On quantum foundations and the assumption that the Lorentz equation of motion defines a fully resolved electrodynamics

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Abstract

The derivation of an equation of motion (EOM) for a particle-like field source is still an open problem for a general classical field theory. However, for the case of Maxwell field theory formulated in Minkowski spacetime, the answer is believed to be known: up to small radiative corrections, the worldline of a charged particle satisfies the Lorentz EOM. This belief is based on the implicit assumption that the mathematical methods used to derive the EOM from the field theory lead to an EOM that is *fundamental* in the sense that the charge-center of the particle is not very much smaller than the spatial resolution of the dynamical description, so that the EOM defines, as far as is possible, a fully resolved dynamics.

Standard methods for deriving an EOM do not satisfy the above assumption. They are resolution limited, leaving open the possibility that the derived EOM is not fundamental. Typically, this is due to use of a point-multipole (eg., Liénard–Wiechert) approximation for the field of the particle that becomes applicable only at some distance from the particle. In the case of Maxwell field theory, it can be proved that the Lorentz EOM is *not* fundamental—it provides only a lowest resolution approximation to an electrodynamics that exhibits multiple length scales.

Evidence for a multiscale electrodynamics can already be found in the physics literature, for example, in Hestenes' Zitterbewegung interpretation of Dirac electron theory. In this essay I collect together this evidence in the form of an ansatz for the quantum physics of the electron. This ansatz has been used to derive a 4th order EOM for the electron that not only models electron spin dynamics, but remains well defined throughout the full range of length scales over which quantum electrodynamics applies.

1 Introduction

The belief that the Lorentz equation of motion (EOM) is the most fundamental EOM that follows from Maxwell field theory has survived almost unchallenged for over a century. Nevertheless, the assumption of being *most fundamental* is wrong. I will explain how this result has been proved and will speculate on what some of its implications might be.

The problem of deriving an EOM from a Lagrangian field theory is a singular perturbation problem of a type known in other contexts to lead to dynamics with multiple time and/or length scales. Such problems are characterized by the highest derivatives in an EOM having coefficients that vanish with ε, a singular perturbation parameter [1, 2]. In the context of charged particle motion, ε is a small length that characterizes the size of the extended charge-center of the particle, and the exact EOM is formally an infinite order ordinary differential equation obtained as a power series in ε by an energy-momentum balance method (but in practice, only the first few terms of this series are simple enough to compute). The predominate response to this singular perturbation problem has been an attempt to avoid it altogether by using a point-particle ($\varepsilon = 0$) approximation and a classical mass renormalization (this method is described in [3] and references therein). The possibility that the dynamics might actually be multiscale has not been recognized and the point-particle renormalization method has been implicitly assumed to lead to a fundamental EOM.

When applied to Maxwell field theory, the point-particle renormalization method leads to the Lorentz EOM.[1] But we shall see that the Lorentz EOM is *not* fundamental to Maxwell theory in the following sense: Maxwell field theory leads to a multiscale electrodynamics for which the 2nd order Lorentz EOM provides only a lowest resolution approximation. The mathematical approximation on which the Lorentz EOM is based fails (for the electron) at the length scale of the reduced electron Compton wavelength $\lambda\!\!\!^{-}_C = \hbar/(m_e c) \approx 3.86 \times 10^{-13}\,\mathrm{m}$, but there exist higher order approximations to the multiscale electrodynamics that are more fundamental and do not fail at this length scale. In particular, we shall see that by modeling the multiscale electrodynamics using a 4th order EOM, one captures electron spin dynamics and, through its electromagnetic field, also the wave-like properties of the electron (Section 2.5).

To derive an electron EOM that is more fundamental than the Lorentz EOM, one should, presumably, use techniques from singular perturbation theory. Instead, we shall take advantage of certain physical insights in order to see what needs to be calculated. This will involve piecing together various results from the literature to form a coherent picture, or ansatz, for the quantum physics of the electron. This is the most interesting aspect of the proof that the Lorentz EOM is not fundamental, and the only part that I will focus on in this essay. The rest of the proof is a long symbolic algebra calculation for the 2nd order Lagrangian of an electron in an external electromagnetic field. The corresponding 4th order Euler–Lagrange EOM is then solved numerically to check that the theory is giving physically sensible results. The Lagrangian and some of these numerical results are given in Section 3, and further details are available in [5].

