Questioning the postulates of quantum field theory

Peter Morgan[*]

Physics Department, Yale University, New Haven, CT 06520, USA.[†]

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There have already been many attempts in the literature to modify the Wightman or Haag-Kastler axioms to be closer to the empirically successful Lagrangian approach to quantum field theory. It is suggested here that insofar as one fundamental difficulty of both Wightman and Lagrangian QFT is the postulate that the quantum field is an operator-valued distribution —a linear map from a linear space of test functions to a linear space of operators—, we are motivated to consider taking a quantum field to be a *non*-linear map from a linear space of test functions to a linear space of operators, an approach that to my knowledge has not previously been proposed. Constructively, some of the non-linear possibilities for the scalar field case are introduced and discussed. The introduction of non-linearity widens the range of well-defined theories enough that they may provide worthwhile effective field models even if they cannot provide ultimately correct models.

I. INTRODUCTION

The reconciliation of the mathematically coherent Wightman axioms with the empirically successful Lagrangian approach to interacting quantum field theory is a longstanding problem. The nature of the mismatch between interacting quantum fields such as QED and the Standard Model of particle physics and the Wightman axioms is not entirely clear, however it does appear that the introduction of products of distributions in Lagrangian QFT is one heart of the difficulty.

It is of course possible to question the postulate that the dynamics is Lorentz invariant, both at very large distances and at very short distances, as has already been done in a number of ways, however we here suppose the dynamics to be Lorentz invariant, essentially ruling out, in particular, the concerns of quantum field theory in curved space-time and of quantum gravity. We also here accept the utility of Hilbert space for modeling evolving statistics of multiple experimental datasets, taking the view that quantum field theory is, mathematically, an elaborate form of signal analysis in the presence of a Lorentz invariant quantum noise, and remaining ontologically neutral to whether there is no hidden signal, whether there are signal components that have not been measured that contain whatever conspiracies or correlations are needed to reproduce the deterministic evolution of statistics that we observe, or whether there is any explanation at all for the effectiveness of quantum field models.

A fundamental postulate of quantum field theory is that a quantum field $\hat{\phi}(x)$ is an operator-

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[*]Electronic address: peter.w.morgan@yale.edu
[†]URL: http://pantheon.yale.edu/~PWM22

valued *distribution*, so that the operators of the theory are obtained by "smearing", for a well-behaved linear space of "test" functions,

$$\hat{\phi}_f \triangleq \int \hat{\phi}(x) f(x) \mathrm{d}^4 x. \tag{1}$$

Notoriously, products of distributions are mathematically problematic, so that although we may straightforwardly introduce integer powers of operators, such as $(\hat{\phi}_f)^n$, we cannot in general introduce integer powers of distributions, such as $\hat{\phi}(x)^n$. We will therefore work only with operators such as $\hat{\phi}_f$, for a well-behaved linear space of test functions, allowing, however, for the non-linear case, that $\hat{\xi} : f \mapsto \hat{\xi}_f$ may be a non-linear functional of the test functions, so that in general $\hat{\xi}_{f+g} \neq \hat{\xi}_f + \hat{\xi}_g$ and $\hat{\xi}_{\lambda f} \neq \lambda \hat{\xi}_f$, essentially denying the linearity that is implied by Eq. (1). Nonetheless, the action of the algebra of operators on Hilbert space vectors is taken to be linear, $(\hat{\xi}_f + \hat{\xi}_g)\psi = \hat{\xi}_f \psi + \hat{\xi}_g \psi$ and $(\lambda \hat{\xi}_f)\psi = \hat{\xi}_f \lambda \psi$. Streater [1, §3.4] discusses a number of ways in which the Wightman axioms have been questioned, but there is no mention of the introduction of non-linearity of this form.

