A Redefinition of the Notion of Truth
FQXi Essay Contest 2020

Paul McKarris

April 23, 2020

Abstract

The Godel incompleteness metatheorem seems to bring a sense of discomfort. This discomfort appears to stem from the notion of truth that originates from model theory. This notion of truth is in some sense not precise. By accepting a simple statement, one can conclude that the notion of truth has to change. In this new view of truth, we will explore how physics could be done.

Preface

The ideas developed in this essay do not reflect my education or any person affiliated with me, those are all primary self-thoughts. I will assume the reader has some basic knowledge of logic, model theory and computability. Throughout the essay, references will be make to the appendices found at the end of the essay to make the text more readable for people that are already familiar with the subject. The essay begins with a short introduction to mathematical logic, model theory, computability; while stating important results from these subjects. I am taking advantage of the introduction to establish some definitions in **bold** that might differ from the reader's previous knowledge of the subject. After the introduction I argue why our "usual" notion of truth should be changed and how this all relates to physics.

1 Introduction

Mathematical logic was developed to formalize mathematical argumentation. Formalizing is reducing mathematics to manipulations of symbols in a language (syntax). A language is class of symbols. The word class is a collection of objects. We will not be using the word set to describe a collection of objects as it has a precise meaning later. In short to formalize mathematics, a language has to be chosen, from which we can write **strings**, which are concatenation of symbol from the language. We then have to prescribe a **deductive system** which is a class of strings together with deduction rules. Furthermore, we will have to prescribe another class of strings called **axioms**. In summary, we choose a language, deductive system H and an axiom Φ. We write $\Phi \vdash_H \varphi$, and say either "Φ **proves** φ" or "φ is a **theorem** of Φ in H" when there is a finite sequence of strings ending with the sentence φ where each step of the sequence is a string from the axioms, a string from the deductive system, or an application of a deductive rule applied to precedent steps in the sequence. For longer strings, we introduce a **definition**, which represents the string using a word.

To match what we do in mathematics to our intuition, we will fix the language to be first order logic. A **sentence** is a special type of string in first order logic which has no free variable. First order logic allows you to interpret sentences, in other words translate the sentence into a meta-language, like English (ordinary mathematics) and therefore do semantics. Hilbert system is a deductive system in first order logic that we will fix since when interpreted it matches what most people do when doing augmentations (sound system). For more details about first order logic and Hilbert system please refer to the appendix A. From now on, axiomatic systems will only consist of sentences, because they are interpretable. An axiomatic system Φ is **consistent** when there is no string ϕ such that $\Phi \vdash \phi$ and $\Phi \vdash \neg\phi$. I have dropped the H in $\vdash$ since the deductive system is fixed.

Now given an axiomatic system Φ, we write $M \vDash \Phi$, M **models** Φ, or M is a **model** of Φ or Φ is **true** in M when all sentences of Φ are interpreted as true in the structure M. This is done semantically, in line with how mathematical arguments are usually made. For more details about interpretation and model theory please refer to [1, 2]. M is thought as intuitively as a structure. For example, $\mathbf{N} = (\mathbb{N}, +, \times, 0, 1, <)$ is a model of Peano axioms PA (i.e $\mathbf{N} \vDash PA$).

We want to argue facts about formal proofs[1]. To capture the fact that a proof is a finite sequence with "primitive" steps, like the application of an algorithm, we will introduce the notion of **computable**. There is a type of function called **recursive function**, also called **computable functions**. Those functions postulated by the Church-Turing thesis capture what we mean by an algorithms; something that can be done by a computer (Turing machine). Given a sequence of sentences, checking that it is a proof can be done by a computer, and so it is computable. It has been shown (argued) that finding the proof of a sentence is not computable. Those type of facts are called **metatheorems**[2], they infer statements about the formal proofs. Please refer to [3, 4, 5] for more details on compatibility, recursive functions, Turing machines and Church-Turing thesis. A class of sentences is said to be **recursively axiomatizable** when an algorithm can check if a sentence is in the class or not. Another example of a metatheorem is the Godel incompleteness metatheorem which says:

> Given a recursively axiomatizable consistent axioms Φ which extends Peano axioms, there is a sentence ψ such that $\Phi \nvdash \psi$ and $\Phi \nvdash \neg\psi$.[3]

We will call such ψ to be an **independent sentence** of Φ. The statements seems restricted to only Peano axioms. But it will also be valid for any recursively axiomatizable consistent axiomatic system that has the capability of expressing Peano axioms with definitions and are consequences of that axiomatic system. For example this will also be applied to Zermelo–Fraenkel axiomatic system.