The significance of the *wrong assumption* for the foundations of quantum mechanics derives from how the Lorentz EOM has been used to distinguish classical motion of the electron from its quantum behaviour. The assumption that the Lorentz EOM is fundamental to Maxwell field theory has forced us into believing something we should not have: that the experimentally observed failure of the Lorentz approximation at the electron Compton wavelength scale indicates a breakdown of Maxwell theory and the impossibility of maintaining a realistic description of the electron at length scales smaller than the electron Compton wavelength.

The 4th order EOM referred to above is well defined throughout the full range of length scales over which quantum electrodynamics (QED) applies. The EOM can be derived for an electron model with a charge-center that is arbitrarily small, while still modeling the observable properties of the electron (charge, mass, spin, and magnetic moment). The theory is therefore trivially consistent with any experimental least upper bound on the size of electron charge structure, which is currently around $10^{-19}\,\mathrm{m}$ [6]. The finite mass renormalization that is implied by such a small electron charge-center is inherent in multiscale electrodynamics simply through fitting the free parameters of the theory. This is quite unlike the problematic infinite mass renormalization that needs to be imposed in order to derive the Lorentz EOM using a point-particle approximation.

These results suggest that underlying QED there may exist a realistic electron theory. If so, then this underlying theory could have important technological applications: it says that the spin-phase of an electron can be experimentally detected and manipulated. This possibility is precluded in quantum theory by the exact $U(1)$ phase symmetry that is built into the complex formalism, and that corresponds to an *approximate* spin-phase invariance in the underlying realistic theory.[2]

[1] This method was introduced by Dirac to derive the Lorentz–Dirac equation [4], which is a version of the Lorentz EOM that includes a radiation reaction correction. The fact that the Lorentz EOM is also derived by the same calculation, taken to lower order accuracy, is rarely mentioned because it was already the expected result.

[2] The experiment reported in [7] is designed to detect electron Zitterbewegung by breaking this symmetry.

2 The electron physics ansatz

If there exists a Lorentz invariant Underlying Realistic Theory (URT) of the electron that is more fundamental than QED (and hence agrees with the predictions of quantum mechanics) then on the basis of Bell's theorem the URT must be retrocausal [8]. The ansatz to be assembled in this section is retrocausal in two respects: (1) through future boundary conditions that select an ensemble of worldline solutions appropriate for a quantum measurement; (2) through the advanced electromagnetic field of the electron. Together, these retrocausal mechanisms should allow enough scope for explaining all quantum weirdness (compare, for example, [9]).

In this Section I assume the URT exists and simply describe how the pieces will fit together. This ansatz maps out a research program for developing the URT, and it provides a context for understanding how the Lorentz EOM is not fundamental.

2.1 Quantum interpretation and the quantum wavefunction

The quantum wavefunction is taken to be an epistemic construct that describes the ensemble of electron worldline solutions that are relevant to the experiment at hand via a Hamilton–Jacobi like approximation of the URT dynamics. This ensemble of worldlines consists of all solutions of the electron EOM that have initial and final worldline data compatible with "state preparation" and the "measurement" that is to be made. Statistical probabilities for measurement outcomes are identified with corresponding relative volumes in the URT solution space. An *interpretation* of quantum mechanics constitutes a derivation of quantum mechanics from the URT.

This independence of the URT from its quantum interpretation (QED, Dirac, or Schrödinger) allows the URT to be developed and tested in a straightforward manner as a modeling problem for the electron EOM. For example, any proposed EOM can be tested numerically by simulating the $g-2$ precision experimental tests of QED [10]. If QED derives from an URT, then the QED results for the anomalous magnetic moment of the electron should be reproducible to comparable accuracy by numerically integrating the electron EOM and then extracting from the numerical solution the required information about spin precession rates and/or energy differences associated with transitions between limit cycles. Single electron numerical experiments of this type provide a test-bed for developing a viable URT before having to seriously tackle the interpretation problem.