One immediate consequence of non-linearity is that a Hilbert space that supports a representation of the Poincaré group will not be separable (that is, there will not be a countable basis). Allowing non-separable Hilbert spaces is possible, and in time mathematical control of non-separable Hilbert spaces may improve, but it is also possible to require only that the construction of the Hilbert space of a quantum field theory is Lorentz and translation covariant (preferably manifestly so), which is mathematically much weaker than requiring that the Hilbert space support a representation of the Poincaré group. Requiring only covariance is compatible with model-building in Physics, which does not require that every possible model that is related to a given model by Lorentz transformations or by translations is encompassed by a single Hilbert space, only that a new Hilbert space can be readily constructed for any given Lorentz transformation or translation.

We will introduce two very different non-linear constructions, one that approximately follows the method that is used by Lagrangian QFT to construct an interacting quantum field, in Section IV, and another, in Section V, that modifies the free field algebra of creation and annihilation operators. It is likely that other ways to introduce non-linearity into 3+1-dimensional quantum field theory will be invented in future. Section II describes some standard interacting QFT, then Section III presents the same material in a way that motivates the construction in Section IV. It seems preferable when modeling laboratory-scale Physics for us not to engage too heavily with extra dimensions or noncommutative and other relatively exotic geometry until non-linear constructions in 3+1-dimensional space-time are exhausted.

II.   SOME STANDARD QUANTUM FIELD THEORY

A well-known textbook way to construct an interacting scalar quantum field is to introduce a time-dependent transformation of a free quantum field $\hat{\phi}(x)$ [2, §6-1-1],

$$\hat{\xi}(x) \triangleq U^{-1}(x_0)\hat{\phi}(x)U(x_0) \qquad \text{where } U(\tau) \triangleq \mathsf{T}\left[e^{-\mathrm{i}\int\limits_{-\infty}^{\tau}\hat{H}(y)\mathrm{d}^4 y}\right],$$

$$= \mathsf{T}^{\dagger}\left[e^{-\mathrm{i}\int\limits_{-\infty}^{\infty}\hat{H}(y)\mathrm{d}^4 y}\right]\mathsf{T}\left[\hat{\phi}(x)e^{-\mathrm{i}\int\limits_{-\infty}^{\infty}\hat{H}(y)\mathrm{d}^4 y}\right], \tag{2}$$

where $\hat{H}(y)$ is a local operator, constructed as a sum of normal-ordered products of $\hat{\phi}(y)$ and its derivatives.

A standard first step is a heuristic derivation of the time-ordered Vacuum Expectation Values (VEVs) for the interacting field,

$$\langle 0|\,\mathsf{T}\left[\hat{\xi}(x_1)\cdots\hat{\xi}(x_n)\right]|0\rangle = \frac{\langle 0|\,\mathsf{T}\left[\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)e^{-\mathrm{i}\int\limits_{-\infty}^{\infty}\hat{H}(y)\mathrm{d}^4 y}\right]|0\rangle}{\langle 0|\,\mathsf{T}\left[e^{-\mathrm{i}\int\limits_{-\infty}^{\infty}\hat{H}(y)\mathrm{d}^4 y}\right]|0\rangle}, \tag{3}$$

which in turn allows the construction of Feynman integrals and the corresponding Feynman diagrams. The time-ordering of the left-hand side of this equation compromises its connection to the Wightman axioms, for which the foundation is a set of *non*-time-ordered VEVs that are distributions and that satisfy six sets of Conditions (Cluster Decomposition, Relativistic Transformation, Spectrum, Hermiticity, Local Commutativity, and Positive Definiteness), which would allow the use of the Wightman reconstruction theorem to construct a Wightman field [3, §3-4].