Now that we have a common ground for our discussion. This essay will first explore something that seem to be a paradox, which will lead us to rethink what we mean by truth. I will then give a new view about truth that will solve this "paradox". I will then explain how physics should be done in this new view.

2 Redefining Truth

For the sake of illustration, consider the true statement about the natural numbers, "All natural numbers bigger then 1 have there squares larger then itself". What a mathematician would do to confirm that the statement is true is assume some basic knowledge about the natural numbers[4](premises, dogmas) and then argument from the premises that the statement is true.

[1] I emphasize this sometimes by adding the word "formal" to remind the reader that this is syntax and not semantics. What is done semantically is referred to as argumentation.

[2] I want to differentiate between theorem, which are syntactic with formal proofs, and metatheorems, which are done semantically about the formal system.

[3] $\Phi \nvdash \psi$ means that there is no finite sequence that finishes with ψ, i.e not $\Phi \vdash \psi$.

[4] This could be Peano axioms interpreted

A mathematician would definitely not have a circular argument or regressive argument. We have to always start from dogmas, which are our premise. This can all be seen with Agrippa's trilemma[6]. You can convince yourself[5] that this whole argumentation of checking that a statement is true can be reduced to syntax. This would be done by translating the premises to be axioms, the arguments to be deductive system (Hilbert system or anything equivalent which captures our augmentations) and the statement to a sentence.

So the statements "All natural numbers bigger then 1 have there squares larger then itself" would be translated to $\forall x(1 < x \to x \times x < x)$. And if the statement is true assuming only the interpretation or Peano axioms, you should hopefully be convinced that $PA \vdash \forall x(1 < x \to x \times x < x)$.

Now observe that the consequent of the incompleteness metatheorem is false if we don't assume recursively axiomatizable axioms. To illustrate that, consider the $Th(\mathbf{N})$ which is the class of all true statement about $\mathbf{N}$ translated to first order logic. We have automatically for any ψ sentence, that $Th(\mathbf{N}) \vdash \psi$ or $Th(\mathbf{N}) \vdash \neg\psi$ because any statement is either true or its negation is true. So this doesn't satisfy the consequent of the incompleteness metatheorem. What this metatheorem implies is that there is no recursively axiomatizable consistent axiomatic system that capture all of $\mathbf{N}$. We have just seen in previous paragraph that every argument about the truth of a sentence can be done formally. So we have a formal system that mimics the semantics but has true statements that cannot be proved (Paradox?). Does that mean that to know all truths about the natural numbers we should not consider recursively axiomatizable axioms?

We could consider them but this would not be practical since we would have too many dogmas to justify all the truths about the natural numbers. This should also not be envisaged since we do not know which dogmas to include to match our intuition. It also doesn't seem right since whenever a mathematician argues about the truth of a statement, they don't assume a non-recursive axiomatizable class of basic knowledge about the natural numbers. Rather, they usually only assume the interpretation of Peano axioms or a recursively axiomatizable extension.

Another way to solve this "paradox" would be to maybe doubt[6] the legitimacy of the Godel incompleteness metatheorem since this metatheorem was proved using semantics. Semantics are based on our intuition and so how could we know that what we are doing makes sense? How could even talk about computable functions if we don't have sets? How can we know our arguments are consistent? If we start doubting our intuition too much then we wouldn't be able to write down anything, explain what a proof is or even explain what a language is. We would then fall into paradoxes such as "What the Tortoise Said to Achilles" by Lewis Carroll[7] or even worse follow Pyrrho in hardcore scepticism and its impediments. So I don't it is reasonable to doubt that much. Those methatheorems have some dimension of "truth", so maybe lets just doubt but less so.

What follows naturally from this discussion is thinking about what it means for a statement to be true. Consider this following statement:

> Every even integer greater than 2 can be expressed as the sum of two primes.