2.2 Zitterbewegung and Hestenes' formulation of the Dirac wave equation

The approximation of the URT dynamics that is afforded by the Dirac wavefunction is evident in Hestenes' Zitterbewegung interpretation of quantum mechanics [11]. Hestenes has given a real Clifford algebra formulation of the Dirac wave equation and shown that the Dirac wavefunction defines an orthonormal Lorentz frame field that can be used to construct a congruence of null helices on spacetime. Each null helix is interpreted as a possible worldline for the electron charge-center. The helical structure of the worldline describes an approximately circular spin motion of the charge-center known as the electron Zitterbewegung.

In Hestenes' account of Dirac electron theory the circular spin motion of the charge-center of the free electron is at exactly twice the Compton angular frequency, $\omega = 2m_{\mathrm{e}}c^2/\hbar$, and the radius of the spin motion is equal to half the reduced electron Compton wavelength, $r = \lambda\!\!\!^{-}_C/2$. The electron spin angular momentum then has magnitude $m_{\mathrm{e}}r^2\omega = \hbar/2$, and the velocity of the charge-center has magnitude $r\omega = c$.

The idea that the worldline of the electron charge-center should be null seems difficult to work with when modeling an extended charge-center. So instead, we shall suppose that the derivation of Dirac electron theory from the URT involves taking the worldline to be null as a simplifying approximation. We suppose that in the URT the worldline of the charge-center is timelike and the above relations that describe the electron Zitterbewegung are approximate. In particular, the rotational speed of the charge-center in the momentum rest-frame of the electron will be just under the speed of light.

2.3 Classical spin models and 2nd order Lagrangian mechanics

The Dirac theory does not explain what causes the Zitterbewegung, so we shall assume the EOM of the electron is, to a first approximation, the Euler–Lagrange equation corresponding to some worldline action integral. A 2nd order Lagrangian will be needed to describe the worldline of the electron, whereas most physicists' intuition for relativistic mechanics is based on 1st order Lagrangian theory. A brief summary of the differences follows.

Mechanics based on 1st order Lagrangian theory is unusually simple. It is the only Lagrangian mechanics in which linear momentum and velocity are proportional, and in which particles have vanishing intrinsic spin angular momentum. These properties are specific to 1st order Lagrangians. In mechanics based on higher order Lagrangians, linear and angular momenta are defined via the Noether theorem and are associated with the Killing vectors of Minkowski spacetime, just as in the 1st order case. However, for a Lagrangian of order k, the Noether momenta involve derivatives of the worldline functions up to order $2k-1$. Thus, for a 2nd order Lagrangian the linear and angular momentum of the particle are 3rd order expressions. Moreover, for $k \geq 2$ the angular momentum splits into the familiar orbital part as well as a non-trivial spin part. The orbital momentum is always defined with respect to some specified reference point, whereas the spin is an intrinsic angular momentum that is independent of any choice of origin.

Variational problems defined by 2nd order Lagrangians that depend on the extrinsic curvature of the worldline have been studied as classical spin models [12, 13]. It is found that a helical worldline structure, similar to that associated with the Zitterbewegung, is by no means unusual. If the Lagrangian includes a term proportional to κ^2 where κ is the extrinsic curvature (ie., the 1st Frenet curvature, or magnitude of the proper acceleration) then the spin angular momentum of the particle manifests as helical structure for the worldline solutions of the Euler–Lagrange EOM.

2.4 Maxwell field theory and the electron Lagrangian

If the electron is modeled as having a suitably non-singular extended charge-center then the action integral for the Maxwell field fully determines the electromagnetic part of the action for the electron. In other words, the electromagnetic part of the electron Lagrangian can be *calculated* rather than *postulated*. A similar calculation using a point-particle model for the electron is impossible because the Maxwell action integral is divergent for the Liénard–Wiechert field of a point-like charge source. The extended charge model that I have used is the charged sphere, where the charge distribution is taken to be rigid and uniform as defined in [14].

An action integral for the charged sphere electron model is found as follows. A series expansion for the electromagnetic field near to the surface of the charged sphere in arbitrary relativistic motion is calculated by using a Green's function method to solve the Maxwell equations with the corresponding charged sphere current source. After substituting this general field solution into the Maxwell action integral, the divergence theorem is used to evaluate the electromagnetic field action as an integral over the worldtube of the sphere, plus some irrelevant boundary terms. The worldtube integral is then approximated as an integral over the worldline on which the sphere is centered. This worldline integral defines the electromagnetic part of the action for the electron. To this, one must add a mechanical action to hold the electron together (the origin of the so-called Poincaré stresses). The later could be taken to be the action for an area minimizing membrane as in Dirac's extended electron model [15] or, for example, it might describe a spherical thin shell modeled using relativistic elasticity theory [16].

