A Derivation of the Speed of Light Postulate

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Abstract: The speed of light postulate, one of the two fundamental assumptions upon which the special theory of relativity (SR) is based, still seems as mysterious today as when it was first introduced. This paper presents a derivation of the postulate based on three axioms:1) A definition of *motion in proper time* that captures most of the mathematical aspects of SR except for the speed of light postulate itself, 2) An axiom concerning the nature of the existence of entities which do not age, and finally 3) an axiom which defines existence in a spacetime to be transitive.

The second axiom is motivated by an apparent paradox inherent in the structure of SR, called the existence paradox, pertaining to entities which do not age. The duration of existence of such entities in their proper frame is precisely equal to zero, which is consistent with non-existence, and inconsistent with the empirical evidence for the existence of such entities. The resolution of this apparent paradox is guided by a quasi-philosophical principle, called the Principle of least Speciality. The second axiom presents a resolution to the existence paradox that follows this principle while at the same time providing the essence of the explanation for the speed of light postulate. Given the three axioms, it is straightforward to show that the speed of light postulate is a logical consequence. Furthermore, this framework implies a direct experimentally testable prediction that is, according to currently prevailing views, unexpected.

I. Introduction

It is well known that Albert Einstein introduced the Special Theory of Relativity (SR), one of the most successful theories in all of science, essentially using only two basic postulates:

- The Principle of Relativity: The laws of physics are the same in all inertial frames of reference
- The constancy of the speed of light: The speed of light has the same value in all inertial frames of reference independent of the motion of the source or the observer.

The second postulate ('speed of light postulate') still seems just as mysterious today as when it was proposed over a hundred years ago because it seems to defy our intuitive ideas of motion. Any approach that could be used to resolve this apparent mystery will be based on a certain preconceived notions, or biases, and it is best if these are clearly identified in the beginning. The principal biases that underlie this paper are the ideas that the laws of nature do not violate the laws of logic, and that it is within our ability to comprehend them. From the first bias follows the view that there must be a logical explanation which can explain why this postulate is correct. From the second it follows that such a logical explanation should also be understandable, i.e. physically make sense to us. This paper presents a derivation of the speed of light postulate that is meant to provide such an explanation.

II. The Starting Assumption

Instead of following the development and formulation of SR from the above postulates, an axiom will be introduced which shares much of the mathematical content of SR except for the speed of light postulate.

Axiom I

(1)
$$v_{\tau} \equiv \sqrt{c^2 - v^2} = \frac{c}{\gamma}$$

Where c is the speed of light, v is the magnitude of a constant velocity in space, which, just as in the standard formulation of SR is assumed to be *inertial* motion, $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ and v_τ is defined as *motion in*

proper time. The interpretation of $c\gamma^{-1}$ as v_{τ} may be unfamiliar and warrants further explicationⁱ. Consider that, according to the standard formulation of SR, γ^{-1} can be regarded as an 'aging parameter', for it relates, via the Lorentz Transformations, the inertial motion in space of a moving frame directly to the rate at which time is measured to pass in it by an observer in a stationary frame (which, for convenience, will be referred to by the shorthand of 'aging'). In fact, for the two extreme values of γ^{-1} we have

No inertial motion in space $\longrightarrow v = 0 \longrightarrow \gamma^{-1} = 1 \longrightarrow \text{Maximal aging}$ Maximal inertial motion in space $\longrightarrow v = c \longrightarrow \gamma^{-1} = 0 \longrightarrow \text{No aging}$

To prove that (1) is consistent with SR, multiply each side by dt and square:

(2)
$$v_{\tau}^2 dt^2 = c^2 dt^2 - v^2 dt^2 = c^2 dt^2 - dr^2 = ds^2$$

Where $(vdt)^2 \equiv dr^2$ and ds^2 is the spacetime interval, and therefore

(3)
$$v_{\tau}^2 = \frac{ds^2}{dt^2} = \left(c\frac{d\tau}{dt}\right)^2 = \left(\frac{c}{\gamma}\right)^2$$

consistent with the definition given in (1). Thus, v_{τ} is a perfectly legitimate and measurable quantity: To measure it, compare the clock in a rest frame to a clock in a moving frame (using a valid standardized procedure for comparing clocks).