A standard alternative is to construct an S-matrix, which time-evolves vector-valued distributions such as $\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle$ (or the equivalent in wave-number coordinates), which form an improper basis for the free field Hilbert space of initial states, using the formal construction $\mathsf{T}\left[e^{-\mathrm{i}\int\limits_{-\infty}^{\infty}\hat{H}(y)\mathrm{d}^4 y}\right]$ as the evolution operator, then use the inner product with a final state, a vector in a Hilbert space that is unitarily equivalent to the free field Hilbert space of initial states. We obtain constructions such as

$$\langle 0|\,\hat{\phi}(y_m)\cdots\hat{\phi}(y_1)\mathsf{T}\left[e^{-\mathrm{i}\int\limits_{-\infty}^{\infty}\hat{H}(y)\mathrm{d}^4 y}\right]\hat{\phi}(x_1)\cdots\hat{\phi}(x)_n)|0\rangle, \tag{4}$$

which allow us to compute transition probabilities. This, however, also cannot be used to construct a Wightman field.

Neither the time-ordered VEVs construction nor the S-matrix construction is well-defined, because, at least, the integral $\int_{-\infty}^{\infty} \hat{H}(y)\mathrm{d}^4 y$ does not exist, however regularization and renormalization allows these and other constructions to make contact with experiment, which is unsurprisingly taken by most Physicists to be more important than making contact with the Wightman axioms.

III.   A DIFFERENT CONSTRUCTION

Here we directly construct non-time-ordered VEVs for the interacting quantum field by noting that the components of $U(\tau)$ that are space-like separated from x commute with $\hat{\phi}(x)$, so that the interacting field $\hat{\xi}(x)$ can also be written Lorentz invariantly as

$$\hat{\xi}(x) = \mathsf{T}^\dagger\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right] \hat{\phi}(x)\mathsf{T}\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right], \qquad \text{where } \mathcal{L}(x) \triangleq \int_{\blacktriangle(x)} \hat{H}(y)\mathrm{d}^4 y \tag{5}$$

and $\blacktriangle(x) = \{y : (y - x)^2 \geq 0 \text{ and } y_0 < x_0\}$ is the region of space-time that is light-like or time-like separated from and earlier than x. Furthermore, the adjoint action of $\hat{\phi}(x)$ on a time-ordered expression is a derivation, because time-ordering ensures commutativity, so that, taking a quartic scalar interaction with Hamiltonian density $\frac{\lambda}{4!}:\hat{\phi}^4(y):$ as an example,

$$\left[\hat{\phi}(x), \mathsf{T}\left[\left(\int :\hat{\phi}^4(y): \mathrm{d}^4 y\right)^n\right]\right] = \mathsf{T}\left[\int 4n\mathrm{i}\Delta(x - z):\hat{\phi}^3(z): \mathrm{d}^4 z \left(\int :\hat{\phi}^4(y): \mathrm{d}^4 y\right)^{n-1}\right],$$

where

$$\mathrm{i}\Delta(x - z) = -\mathrm{i}(G_{\mathrm{ret}}(x - z) - \mathrm{i}G_{\mathrm{adv}}(x - z)) = [\hat{\phi}(x), \hat{\phi}(z)]$$

is the free field commutator and $G_{\mathrm{ret}}(x - z)$ and $G_{\mathrm{adv}}(x - z)$ are the retarded and advanced Green functions [2, §1-3-1]. For the interacting field, therefore, we have the construction

$$\begin{aligned}
\hat{\xi}(x) &= \mathsf{T}^\dagger\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right] \hat{\phi}(x)\mathsf{T}\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right] \\
&= \mathsf{T}^\dagger\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right] \left(\mathsf{T}\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right]\hat{\phi}(x) - \mathsf{T}\left[\tfrac{\mathrm{i}\lambda}{3!}\int_{\blacktriangle(x)} \mathrm{i}\Delta(x-z):\hat{\phi}^3(z):\mathrm{d}^4 z\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right]\right) \\
&= \hat{\phi}(x) - \mathsf{T}^\dagger\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right] \mathsf{T}\left[\tfrac{\mathrm{i}\lambda}{3!}\int_{\blacktriangle(x)} \mathrm{i}\Delta(x-z):\hat{\phi}^3(z):\mathrm{d}^4 z\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right] \\
&= \hat{\phi}(x) - \tfrac{\mathrm{i}\lambda}{3!}\int_{\blacktriangle(x)} \mathrm{i}\Delta(x-z)\mathsf{T}^\dagger\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right] \mathsf{T}\left[:\hat{\phi}^3(z): \mathrm{e}^{-\mathrm{i}\mathcal{L}(x)}\right] \mathrm{d}^4 z, \\
&= \hat{\phi}(x) - \tfrac{\mathrm{i}\lambda}{3!}\int_{\blacktriangle(x)} \mathrm{i}\Delta(x-z)\mathsf{T}^\dagger\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(z)}\right] :\hat{\phi}^3(z): \mathsf{T}\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(z)}\right] \mathrm{d}^4 z,
\end{aligned}$$