An electron action obtained by the above method is given in Section 3. The point to be made here is that the electromagnetic self-energy part of the Lagrangian is found to be (in SI units),

$$\frac{e^2}{8\pi\epsilon_0}\left(\frac{1}{r_\circ}+\frac{4}{9}\,\kappa^2 r_\circ + O\big({r_\circ}^2\big)\right), \tag{1}$$

where $r_\circ$ is the radius of the charged sphere. That is, the electron Lagrangian *does* contain a κ^2 term when the series is taken far enough, and moreover, such a term follows directly from Maxwell field theory (although its numerical coefficient is likely model dependent). To get a feel for why this is, an analogy from elastostatics may help: the approximation of the elastic energy of a 3-dimensional thin rod by the curvature dependent Kirchhoff energy integral over the 1-dimensional central line of the rod [17]. Apart from a signature change in the background metric, this elastostatics approximation is conceptually similar to the approximation of the worldtube action integral for the sphere as an integral over the worldline of the sphere center.

A term proportional to $e^2\kappa^2 r_\circ$ in the Lagrangian may seem too small to be significant, especially if $r_\circ \lesssim 10^{-19}\,m$ (cf. Section 1), but here the nature of the EOM problem as a singular perturbation problem becomes apparent. Consider a prototype free particle action integral $I=\int_{\mathcal{C}}\left(1+\lambda^2\kappa^2\right)\mathrm{d}\tau$, where $\mathcal{C}$ is a timelike worldline in Minkowski spacetime, τ is a proper time parameter for $\mathcal{C}$, and λ is a small perturbation parameter. For $\lambda=0$ the critical values of I are maxima, corresponding to $\mathcal{C}$ being a timelike geodesic, whereas for $\lambda\neq 0$ (but arbitrarily small) I has critical values that are minima, with $\mathcal{C}$ being close to a null helix [12]. In the latter case the tendency for $\mathcal{C}$ to try to minimize the length term in I by becoming exactly null is eventually balanced against a corresponding blow-up of the square curvature term. The variational problems for $\lambda=0$ and $\lambda\neq 0$ are therefore fundamentally different and the κ^2 term in the electron Lagrangian can never be so small as to be insignificant.

2.5 The electromagnetic pilot-wave and bouncing droplets

If the circular spin motion of the electron charge-center is to persist indefinitely then conservation of energy-momentum requires that the electromagnetic field of the free electron be exactly semi-retarded plus semi-advanced. The rate at which electromagnetic energy-momentum is emitted by the free electron via its retarded field is then exactly balanced by the rate at which energy-momentum is absorbed by the electron via its advanced field.

The retarded and advanced components of the field of the circulating electron generate, respectively, outgoing and incoming wavefields that have an approximate spherical symmetry. The sum of these two electromagnetic waves has a quite complicated geometry, but for the purpose of visualization it can be thought of as a spherical standing wave that is always centered on the electron. Boundary conditions and reflections of this standing wave will influence the electron motion, so in this sense it is aptly described as the *electromagnetic pilot-wave* of the electron.

The role of advanced waves in explaining wave-particle duality in terms of a pilot-wave is beautifully illustrated in the bouncing droplet system [18]–[23]. In this macroscopic fluid dynamics system, the Faraday instability is coaxed into simulating the time-symmetric radiation field of a particle carrying a de Broglie clock. The retrocausal aspect of this analogue quantum system has not been emphasised before, so I shall do so here.