Is axiom I by itself sufficient to derive the speed of light postulate? To address this, consider the following "triangle argument": square (1) and solve for c^2 to obtain

(4)
$$c^2 = v^2 + v_{\tau}^2$$

This is just c^2 multiplied by the familiar relation $\beta^2 + \gamma^{-2} = 1$. Geometrically, the three motions can therefore be visualized as the sides of a right triangle with hypotenuse c. The speed of light postulate says that c is invariant under a transformation $v \to u \neq v$. But this is not the only possibility: instead of c, the invariant side could have been v_τ . Then, any transformation $v \to u > v$ would also increase c, and an infinite iteration of such transformations could even lead to frames in which $c = \infty$. Conversely, under the condition v_τ invariant there is only one rest frame in which c has a minimum value, namely $c = v_\tau$, which corresponds to the geometric picture of the triangle collapsing into a line. An observer in this rest frame could directly determine the rate at which he ages by measuring the speed of photons. That makes this rest frame different from all others and gives empirical justification for considering it one of 'absolute rest'. Furthermore, v_τ invariant means that all inertial observers in relative motion age at the same rate. Proper time therefore merges with coordinate time into 'absolute time'. These are, of course, key characteristics of a *Newtonian* Universe. Thus, while it is commonly assumed that SR reduces to this in the limit $c \to \infty$, by this argument it is more accurate to state that both converge to the same framework in that limit.

While this argument falls short of a rigorous proof, it does strongly suggest that (1) by itself is insufficient to derive the speed of light postulate (at least for finite values of c) but is useful in seeing the difference between pre-relativistic and relativistic concepts from a new perspective.

Finally, consider the following statements:

- 1. v = c is invariant in all inertial frames of reference independent of the motion of the source or the observer
- 2. $v_{\tau} = 0$ is invariant in all inertial frames of reference independent of the motion of the source or the observer.
- 3. $ds^2 = 0$ is invariant in all inertial frames of reference

ⁱ One using no equations can be found in L. Epstein's *Relativity Visualized* pp. 78-87, Insight Press, 1997

Statement 1 is the original statement of the postulate. Statement 2 follows from statement 1 and inserting v = c into (1). Statement 3 follows from statement 2 and inserting $v_{\tau} = 0$ into (2). So, by defining the basic geometric structure of spacetime, Axiom I helps us conceptualize the speed of light postulate in different logically equivalent ways, but it does not appear to explain why nature obeys it.

A problem like this sometimes requires a profoundly new idea, highly unfamiliar at first, after the introduction of which the resolution of the problem becomes almost trivial. The following section will present the motivation for such an idea.

III. An (Apparent) Existence Paradox

It will now be shown that there is an apparent paradox involving a difference between the ontological status of physical existence of entities traveling at speeds less than that of light and of photons inherent in the mathematical structure of axiom I, and therefore SR.

In order to highlight this, let us consider an object that has a constant motion in proper time $v_{\tau}=0$ in a particular frame, which will be called S_0 . Then, according to (1), v=c, so the object is a photon (using this term as a convenient stand-in for any entity determined to travel at the speed of light), and

$$(5) x_f - x_i = \int cdt = ct = x_{\text{max}}$$

Where x_{\max} is the maximum distance that could be traversed in coordinate time t. The duration for which (5) is supposed to hold in terms of the coordinate time in S_0 is stipulated to be finite and is given by

$$(6) t_f - t_i = \int dt > 0$$

On the other hand, the amount by which that object is observed to have aged (as indicated by the change in proper time τ) in S_0 is zero:

(7)
$$\tau_f - \tau_i = \int \frac{v_\tau}{c} dt = 0$$

But (7) must hold for any coordinate times during which the photon is observed to exist (assuming $v_{\tau}=0$ throughout the coordinate time interval). In other words, one can push t_i far enough into the past and t_f far enough into the future that one's observation of τ_i corresponds to the instant the photon comes into physical existence and one's observation of τ_f corresponds to the moment it ceases to physically exist, respectively. These events correspond to what we would call the emission and absorption of a photon, respectively. Since they are exactly the same instant in the photon frame $v_{\tau}=0$ observed over the entire duration in coordinate time means that the duration a photon 'observes itself' to exist is precisely zero.