Let φ be the translated statement to first order logic. This translation is done in appendix B. What would a mathematician start by doing[7] to see if the statement is true? They would try to plug in a couple numbers and get some intuition to see if they can quickly find a counter example. If they find a counter example, then not every even integer greater than 2 can be expressed as the sum of two primes. And you can be convinced that $PA \vdash \neg\varphi$. Now lets say they don't find a counter example. In this case, the mathematician would start doubting

[5]It is unfortunate if you are not convinced as the essay relies on that.
[6]Lets be Descartes
[7]This is just me speculating

the truth of this statement. They would try to find arguments to explain why it is true, and would then realise that it is not easy. It is possible that the statement, also know as Goldbach's conjecture, is false but only for numbers larger than $4 \cdot 10^{18}$, [8]. We would never know if the statement is true or if its negation is true unless there is an argument or a counter example. Maybe the statement does not assume enough dogmas about the natural numbers. We have just seen that the fact that a statement is true relies on the existence of an argument, or a counter example. Both of which can be translated to a proof or proof of its negation[9]. Perhaps Goldbach's conjecture is one of the independent statements of Peano's axioms. It doesn't make sense to talk about truth in the absolute. In this view, something is true if it is provable. The notion of truth in models is too vague and relies too much on our intuition. So then notion of truth would make more sense if it were the same as provable. This causes a question arise, how can we talk about the natural numbers more presicely? Is there a better way to capture our intuitive understanding of the natural numbers, while continuing to be precise? We could define the natural numbers as a sentence in first order logic with more "powerful" axioms. Those axioms have to be able to be strong enough to define $\mathbf{N}$, $+$, $\times$, 0, 1, so that we can at least express Peano's axioms in those new axioms. An example of that would be Zermelo-Frankel axioms (ZF). We know from the incompleteness metatheorem that there are independent sentences in ZF. It may be that all sentences concerning the natural numbers are either provable or its negation provable[10]. Now let's see how this could all be related to physics.

3 Relationship to Physics

What do physicists do when they do physics? They want to explain and make predictions about the world. How do we explain something? They do it in a language like English, sign language or mathematics for example. When talking about the world we want it to be in a precise way such that there is no ambiguity and we also want it to be powerful enough. If we choose the English language, we will have a hard time expressing what the path of a particle is or the values of the electric fields; this is why we use mathematics and we have seen before that mathematics can be reduce to first order logic. Most people would use a Hilbert system or an equivalent system[11] as their deductive system since it matches well what we do when arguing. From that most people use ZF or ZFC as axioms because of their power to express most objects in physics. When we talk about expressing our world, we have to define what we mean by world? What we mean by that is physical notions we have by observing. For example it could be the universe, a quantum system, electromagnetic field, etc. The act of relating a physical notion to a string in some language is called a **postulate**. You would think that a postulate is the same as a definition but the difference is that with a postulate we have a physical notion for which we can go and do observations on. For example in General Relativity we have:

Postulate 1. ***Spacetime*** *is a 4-dimensional topological manifold with a smooth atlas carrying a torsion-free connection compatible with a Lorentzian metric and a time orientation satisfying the Einstein Field Equations.*

Notice that what is postulating Spacetime is an English sentence and not a sentence in first order logic. This is true, however each word of the postulate is defined in first order logic. If we would unravel the definitions in the postulate, we would have a very long string. The purpose

[8]It has been verified to be true up to that threshold. http://sweet.ua.pt/tos/goldbach.html

[9]This is what I asked you to believe earlier.

[10]This is a big speculation since I haven't found anything concerning it.

[11]Some people use weaker versions. Equivalent means that it is sound and complete.

of using definitions is practicality and readability. Of course a purist would not have definitions but we are human and it is easier for us when we group sentences together to form definitions.

When a physicists develops a new theory they will not write it down formally, however, in my opinion, a theory should have to capability to be expressed in this way. This would ensure that things are properly defined. We could also change our deductive system or axioms, as those are definitely not fixed. It would be hard to think of using another language then first order logic since as humans we need to be able to interpret the sentences.

4 Conclusion

This essay tried to convey that the way we argue the truth of a statement in a model is not precise enough. By accepting that any argument a mathematician would make could be completely mimicked in first order logic, we changed our notion of the truth to be provable. Since all models are sets, we can define them within ZF and talk about them within ZF. Physical notions can be postulated, in the same way models are define, as sentences in ZF. Coming back to our assumption about the arguments, there might be some arguments that cannot be written formally, like infinitely long proofs or sentences. This essay ignores this fact. Many texts explain infinitary logic but it never convinced me, since I think we should have the capacity write to everything down explicitly.