The bouncing droplet system combines two fluid phenomena, both associated with a liquid bath driven by a vertical oscillation. The first is that a droplet can be kept bouncing indefinitely on a bath of the same liquid without it coalescing with the bath. The second is that there exists a critical acceleration for the vertical oscillation of a liquid bath, known as the Faraday threshold, beyond which the planar surface of the bath is unstable to perturbation and develops into a regular pattern of Faraday waves. The group headed by Couder (cited above) has experimentally demonstrated bouncing droplet systems for which there exists a regime, just below the Faraday threshold, where the outward travelling surface wave generated by the impact of a bouncing droplet is just sufficient to trigger the Faraday instability in the vicinity of the droplet. The net result is a parametrically amplified standing wave that is generated in response to a sequence of the droplet's most recent impacts. If this sequence is not too long (that is, the system has a short path-memory [22]), then the standing wave that is generated is approximately circular and centered on the droplet. The bouncing droplet—de Broglie's clock carrying particle—then has the appearance of being the source of a semi-retarded plus semi-advanced radiation field, the components of which interfere to form the pilot-wave of the droplet.

Bouncing droplet experiments have demonstrated analogues of single-particle diffraction and interference [18], quantum tunneling through a barrier [19], and quantized orbits for particles in bound states [20].[3] These analogue quantum effects are all due to the influence of the pilot-wave that accompanies the droplet as it travels over the surface of the liquid bath. The electromagnetic pilot-wave should similarly account for the wave-like properties of the electron.

Note that the electromagnetic pilot-wave is part of the URT but the wavefunction is not (cf. Section 2.1). This situation is very different to that in Bohmian mechanics in which the wavefunction is given ontic status and acts on the particle via a quantum potential [24]. In this respect, our ansatz for electron physics is closer to de Broglie's theory of the Double Solution, in which a clear distinction was maintained between a "statistical Ψ-wave" and a physical "u-wave" [25]. That the pilot-wave and wavefunction are two different waves is nicely explained in [23] with reference to the diffraction of bouncing droplets.

2.6 Radiation dynamics and solitonic electron properties

In Section 2.5 an argument was given for taking the electromagnetic field of the *free* electron to be semi-retarded plus semi-advanced, but there also exist conditions under which the retarded and advanced fields of the electron are not in balance. For example, to be able to associate discrete energy levels with limit cycles of the dynamics, the electron needs to be able to radiate net energy–momentum. Moreover, since transitions between energy levels can go either way, the electron must at times be able to both emit and absorb net energy–momentum. None of this is possible if we stay strictly within Maxwell field theory, for then our model electron (whatever its details) will necessarily have an electromagnetic field that has a fixed mixture of retarded and advanced components due to linearity of the Maxwell equations.

So we have the problem that the electron must radiate but Maxwell theory does not tell us why it does so. A second problem is that the mass and spin of a particle in 2nd order Lagrangian mechanics are dynamical quantities [13]. Although initial values for the mass and spin of the free particle can be set to those of the electron, these values will generally differ from those of the electron after any interaction of the particle.

[3] Some of the experiments in which quantized orbits are reported involve systems with *long* path-memory and consequently a dynamics that is far from being time-symmetric. Such systems are interesting in their own right, but I believe are not so directly relevant as analogue quantum systems.

A solution to both these problems is to suppose the electron is a soliton in some nonlinear field theory to which Maxwell theory is a linear approximation. The electron then radiates net linear and angular momentum in such a way as to preserve its soliton identity, including, in particular, its free particle mass and spin. To model an electron with solitonic mass and spin we can modify the Euler–Lagrange EOM by adding a radiation reaction force with a coefficient that vanishes when the mass and spin of the particle are equal to those of the free electron. It is clear that any such modification will be somewhat ad hoc, but it also seems likely that even a crude model for the radiation dynamics may suffice provided it respects conservation of energy-momentum. After all, quantum mechanics works but says almost nothing about the process of emitting or absorbing a photon, or of how an electron makes a transition from one energy level to another.

The origin of the Planck relation, $\Delta E = \hbar\omega$ for photon frequency ω, lies with the nonlinearity of the electron EOM. A transition from one energy level to another will be accompanied by a change in the Zitterbewegung frequency due to the changed environment of the electron. It is to be expected that the Fourier transform of the functions defining the worldline of the electron will be dominated by components at the initial and final Zitterbewegung frequencies and at simple multiples of the sum and difference frequencies. These components of the electron motion will be associated with corresponding frequency components in the radiation field of the electron, suitably advanced or retarded in accordance with energy-momentum conservation.