ⁱⁱ Here it is obviously assumed that it is sensible to speak of such a thing as a 'photon frame'. One might object that since no observer in spacetime can transform to such a frame even in principle this assumption is questionable. However, there is difference between not being able to transform to a frame and dismissing altogether the possibility that it exists simply because one cannot transform to it. Claiming that it is not sensible to speak of a photon frame seems tantamount to either claiming that photons have no frames, or that photons do not exist. The former would provoke the extremely difficult question of exactly *how it is possible for something to exist without having a frame associated with it,* and the latter would be immediately dismissed due to the overwhelming evidence for the existence of photons.

Now, in any other context, a duration of physical existence of precisely zero is considered to be logically equivalent to *non-existence*. Here the definition of 'physical existence' is taken to be a qualitative and operational one, namely 'capacity to produce observational consequences'.

There are abundant observational consequences for which the existence of photons seems to be the most plausible explanation and the one most consistent with experimental data.

The existence paradox, then, is this: How is it possible that entities traveling at the speed of light 'observe' their own duration of existence to be precisely equal to zero, and yet produce observational consequences evidencing that they do exist?

Notice that if we lived in a Universe in which universally $c=\infty$, then this apparent paradox would not occur: The proper time would always be equal to the coordinate time because then $\gamma^{-1}=1$ in every inertial frame. Photons would not be measured to spend any finite time in transit in any frame, and this would likely obviate the need for postulating their existence altogether, since any interaction involving them could be treated as a contact interaction, no matter how far apart. In the standard formulation of SR the velocity dependence of the Lorentz factor arises precisely because the speed of light postulate is applied to a speed less than infinity, so this suggests that there is a deep connection between this postulate and the nature of the physical existence of photons.

IV. A Solution

In the previous section, it was demonstrated that the speed of light postulate is closely related to a distinction between the nature of existence of massive particles and of photons. Understanding this distinction more deeply would therefore appear to be the key in understanding the physical origin of the speed of light postulate.

In order to resolve this issue, one needs to examine any unspoken assumptions that went into the above analysis. As an aid in this endeavor, we will formulate and use a principle that amounts to a rule for selecting fundamental assumptions that go into a framework.

The Principle of Least Speciality

Of alternative fundamental assumptions of which one must be chosen to derive an explanation for a physical phenomenon, that one is to be preferred which assigns to that phenomenon, and closely related aspects of nature, the least special status, unless there exists a logical reason for preferring a more special explanation.

The idea behind this principle is not really new, as there are many examples in physics in which fundamental ideas follow this implicitly: the homogeneity of time and the isotropy and homogeneity of space absent gravitational fields, the equal-magnitude contribution of all paths in path integrals, the fundamental assumption of statistical mechanics, the cosmological principle, to name some, can all be thought of as fundamental assumptions that are consistent with this principle. Indeed, the principle of relativity itself follows it: One might imagine alternative theories in which there exist one or more special inertial frames in which the laws of physics are different from all the others, but constructed in such a way that they yield the same predictions as SR. The aether theories at the turn of the 20th century are examples of just those kinds of theories. Even if there was no way ever to distinguish between them and SR, the principle introduced here would explicitly give a criterion for preferring SR over them iii.

Now, an important unspoken assumption that was contained in the existence paradox was that spacetime is all there is in which things can physically exist. This assumption is amenable to a check by the Principle of Least Speciality, for stated in this way, there are two possibilities: Either spacetime is or is not the sole continuum in which all things physically exist, where by continuum a spacetime is meant that consists of n length dimensions and one time dimension. The first possibility can be regarded to be the current default assumption (some recent trends in theoretical physics notwithstanding). The second possibility treats spacetime as one of a set of more than one continua in which things can physically exist. Now, as observers in spacetime it is

ⁱⁱⁱ Of course, had evidence for the existence of an aether been found, then there would be a logical explanation for the special status of certain inertial frames, and the principle would have preferred an aether theory over SR.