$$
\begin{aligned}
&= \hat{\phi}(x) - \tfrac{\mathrm{i}\lambda}{3!} \int_{\blacktriangle(x)} \mathrm{i}\Delta(x-z) :\hat{\xi}^3(z): \mathrm{d}^4 z \quad \left[:\hat{\xi}^3(z): \triangleq \mathsf{T}^\dagger\!\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(z)}\right] :\hat{\phi}^3(z): \mathsf{T}\!\left[\mathrm{e}^{-\mathrm{i}\mathcal{L}(z)}\right]\right]\\
&= \hat{\phi}(x) - \tfrac{\lambda}{3!} \int G_{\mathrm{ret}}(x-z) :\hat{\xi}^3(z): \mathrm{d}^4 z,
\end{aligned}
\tag{6}
$$

where the restriction to the backward light cone $\blacktriangle(x)$ is equivalent to replacing the propagator $\mathrm{i}\Delta(x-z)$ by $-\mathrm{i}G_{\mathrm{ret}}(x-z)$. Insofar as we can take $:\hat{\phi}^3(z):$ formally to be an infinite multiple of $\hat{\phi}(z)$ subtracted from $\hat{\phi}^3(z)$, we can take $:\hat{\xi}^3(z):$ formally to be the same infinite multiple of $\hat{\xi}(z)$ subtracted from $\hat{\xi}^3(z)$.

$\hat{\phi}(x)$ satisfies the homogeneous Klein-Gordon equation, $(\Box + m^2)\hat{\phi}(x) = 0$, and $G_{\mathrm{ret}}(x-z)$ satisfies $(\Box + m^2)G_{\mathrm{ret}}(x-z) = \delta^4(x-z)$, so, generalizing from the quartic scalar interaction, $\hat{\xi}(x)$ satisfies the differential equation

$$
(\Box + m^2)\hat{\xi}(x) + :F(\hat{\xi}(x)): = 0,
\tag{7}
$$

where normal-ordering for $F(\hat{\xi}(x))$ is defined as it was for $\hat{\xi}^3(z)$, in terms of normal ordering for the free field.

The above construction shows that, apart from the mathematical necessity to regularize and renormalize, to construct an interacting field we replace $\hat{\phi}(x)$ at a point by a complex of operators at points of the backward light cone of x, using the propagator $G_{\mathrm{ret}}(x-z)$. Once we have constructed $\hat{\xi}(x)$ as a functional of $\hat{\phi}(z)$, using $G_{\mathrm{ret}}(x-z)$ to propagate to all points in the backward light cone, all contractions of the resulting objects, in accordance with Wick's theorem, use the positive frequency propagator $\mathrm{i}\Delta_+(x-y)$. This contrasts with the time-ordered VEVs of Eq. (3), which use only the Feynman propagator; and with the S-matrix construction of Eq. (4), which uses the positive frequency propagator for the operators used to construct the initial and final states, and uses the Feynman propagator between all pairs of points that occur within the time-ordering. Nonetheless, all three constructions are quite direct consequences of Eq. (2).