3 Proof of concept

The procedure outlined in Section 2.4 has been used to derive the following action integral for an electron in an external electromagnetic field $F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta}$ [5],

$$I[z] \;=\; \int_{\mathcal{C}} \left(m_{B} c \left(1 + (\lambda_{B}\kappa)^2\right) - qA_\alpha \dot{z}^\alpha + \frac{q{r_\circ}^2}{6}\, F_{\alpha\beta}\, \dot{z}^\alpha \dddot{z}^\beta \right) \mathrm{d}s\,. \tag{2}$$

Here, the worldline $\mathcal{C}$ of the charge-center of the electron has equation $x^\alpha = z^\alpha(s)$, where x^α are Minkowski coordinates with metric $\mathrm{diag}(-1, 1, 1, 1)$, the worldline coordinate s is such that $\dot{z}^\alpha \dot{z}_\alpha = -1$, and where an overdot denotes differentiation with respect to s. The free parameters in (2) are the sphere radius $r_\circ$ and the two constants m_B and λ_B (with dimensions of mass and length respectively). The electron charge is $q = -e$. The electromagnetic self-energy terms in (1) have been combined with those of a mechanical Lagrangian that is assumed to be the sum of two terms, a constant plus a κ^2 term (the result being the m_B and λ_B terms in (2)).

The action integral (2) was derived under the assumption that all of the spin angular momentum of the electron arises from the circular Zitterbewegung motion of the charge-center. Accordingly, the charged sphere was taken to be *non-rotating*, in the sense that any frame field along $\mathcal{C}$ that is at rest with respect to the surface charge on the sphere is Fermi–Walker transported along $\mathcal{C}$. In terms of the Kirchhoff thin rod analogy from Section 2.4, non-rotation of the sphere corresponds to a worldtube that can bend but not twist. It is likely that such an approximation could only be consistent while the electron is radiating zero net angular momentum. If so, this precludes the possibility of modifying the EOM to introduce a radiation dynamics as described in Section 2.6 (attempts to do so were not encouraging).

Despite these shortcomings, the 4th order Euler–Lagrange EOM that follows from (2) can still be used to model an electron that has appropriate values for its charge, mass, spin, and magnetic moment. Moreover, the sphere radius remains a free parameter that can be chosen arbitrarily small, with the worldline solutions converging to a null worldline as $r_\circ \to 0$ [5] (the parameters m_B and λ_B then depend on $r_\circ$). Numerical solutions show that the worldline of the charge-center spirals around an average worldline that is usually well approximated by a solution of the Lorentz EOM. Similarly, the evolution of the spin vector (which is orthogonal to the plane of the Zitterbewegung motion) is well approximated by a solution of the Bargmann–Michel–Telegdi (BMT) equation. Figure 1 shows numerical simulations for cyclotron motion of an electron in a uniform magnetic field. Figure 1(d) shows how the spin precession method used in experimental tests of QED [10] can also be used in numerical simulations to measure the anomalous magnetic moment of the electron. In this simple model the electron g-factor is put in by hand, but it may be possible to derive an EOM that *predicts* the electron g-factor by removing the non-rotation constraint and introducing a suitable radiation dynamics.

4 Conclusion

The Lorentz EOM fails at the electron Compton wavelength scale not because a realistic description of the electron ceases to be possible, but because the Lorentz EOM is only the lowest resolution description of a multiscale electrodynamics. This has been proved by deriving a more fundamental EOM from Maxwell field theory using a simple charged sphere model of the electron. This result needs to be reconciled with quantum mechanics. If the way to do this is via the URT, as described in Section 2, then perhaps the only true difference between the classical and quantum worlds is the relative importance of retrocausality.