natural for us to assume that our continuum of three dimensions of length and one dimension of time is all there is to the universe; that is, when we speak of the 'existence' of something, we really mean 'existence in our spacetime' but leave out the qualifier because, according to this view, it is redundant. In that respect, we are perhaps not that different from our ancestors who, centuries ago, thought that, say, the sun and some of the planets were essentially all there was to the universe. But other than the fact that we as observers have not recognized clear evidence for the existence of other continua (which is not necessarily the same as saying that there is no evidence), there is no explanation for why our spacetime should be special in this way i.e. we know of no reason why the existence of other continua is forbidden. The Principle of Least Speciality tells us that, because the first possibility assigns to spacetime a greater special status than the second, there is either an as yet unknown logical explanation for this or we should prefer the second possibility. We will assume that such a logical explanation does not exist and therefore only examine the consequences of the latter. If one does allow for the second possibility, that spacetime is not all there is in which things can exist, then one can immediately consider an explanation for the existence paradox that is not available under the first: one can consider the idea that *photons physically exist outside of our spacetime*.

If there was an entity which physically existed outside of our spacetime, it would not be a surprise that observational evidence would indicate its proper time to be precisely zero (recalling that "proper time" in this context for such entities is forced upon us by observational consequences that propagate at speed \mathcal{C}). In fact, if its proper time were finite, then this would seem to directly contradict the idea with which we started, since 'aging' in spacetime seems to imply 'existing' in spacetime as it leads to the 'observation' of one's existence in spacetime over a duration greater than zero.

We should also not be surprised to find that a hypothetical observer in that entity's frame would measure *our* proper time to be precisely zero (even though we age), since we physically exist in spacetime and not in that observer's continuum. To the extent that SR allows one to make any intelligible statements about a photon's frame, this appears to be exactly what SR predicts, since we, too, are moving relative to a photon at speed \mathcal{C} . This relationship between a spacetime observer's and a photon's frame brings us to the speed of light postulate: Its statement amounts to the assertion that no observer in spacetime can transform to the rest frame of a photon, and no photon frame can transform to the rest frame of an observer in spacetime. If photons really exist outside of spacetime, then it *must* be so, since transforming one's frame to the rest frame of the other would require changing the continuum in which one exists! We have now touched upon a possible explanation for the speed of light postulate, and to resolve it, the assumption just discussed will be formalized as a second axiom:

Axiom II

If all possible observational consequences are consistent with a zero rate of aging of an entity in the frame of an observer $(v_{\tau} = 0)$, then that entity exists in a continuum other than that of the observer, otherwise it exists in the same continuum as that of the observer.

Note that only in light of Axiom I $v_{\tau}=0$ becomes equivalent to v=c. It is this equivalence which makes the notion of 'observing' something that exists outside of one's continuum coherent. Measuring opposite changes in the energy levels of, say, two electrons (which age and therefore exist in the same continuum as the observer) a time t and distance ct apart is sufficient for this purpose, so the 'observation' is actually quite indirect. Nevertheless, according to this framework, the more fundamental property of photons in terms of assigning them to a continuum distinct from ours is that they do not age in spacetime, not that they are found to travel at c. Notice that axiom II also ensures that once something is found to travel at c, it must travel at that speed for all future '(unless it ceases to exist), because a change in its speed amounts to a change

^{iv} Interestingly, the Principle of Least Speciality, when applied to the speed of light postulate itself, leads to one of two conclusions: either the postulate is wrong, or there is a logical explanation for it (the third possibility, that the postulate is correct but has no logical reason, is ruled out by this principle). Since empirically it has been shown to be correct without exception, one is left to conclude that there must be a logical reason for it, one of the exact underlying biases of this paper.

^v Of course, this is assuming identical conditions, i.e. speed in vacuum

in the continuum in which it exists. Thus, whereas in eqn. (7) we had to insert the constancy of c for the duration of the integral in coordinate time as a separate assumption, it now follows from the second axiom. Since we have now introduced the possibility of observers and events existing in distinct continua, we must also introduce an assumption regarding the transitivity of existence in a continuum to properly consider cases involving more than two entities.

Axiom III

If an entity A exists in the same continuum as an entity B, and B exists in the same continuum as an entity C, then A exists in the same continuum as C.

Given Axioms I-III, it is straightforward to derive the speed of light postulate.

V. The Derivation

We shall consider two distinct scenarios which can be generalized to any situation in which observers and events move at constant velocities relative to one another.

Let there be two observers, O' and O, such that O' is in relative motion to O at constant velocity with magnitude u < c in the direction of the positive x-axis. By axioms I and II O' and O therefore exist in the same continuum and in the frame of O', O has a relative velocity $f(-\bar{u}) = -\bar{u}$ (under the auxiliary assumption of symmetry of motion in space).