Appendices

A First order logic and Hilbert system

Those are the symbols of first order logic:

$$\neg, \rightarrow, \forall,), (, a, b, c..., A, B, ..., Z, x_1, x_2, .., y_1, ...$$

It may additionally include symbols depending on the axioms used. For example, in Peano's axioms there will additionally be $\times, +, 0, 1, \equiv, >$ and similarly you will find $\in$ in Zermelo-Frankel axioms. All symbols that are not $\neg, \rightarrow, \forall,), ($, or symbols from the axioms are called **variables**. The comma "," and the "..." are not part of the language and are simply there to differentiate symbols. The "..." indicates that any variable you can think of can be added to the language. With those symbols we can form a strings, which are concatenation of symbols, for example "$d\forall) \in ab \rightarrow hE$". **Sentences** are special type of strings that have no free variable, refer to [1, 2, 5] page 24 for explanations. I will usually denote sentences with Greek letter $(\varphi, \psi, \chi...)$, those letter should not be used in the language so that there is no ambiguity.

The Hilbert system consists of the following sentences:

- $(\varphi \rightarrow (\psi \rightarrow \varphi))$
- $((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)))$
- $((\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi))$
- $(\forall x\varphi \rightarrow \neg\forall x\neg\varphi)$
- $(\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x\psi))$

and the following inference rule

- modus ponens : $\varphi, (\varphi \rightarrow \psi)$ outlets ψ

So the five strings and modus ponens constitute a Hilbert system. A **definition** is a way to write a string in a shorter form. Common ones include:

Definition A.1.

- $\exists x$ is the string $\neg\forall x\neg$
- $(\phi \wedge \psi)$ is the string $\neg(\phi \rightarrow \neg\psi)$
- $(\phi \vee \psi)$ is the string $(\neg\phi \rightarrow \psi)$

Definitions should be new symbols from the language so that there is no ambiguity. So the axiom $(\forall x\varphi \rightarrow \neg\forall x\neg\varphi)$ will become $(\forall x\varphi \rightarrow \exists x\varphi)$. Definitions are just to abbreviate strings. I will also give an example of a theorem which is proved without using any axioms. Those are called **tautologies**.

Theorem A.1. $\vdash \phi \rightarrow \phi$

Proof.

1. $(\phi \rightarrow (\phi \rightarrow \phi \rightarrow \phi) \rightarrow (\phi \rightarrow \phi \rightarrow \phi) \rightarrow \phi \rightarrow \phi))$ (axiom)
2. $\phi \rightarrow (\phi \rightarrow \phi \rightarrow \phi)$ (axiom)

3. $(\phi \to \phi \to \phi) \to \phi \to \phi))$ (MP on 1 and 2)

4. $\phi \to \phi \to \phi)$ (axiom)

5. $\phi \to \phi$ (MP on 3 and 4)

□

For more further reading on the subject here some suggestions: [1, 2, 5, 8]

B Goldbach conjecture

The Goldbach conjecture states:

> Every even integer greater than 2 can be expressed as the sum of two primes.

To translate this into first order logic we will first consider the below definitions.

Definition B.1.

- $(\phi \wedge \psi)$ means $\neg(\phi \to \neg\psi)$
- $x|y$ means $\exists w(x \times w \equiv y)$
- $prime(x)$ means $(1 < x) \wedge \forall v((v|x) \to ((v \equiv 1) \vee (v \equiv x)))$
- 2 means $(1 + 1)$

So the Golbach conjecture in first order logic reads:

$$\forall x(((2|x) \wedge (2 < x)) \to \exists y \exists z((prime(y) \wedge prime(z)) \wedge (x \equiv (y + z))))$$

References

[1] H.-D. Ebbinghaus, J. Flum, and W. Thomas, *Mathematical Logic*. Springer, 2 ed., 1994.

[2] E. Mendelson, *Introduction to Mathematical Logic, Fourth Edition*. Discrete Mathematics and Its Applications, Taylor & Francis, 1997.

[3] C. H. Papadimitriou, *Computational Complexity*. Addison-Wesley, 1994.

[4] R. Soare, *Turing Computability: Theory and Applications*. Theory and Applications of Computability, Springer Berlin Heidelberg, 2016.

[5] S. Kleene and M. Beeson, *Introduction to Metamathematics*. Ishi Press International, 2009.

[6] B. Morison, "Sextus empiricus," in *The Stanford Encyclopedia of Philosophy* (E. N. Zalta, ed.), Metaphysics Research Lab, Stanford University, fall 2019 ed., 2019.

[7] L. Carroll, *What the Tortoise Said to Achilles*, (accessed 19-April-2020). https://en.wikisource.org/w/index.php?title=What_the_Tortoise_Said_to_Achilles&oldid=6456797.

[8] N. Bourbaki, *Éléments de mathématiques : Théorie des ensembles*. Springer, 2006.