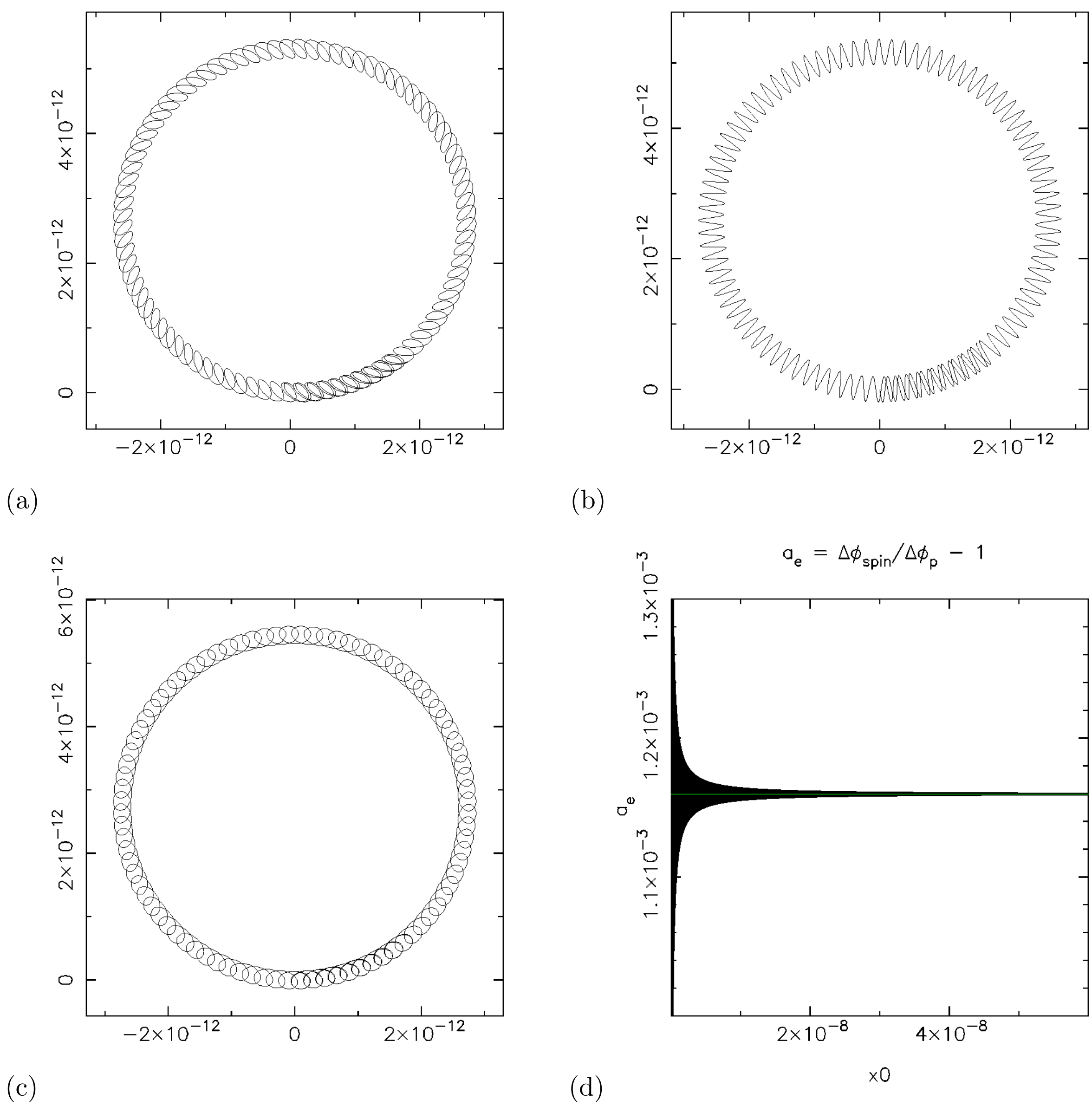

Figure 1: (a)–(c) Numerical solutions of the Euler–Lagrange EOM that follows from the action (2). Orbits of an electron in a uniform magnetic field for three different orientations of the spin vector. The Dirac value $g = 2$ corresponds to zero relative precession between the spin direction and the 3-momentum. In these numerical simulations a formula for the magnetic dipole field of the free electron was used to choose parameters for the electron model so that $g = 2(1 + a_{\mathrm{e}})$, where $a_{\mathrm{e}} \approx 0.001159$ is the electron magnetic moment anomaly. (d) The electron magnetic moment anomaly can be measured numerically by calculating the rate at which the spin precesses with respect the direction of the 3-momentum, $a_{\mathrm{e}} = \Delta\phi_{\mathrm{spin}}/\Delta\phi_p - 1$. The graph shows spin precession data for a numerical simulation for which $a_{\mathrm{e}} = 0.001160 \pm 0.000001$.

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