1) Suppose O 'emits' an object p_1 that travels at some speed $v_1 < c$ along the positive x-direction in the frame of O (Note that this language of a classical trajectory is now explicitly meant to be short-hand for the more correct, but more cumbersome description according to which we are only considering the duration between the instant of 'emission' and that of 'absorption' at two distinct locations). What speed along the same direction would O' observe for p_1 ?

Since $v_1 < c$, $v_{\tau 1} > 0$ by axiom I, and p_1 exists in the same continuum as O by axiom II. Since O exists in the same continuum as O', p_1 must also exist in the same continuum as O' by axiom III. O' would therefore be expected to observe p_1 to travel at speed $|f(\vec{v}_1 - \vec{u})| < c$ in the frame of O', where

 $f(\vec{v}_1 - \vec{u})$ is an as yet unidentified function of $\vec{v}_1 - \vec{u}$. The precise mathematical expression of $f(v_1 - u)$ can be derived once the speed of light postulate has been derived.

2) Suppose O 'emits' an object p_2 that travels at speed $v_2 = c$ along the positive x -direction in the frame of O. What speed would O' observe for p_2 ?

Since $v_2=c$, $v_{\tau 2}=0$ by axiom I, and p_2 exists in a continuum distinct from that of O by axiom II. Since O exists in the same continuum as O', p_2 must also exist in a continuum distinct from that of O' by axiom III. O' could not observe the speed to be $|f(\vec{v}_2-\vec{u})| < c$ because by axiom I that would mean that in his frame, $v_{\tau 2}>0$, which by axiom II would imply that p_2 existed in the same continuum as O', contradicting the earlier result. The only consistent observation would be that in the frame of O', $v_{\tau 2}=0$ as well, which by axiom I means that O' must observe p_2 to travel at $|f(\vec{v}_2-\vec{u})|=c$.

To summarize, because $v_{\tau}=0$ signifies existence in a distinct spacetime from that of the observer by axiom II, any other observer who exists in the same spacetime as the first must also observe $v_{\tau}=0$ by axiom III. By axiom I, $v_{\tau}=0$ is equivalent to v=c and therefore the conclusion is clear: If a photon travels at speed c in one inertial frame in spacetime, then, given axioms I-III, it must travel at speed c in all other inertial frames in spacetime. Once this is established, it is straightforward to derive the Lorentz coordinate transformations,

using the same auxiliary assumptions as in the standard formulation of SR (symmetry of motion in space, linearity of transformations), from which one can determine that the function $f(\vec{v} - \vec{u})$ is nothing but the Lorentz velocity transformation

(8)
$$f(\vec{v} - \vec{u}) = \frac{\vec{v} - \vec{u}}{1 - \frac{\vec{v}\vec{u}}{c^2}}$$

where \vec{v} and \vec{u} are assumed to have the same direction.

Relativistic dynamics can be obtained fairly directly by multiplying (4) with $\gamma^2 m_0^2 c^2$, where m_0 refers to the rest mass of an object, yielding

(9)
$$\gamma^2 m_0^2 c^4 = \gamma^2 m_0^2 v^2 c^2 + m_0^2 c^4 = p^2 c^2 + m_0^2 c^4 = E^2$$

Where $p = \gamma m_0 v$ is the relativistic momentum and E is the total Energy of the object. Evidently, then, these three axioms lead to a framework that is essentially equivalent to SR. Indeed, from a geometric perspective this framework says that something which in our four-dimensional spacetime spans zero distance ($ds^2 = 0$) but still produces observational consequences actually exists outside of our spacetime; therefore the null interval it spans is invariant in every inertial spacetime frame.

Note that Axioms II and III can be replaced by alternative axioms of similar logical structure from which the speed of light postulate can still be derived but as far as this author can tell, these do not give a direct *physical explanation* for the speed of light postulate. For example, Axiom II might be replaced by "If an entity is observed to travel at speed v = c, then it has zero mass." Axiom III would then be replaced by: "If an entity has zero mass in one inertial frame, then it has zero mass in all inertial frames." Together, these two axioms directly lead to the speed of light postulate, but they do not provide any additional insight; they are the speed of light postulate broken into two separate axioms. The actual axioms proposed above contain an explanation that is absent in the speed of light postulate proper, and the way to check whether this is a true explanation is to verify whether it also explains the existence paradox, which any true explanation of the speed of light postulate must be able to do. The alternative axioms just mentioned, while consistent with SR, fail on this account.