The distinction between the two propagators $\mathrm{i}\Delta_+(x)$ and $G_{\mathrm{ret}}(x)$, the first being confined to the forward light-cone in wave-number space and the second being confined to the forward light-cone in real space, gives an alternative to conventional discussions of virtual particles, in that we have used the virtual propagator $G_{\mathrm{ret}}(x)$ to construct an interacting field operator as a complex of free field operators that are confined to the backward light-cone. The free field operator products that are introduced by this construction using only $G_{\mathrm{ret}}(x)$ are then reduced by the vacuum state, in accordance with Wick's theorem, to give products of factors that use the positive frequency on-shell propagator $\mathrm{i}\Delta_+(x)$. This alternative discussion of course does not undermine conventional

discussions or computations, but it gives a relatively clear idea of an interacting field operator $\hat{\xi}(x)$ being associated with the backward light cone of x.

It is essential that time-ordering ensures that the complexes of operators in the backward light cones of two space-like separated points do not modify the Local Commutativity of the interacting field, because the product $\hat{\xi}(x)\hat{\xi}(y)$ can be written as

$$
\begin{aligned}
U^{-1}(x_0)\hat{\phi}(x)U(x_0)U^{-1}(y_0)\hat{\phi}(y)U(y_0) &= U^{-1}(y_0)U(y_0)U^{-1}(x_0)\hat{\phi}(x)U(x_0)U^{-1}(y_0)\hat{\phi}(y)U(y_0) \\
&= U^{-1}(y_0)U^{-1}(x_0,y_0)\hat{\phi}(x)U(x_0,y_0)\hat{\phi}(y)U(y_0).
\end{aligned}
\tag{8}
$$

When x and y are space-like separated, we can note that there is a choice of time coordinates in which $x_0 = y_0$, $U(x_0,y_0) \triangleq U(x_0)U^{-1}(y_0) = 1$, or we can, alternatively, note that for arbitrary time coordinates $U^{-1}(x_0,y_0)\hat{\phi}(x)U(x_0,y_0)$ replaces $\hat{\phi}(x)$ by a complex of operators at points of the backward light cone at x that are all at space-like separation from y. Hence, as for the free field $\hat{\phi}(x)$, $\hat{\xi}(x)$ satisfies $[\hat{\xi}(x),\hat{\xi}(y)] = 0$ when x and y are space-like separated.

Regularization and renormalization of this construction is as much a problem as it is for the conventional construction of time-ordered VEVs and of the S-matrix, however what we are constructing is somewhat clearer, and closer to the Wightman axioms in character, because we have largely eliminated the conceptual distraction of time-ordering.

IV. A NON-LINEAR CONSTRUCTION

As well as "smearing" with a test function, we may also take some motivation from signal analysis to construct a convolution of a quantum field with a smooth window function $w(\cdot)$,

$$
\hat{\phi}_w(x) \triangleq [\hat{\phi} \star w](x) = \int \hat{\phi}(x-y)w(y)\mathrm{d}^4y = \int \hat{\phi}(y)w(x-y)\mathrm{d}^4y,
\tag{9}
$$

which satisfies the linear differential equation $(\Box + m^2)\hat{\phi}_w(x) = 0$ and for which we have the commutation relations $[\hat{\phi}_{w_1}(x),\hat{\phi}_{w_2}(y)] = [w_1 \star \mathrm{i}\Delta \star w_2^{(-)}](x-y)$, where $w^{(-)}(z) \triangleq w(-z)$. We could equally well present this convolution in terms of smearing as $\hat{\phi}_w(x) \triangleq \hat{\phi}_{w_x}$, where we define the indexed set of test functions w_x as $w_x(y) \triangleq w(x-y)$, however Particle Physics has become something of an extreme exercise in processing large numbers of electronic signals, where window functions are more familiar than test functions.

