

FIG. 2: **A**: Schematic view of model. **B**: Dynamical functions $f(x)$ with parameters as examples of simulations presented in this paper. Negative feedback function with parameters $\beta = 6$. Straight line has slope of $\alpha = 0.5$. **C**: Dynamics (left) and power spectrum (middle) of delayed dynamical model with stochastic time flow (right). This is an example of the dynamics and associated power spectrum through the simulation of the model given in Eq. (3) with the probability, p, of stochastic time flow varied. The parameters are set as $\alpha = 0.5$, $\beta = 6$, $\epsilon = 0.1$, and $\tau = 5$, and the stochastic time flow parameter p is set to (a) $p = 0$, (b) $p = 0.1$, (c) $p = 0.25$, and (d) $p = 0.45$. We used the initial condition that $x_n = 0.5 (n \leq 0)$, and $n_0 = 0$. The simulation was carried out up to $k = 5000$ steps. At that point the values of x_n for $0 \leq n \leq L$ with $L = 512$ were recorded. Fifty averages were taken for the power spectrum on this recorded x_n. The unit of frequency λ was set as $\frac{1}{L}$, and the power $P(\lambda)$ was in arbitrary units. **D**: SNR $\frac{S}{N}$ at peak height as function of probability p of stochastic time flow. Parameter settings are same as in **C**.

Another direction may be to consider quantum or relativistic models with temporal stochasticity and non-locality. For example, it may be interesting to see whether a quantum extension of stochastic time will lead to alternative understandings of such properties as time-energy uncertainty relations. Quantum mechanics is a temporally local theory, but spatially non-local. In this sense, the difference between time and space is more emphasized compared to classical physics. It is interesting to note that this is in contrast to the theory of relativity, which is another major development of physics in the last century. This contrast in thinking about relations between space and time may be affecting the relations between the two theories. It is still too early to know if temporal stochasticity and non-locality can provide any insight on these issues. Nevertheless, we hope that the point of view presented here will contribute to the current thinking on the nature of time.

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