So, while it may seem like a step back to derive a framework from three fundamental assumptions when it can be derived from just two, the value of this approach lies in that it appears to provide an unambiguous physical explanation for the speed of light postulate.

VI. Some Implications

Perhaps the most significant implication of the explanation presented here is its suggestion that one of the most established theories in all of science points directly to the existence of at least one continuum other than our own spacetime. While this may conjure up currently fashionable notions such as extra dimensions, parallel universes, a 'multiverse' etc., SR does not seem to specify what lies beyond spacetime in any obvious way, and so this paper shall not speculate on that either.

This implication leads to the prediction of a second important null-result associated with SR (the first being Michelson-Morley's) that is at least in principle testable: *The gravitational field produced by photons in transit must be exactly zero* because only objects that exist *in* spacetime are expected to produce local spacetime curvature. This prediction is distinct because it goes against the current prevailing view, which assumes that mass-energy equivalence extends to gravitational mass. Unfortunately, this author is not aware of any relevant experimental tests.

For mass, this explanation ties together inertia, aging and existence in spacetime. By this framework, massive particles cannot be accelerated to the speed of light because they age ($v_{\tau} > 0$ so v < c for all nonzero masses in all inertial spacetime frames), and because they age they exist in spacetime, since aging is prerequisite to observing the duration of one's existence in spacetime to be greater than zero.

An objection can be raised that if photons really exist outside our spacetime, then why do they interact with objects in our spacetime at all? Indeed, the assertion that a photon travels at speed *C in* space seems to contradict the notion that it exists outside of spacetime. To respond to this, consider that no experiment measures the existence of a photon directly; what is measured is a change in the energy level of a massive particle at some location and an opposite change a time *t* later in the energy level of another massive particle located a distance *ct* away (assuming only the vacuum separates the two). This is *interpreted* as, say, the emission or absorption of a photon, from which it is then extrapolated that photons travel in space. Our language, when we speak of photons traveling in space, is a reflection of this extrapolation, not of what has actually been experimentally shown. Indeed, according to the quantum theory of photons, it is incorrect to assume that a photon travels along any determinate path in space. So the objection seems to be reducible to one about the coupling of photons to massive particles the instant their energy levels change, and here it really amounts to an argument for framing the issue and studying it further from the perspective of quantum theory. Given that photons apparently cease to exist at precisely the instants they are 'measured', it is not clear that our current understanding of photons really contradicts the notion that photons exist outside of our spacetime.

Finally, it may become necessary in certain circumstances to now add the qualifier 'in spacetime' when speaking of observers or events in spacetime. For example, the explanation for the speed of light postulate given in this paper suggests that the speed of light postulate itself should now be stated as 'the speed of light has the same value in all inertial frames of reference *in spacetime* independent of the motion of the source or the observer'. That makes its domain of validity explicit, which is important because the domain of validity of the speed of light postulate also defines the domain of validity of the *principle of locality*, which says that nothing travels faster than the speed of light. The principle of locality should now be stated as 'nothing *in spacetime* travels faster than light'. This in turn allows one to approach an understanding of its apparent violations, such as those occurring in Bell's paradox, by asking in what way they might lie outside its domain of validity.

VII. Conclusion

There is a precedent that when an apparent incongruity is noticed in an established theory, its resolution may lead to new theoretical insights. SR itself seems to have been born this way, since Einstein apparently came upon it when he noticed that in Maxwell's theory of electromagnetic fields, observers in relative motion would disagree about the fields they observed even though the physical effects were the same. The apparent incongruity which prompted the explanation given in this paper is what was called the existence paradox. In retrospect, it seems a little odd that, at least as far as this author knows, it had not been pointed out previously. Perhaps this is due to the fact that 'existence' is not currently regarded as a physical, but rather as a philosophical concept. The conclusion to which the explanation of the speed light postulate leads us suggests that incorporating this concept into physics more formally may be prerequisite to a more fundamental understanding of our world.

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