A deformation that follows the construction of Section III requires $\hat{\xi}_w(x)$ to satisfy a non-linear differential equation that depends on the window function,

$$
(\Box + m^2)\hat{\xi}_w(x) + \frac{\lambda[w]}{3!}\left(\hat{\xi}_w(x)^3 - 3\mu[w]\hat{\xi}_w(x)\right) = 0,
\tag{10}
$$

where $\mu[w] \triangleq [\hat{\phi}_w(x), \hat{\phi}_w(x)] = [\hat{\phi}_w(0), \hat{\phi}_w(0)]$ so that the term $-3\mu[w]\hat{\xi}_w(x)$ has the same effect as normal-ordering, and the interaction functional $\lambda[w]$ should be manifestly Poincaré invariant. Given this construction, $\hat{\xi}_w(x)$ in general cannot be a linear functional of w_x. Solutions of this differential equation may be constructed perturbatively as

$$\hat{\xi}_w(z) = \hat{\phi}_w(x) - \frac{\lambda[w]}{3!}\int G_{\text{ret}}(x-z)\left(\hat{\xi}_w(z)^3 - 3\mu[w]\hat{\xi}_w(z)\right)\mathrm{d}^4 z. \tag{11}$$

When $w(\cdot)$ is not close to the Dirac δ-function, there is apparently no guarantee that $\hat{\xi}_w(x)$ commutes with $\hat{\xi}_w(y)$ when x and y are space-like separated enough to make the supports of $\hat{\phi}_w(x)$ and $\hat{\phi}_w(y)$ space-like separated, because we cannot use Eq. (8), but when $w(\cdot)$ approaches the Dirac δ-function, Eq. (10) approaches the conventional Eq. (7), with a potentially infinite $\lambda[w]$, and $\hat{\xi}_w(x)$ approaches the $\hat{\xi}(x)$ of Eq. (5).

The functional dependence of $\lambda[w]$ can be taken to be an explicit way to control the mathematics of regularization and renormalization, with the dependence on $w(\cdot)$ replacing the usual dependence on the renormalization scale; it is apparent that different functional properties of $\lambda[w]$ can be used to tune the variation of VEVs for different choices of w, and that modification of the functional dependence of the mass term $\mu[w]$ may also be useful. In contrast to the usual formalism, however, different window functions may be used for different measurements, representing different non-linear responses for different apparatuses.

V.   NON-LINEAR CREATION AND ANNIHILATION ALGEBRAS

Rather different non-linear constructions can be introduced if we note that the conventional GNS-construction of a Fock space that is based on a *-algebra of creation operators $a_{f_i}^\dagger$ and annihilation operators a_{f_i}, for some countable set of test functions $\{f_i\}$, depends only on the commutator matrix $[a_{f_i}, a_{f_j}^\dagger] = (f_i, f_j)$ being a positive semi-definite matrix. For the free field, this is ensured by (f_i, f_j) being a Gram matrix of the positive semi-definite inner product (f, g) on the test function space, which is diagonal in the wave-number basis, $[a(k), a(k')^\dagger] = (2\pi)^4\delta^4(k-k')2\pi\delta(k^2-m^2)\theta(k_0)$; the factor $\theta(k_0)$ implements the positive spectrum condition. We also require, for locality for a quantum field $\hat{\xi}_f = a_{f^*} + a_f^\dagger$, that $[\hat{\xi}_f, \hat{\xi}_g] = [a_{f^*}, a_g^\dagger] - [a_{g^*}, a_f^\dagger]$ is zero when f and g have space-like separated supports. Extending this construction to the non-linear case, we may also construct the commutator matrix $[a_{f_i}, a_{f_j}^\dagger]$ as a positive semi-definite matrix by introducing Hadamard products such as

$$[a_{f_i}, a_{f_j}^\dagger] = (f_i, f_j)_1 + (f_i, f_j)_2(f_i, f_j)_3 + (f_i, f_j)_4(f_i, f_j)_5 + ... + (f_i, f_j)_6(f_i, f_j)_7(f_i, f_j)_8 + ..., \tag{12}$$

for some set of positive semi-definite inner products $(f_i, f_j)_n$ that satisfy the positive spectrum condition, because the Hadamard product $(M \circ N)_{ij} = M_{ij} N_{ij}$ of positive semi-definite matrices is positive semi-definite. This construction also straightforwardly satisfies locality and the positive spectrum condition.

