Chaos and the Quantum:
Limits of Physics Involving the Physics of Limits

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1. Introduction

Quantum mechanics has always been considered to be the epitome of a linear science. However, because modern chaos theory effectively did not come into being until after Lorenz' meteorological discoveries concerning the "butterfly effect" [1], the founders of quantum mechanics did not have access to it or to much of nonlinear dynamics, so they were forced to fit quantum mechanics into a linear mold [2]. It is quite conceivable that much of the strangeness, many of the paradoxes inherent in quantum mechanics originate from forcing it to fit into such a linear framework. Nonlinear dynamics, especially modern chaos theory, had to await the advent of modern computers and computer graphics to make sense of the unwieldy wealth of numbers that can be so easily generated. As a result, it is only during the last few decades that chaos theory has come to the fore, yet it has successfully permeated most of science — from pure mathematics to chemistry and biology, even to economics and traffic patterns. Its most notable lack of success has been with quantum mechanics; in fact, some physicists question whether or not there is such a thing as "quantum chaos" [3].

Nonlinear behavior does occur in quantum systems, however, and there have been a number of attempts to explain it — however, with only very limited success. These have mostly been concerned with nonlinear perturbations on fundamentally linear systems, "weakly" nonlinear systems in which chaos cannot develop. Mielnik [4] sums up this situation nicely:

> I cannot help concluding that we do not know truly whether or not nonlinear QM generates superluminal signals — or perhaps, it resists embedding into too narrow a scheme of tensor products. After all, if the scalar potentials were an obligatory tool to describe the vector fields, some surprising predictions could as well arise! …the nonlinear theory would be in a peculiar situation of an Orwellian 'thoughtcrime' confined to a language in which it cannot even be expressed. … A way out, perhaps, could be a careful revision of all traditional concepts…

During the last few years increasing evidence has accumulated demonstrating that many of the so-called imponderables generated by the Copenhagen interpretation of quantum mechanics

have surprisingly similar parallels that can be generated by nonlinear dynamics and its extreme form, modern chaos theory [5,6]. Superficially, chaos theory is just as peculiar as quantum mechanics — that is, until one examines it closely, when its "weirdness" is seen to arise naturally out of feedback loops and the taking of limits. And it can arise only in "strongly" nonlinear situations [7], which means that it cannot be applied as a perturbation to linear quantum mechanics. What if quantum mechanics contains fundamental nonlinear, even chaotic elements? Chaos theory is fundamentally deterministic, yet because of extreme, exponential sensitivity to initial conditions (the "butterfly effect"), it must be interpreted statistically. It could provide a bridge between the determinism so dear to Einstein's heart and the statistical interpretation of Bohr. It is conceivable that the incompatibility of their arguments was an artifice — both Einstein and Bohr could have been right in their debates!

Chaos is ubiquitous. In nature nonlinearity and feedback are the rule rather than the exception. Modern deterministic chaos theory promises to change not only the way we do science, but also to change the way we perceive the world. Indeed, some of its adherents claim that it is the third pillar — along with relativity and quantum mechanics — of modern physics. Whether or not this is true is yet to be seen; yet physicists, especially quantum mechanicians, ignore it at their own risk. It is complex, messy, and nonintuitive: it delves into regions where our intuitions fail us just as badly as they do with quantum mechanics — areas such as self-similarity, self-affinity, and physical behavior when taking infinite limits. This essay raises questions rather than answers them. Its primary intent is to examine the question of compatibility between chaos theory and quantum mechanics. However this question has more far-reaching consequences than merely collecting examples of quantum paradoxes having nonlinear parallels. Ultimately, it questions the validity of our beloved linear models, raises doubts about reductionism itself, and even places limits on the validity of how we scientists are accustomed to thinking.

2. Peculiarities of Chaos Theory

2.1. Chaos Can Be Counterintuitive

Chaos is the unpredictable and apparently random behavior that can occur in *simple* nonlinear systems. It originates from within the system itself and is not the result of complex interactions or external influences, and it can arise whenever there is feedback and the system is at least quadratic in nature. Perhaps this can be explained most easily with the simplest example, the logistic map, which was originally studied as a simple model for biological populations:

$$x_{n+1} = Ax_n(1 - x_n)$$

Here x_n is the population of the present generation, x_{n+1} is the population in the next generation, and A is a control parameter, such as the birth rate. (All populations are assumed to be normalized between 0 and 1, with 0 representing extinction and 1, the maximum population, and the iterations are continued a large, effectively infinite number of times.) Without the term in parentheses, this equation is that of compound interest or Malthusian, exponential growth. However, the term $(1 - x_n)$ represents "room for growth," i.e., the difference between the present population and the maximum possible population. It can easily be seen that population growth is smallest both when x_n is close to 0 and when it is near its maximum of 1, and growth is greatest when it lies near 0.5. A value of A less than 1 obviously leads to extinction, whereas, as the birth rate increases above 1, the population growth accelerates. However, something peculiar happens when A reaches the value of 3, as can be seen in Fig. 1, where the ultimate values of x_n are plotted against A. Just above (even if only infinitesimally above) 3, the map bifurcates, with the final value x_∞ oscillating between two different points (period 2). As the value of A continues to
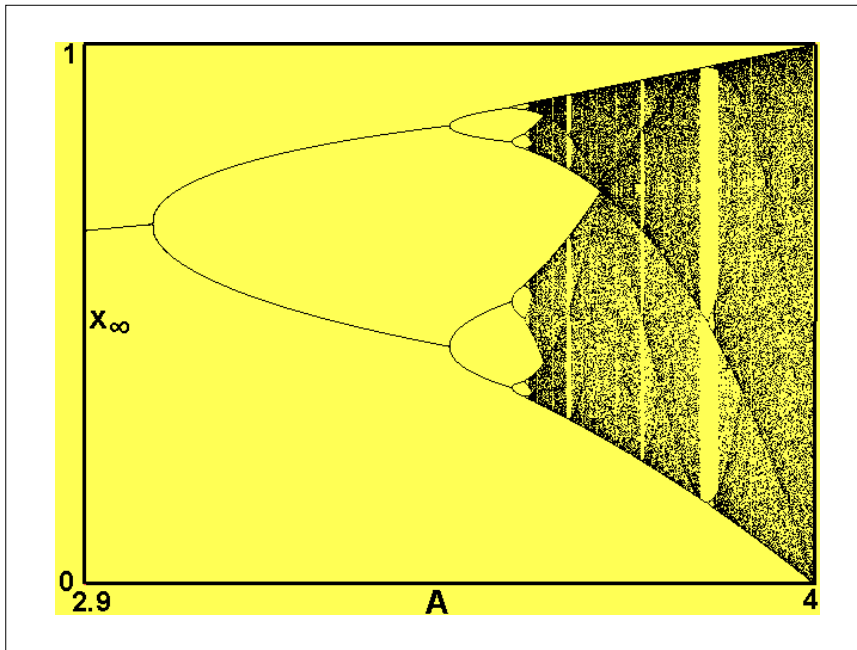

Figure 1

increase, further bifurcations occur, with periods 4, 8, 16, … Finally, for $A > 3.44948…$ chaotic behavior sets in, in which the map never settles down but continues to hit seemingly random values *ad infinitum*. (For values of $A > 4$ the map diverges, with all values heading toward $-\infty$.)

Several things should be noted: First, the equation is quadratic, just about the simplest quadratic equation one can construct; thus, it represents an exceedingly simple physical model having no interferences from the outside. Second, this bifurcation diagram is a fractal, and like most fractals it exhibits self-similar (more correctly, self-affine) behavior. If one blows up a portion of the diagram, the resulting figure is the same as the (here, mirrored) original diagram, and this continues *ad* infinitum. (Zooming in on the Mandelbrot set produces similar, perhaps more fa-

miliar, behavior.) Third, there are windows of stability (closed cycles) in the midst of chaotic behavior — the largest one is the period-3 gap near $A = 3.82$; again, this continues *ad infinitum*.

2.2. Properties of Nonlinear Dynamics and Chaos of Interest to Quantum Mechanics

Nonlinear dynamics and chaos theory produce many more or less nonintuitive effects, far too many to explain in this short essay. And although the literature on chaos theory is vast, books jump from the simple [8,9] to the advanced, with little in between; I have found the book by Hilborn [10] to be relatively accessible yet rigorous. Here I simply summarize a number of properties of nonlinear dynamics and chaos that could have relevance to quantum mechanical thinking.

1) **Innate or quantized modes.** Many *classical* but nonlinear systems exhibit preferred modes. Examples are the resonances in bridges (Tacoma Narrows or, more recently, the Centennial Bridge in London) and regularities in heartbeats and brain waves. Such systems can be represented by *eigenvalue equations*.

2) **Extreme sensitivity to initial conditions — the "butterfly effect."** This was demonstrated by the logistic map in the previous section, and it occurs throughout nature, ranging from weather predictions to predator-prey relations. A particularly clear example is a pendulum having an iron bob and and swinging over three magnets that attract the bob [6, 11].

3) **Basins of Attraction.** Oftentimes different starting conditions can lead to different innate modes. Sometimes these basins are so intimately mixed together that they can be termed "riddled," in which case infinitesimal differences can lead to widely different final values. Again, the pendulum swinging above three magnets illustrates this.

4) **Order in chaos.** The existence of regions of order (periodic behavior) intimately mixed in with chaotic behavior can lead to diffraction-like behavior. For example, chaotic scattering from three or more disks or spheres [12] shows not only extreme sensitivity to initial conditions, but also can produce patterns surprisingly resembling diffraction patterns.

5) **Spontaneous symmetry breaking — nonconservation of parity.** Many nonlinear systems governed by odd-order equations — e.g., cubic or sine maps — can spontaneously break their symmetry. A simple example is the separation of well-mixed powders into separate bands when subjected to nonlinear tumbling [13].