We may also introduce convolution powers of test functions, equivalent to simple powers in momentum space, such as

$$[a_{f_i}, a_{f_j}^\dagger] = (f_i, f_j)_1 + (f_i \star f_i, f_j \star f_j)_2 + ..., \tag{13}$$

which is positive definite because it is again a Gram matrix construction, and which also satisfies locality and the positive spectrum condition. This construction does not directly introduce interactions between different wave-numbers, but it may be combined with the use of Hadamard products as in the example given above, Eq. (12). Note that only convolution powers may be used, convolution polynomials in general result in constructions that are not translation invariant.

We might also introduce smooth functionals of test functions in real space, which would also be a Gram matrix construction and would also satisfy locality, however this construction results in contributions due to negative frequency components of the test functions, which might therefore be understood to contradict the positive spectrum condition. Questioning the positive spectrum condition is somewhat in conflict with the correspondence principle, insofar as frequency is generally taken to correspond linearly to classical energy, however the GNS-construction still results in a vacuum state.

All these constructions result in a vacuum state that is Gaussian, so there is a subsidiary need to construct non-Gaussian vacuum states, which can be achieved using the methods of Section IV, however we might also proceed by regarding a Gaussian state as a starting point for constructing other Lorentz and translation invariant states. The tools of this section can be applied equally well to fermionic quantum fields, albeit of course the corresponding vacuum state is not Gaussian.

Quantum field theory is a higher-order mathematical structure than classical field theory, even more so when non-linearity is introduced, so it seems possible that the construction given in this section may not be expressible in a Lagrangian formalism.

VI. DISCUSSION

There are some high-Mathematical postulates in the Wightman axioms that are given little notice by most Physicists, but that introduce quite forceful constraints on the overall mathematical structure. The postulate that the quantum field is a distribution is one, that the Hilbert space must

be separable is another, and that the Hilbert space must support a representation of the Poincaré group is another. The main suggestion here has been that we might more investigate non-linearity and not requiring the physical Hilbert space to support a representation of the Poincaré group, because the latter is not needed for the ways in which Physics is generally modeled using quantum field theory. Physics often uses only a limited range of frequencies and only requires covariance of a model, not that a model includes every possible frequency.

One aspect of the use in models of only a limited set of test functions is that the test functions that are used in constructing a quantum field model for a given set of experiments effectively determine a maximum length scale and a maximum energy scale. If we take those length scales to determine an infrared regularization and the renormalization scale, then we effectively make the construction of the interacting quantum field theory weakly non-linearly dependent on the whole set of test functions, which is somewhat comparable to the proposals above, provided the non-linearity that is introduced is weak enough. An alternative view of regularization and renormalization is that the auxiliary length scales that are introduced modify the space of test functions [4], so that the quantum field remains a distribution; this choice is effectively an additional postulate for the structure of quantum fields. The effects of Sections IV and V are to apply non-linearity to each individual observable and to pairs of observables, respectively, instead of introducing ill-defined non-linear differential equations such as Eq. (7) and non-linearly re-engineering their meaning by regularization and renormalization, dependent on length scales that are either introduced independently or that are determined by the whole set of test functions.

Regardless of the merits of the particular non-linear constructions that are suggested here, Lagrangian QFT introduces non-linear interaction terms into the Lagrangian, so it seems desirable to consider what form of non-linearity to introduce into the Wightman or Haag-Kastler axioms.

[1] R. F. Streater, *Rep. Prog. Phys.* **38**, 771-846(1975).
[2] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1985).
[3] R. F. Streater and A. S. Wightman, *PCT, Spin & Statistics, and all that* (W. A. Benjamin, New York, 1964).
[4] A. Neumaier, on-line answer http://physics.stackexchange.com/a/22524/1588 (accessed August 22, 2012).