6) **Knife-edge stability of equilibrium states in conservative systems.** This involves the successive breaking of Kolmogorov-Arnol'd-Moser (KAM) tori in integrable Hamiltonian systems [12] and might be termed *a posteriori* extreme sensitivity to conditions.

7) **Emergent Behavior.** The self-organization of systems starting from highly-disorganized initial conditions. This is best exemplified by cellular automata and by evolutionary computer programs. A stunning presentation of emergent computation is given by Hillis [14], in which he shows that it is possible to evolve programs that perform tasks efficiently, at times more

efficiently than those produced by human programmers — but then it is impossible to analyze or understand just how such programs go about performing these tasks!

8) Non-additivity of correlated statistics. Many classical systems exhibit their own version of "entanglement." It turns out that non-ergodic behavior (i.e., preferential population of certain regions of phase space) can masquerade as action-at-a-distance. Among the best-studied of such effects is the idea of nonextensive thermodynamics, in which the entropies of two systems are not additive but contain an "interference" term" [15].

9) Global Interaction of Attractor Basins in Phase Space. This can easily lead to apparent action-at-a distance, and systems subject to it can be described by *fractional* calculus (which understandably is a field ripe for development). An excellent overview of these ideas can be found in the book by West, Bologna, and Grigolini [16].

10) Universality in Chaos. Many diverse chaotic systems, even from seemingly unrelated fields, have been found to obey identical laws. Feigenbaum termed this "universality" in chaos [17]. This has proved to be both a blessing and a curse: a blessing because chaotic systems fall into classes, meaning that one does not have to start from scratch for each new system; a curse because one cannot determine a specific mechanism for a given system just from knowing its quantitative behavior.

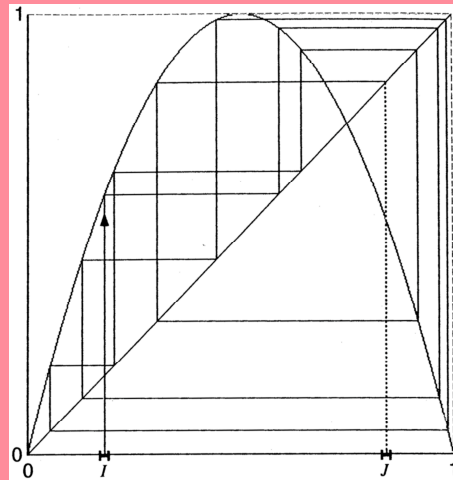## 3. Quantum Paradoxes Having Nonlinear Parallels

Obviously, this is an immense, complicated — and necessarily controversial — subject. Here I primarily raise questions to whet your appetite. For more details you can peruse my papers on the subject [5,6]. I would, however, like to present two relatively straightforward examples in slightly more detail.

3.1. Mocking up radioactive decay via a prisoner-escapee routine using the quadratic (logistic) iterator. One of the problems that has plagued scientists ever since radioactive decay was discovered is how to explain its exponential, statistical nature. It is impossible to determine when a given radioactive nucleus will decay; yet given enough nuclei to make a statistically significant sample, their decay follows a straightforward first-order exponential law.

A common explanation is to invoke an analogy with actuarial data, say, life-expectancy tables. Again, it is difficult to predict a particular person's lifetime, but insurance companies make their profits by relying on the precise results on life expectancies of large populations. However, this is a false analogy: A human population is quite diverse, and so are the causes of death. It seems clear that this is an exercise in complexity, where a myriad of interactions, many of them external, influence a complex system. On the other hand, to paraphrase Gertrude Stein, "A nu-

clide is a nuclide is a nuclide!" One of the fundamental tenants of quantum mechanics is that identical bodies or particles are truly identical.

Among the ideas proffered by the earlier detractors of the orthodox Copenhagen interpretation of quantum mechanics was the interaction of the microscopic world with an even smaller submicroscopic background [18]. One does not need to go this far, however, to find a possible parallel. Any uncertainty, *however minute*, in the in the initial conditions of a given nucleus could affect its subsequent behavior *provided the system is in a region of phase space subject to extreme sensitivity to initial conditions*. Let's give it a try. Following Ockham's Razor, we choose the simplest case, the logistic map, and perform a prisoner-escapee routine, as depicted in Fig. 2. Here 10,000 points were chosen randomly in an initial interval, $[0.2, 0.2 + 10^{-11}]$; these represent the initial state. The iterations then represent whatever physical process is necessary for the


Graphical iteration of the logistic map, demonstrating the "prisoner-escapee" procedure for generating an exponential decay law. Random points are selected from the initial interval *I*, which represents the initial decaying state. A vertical line up to the parabola locates the next point. A horizontal line over to the diagonal (where the ordinate equals the abscissa) takes this point into position for input into the next iteration. The procedure is continued until the trajectory "escapes" into the interval *J*, which corresponds to the final state. A record of the number of surviving states is kept, and these are plotted against the number of iterations to obtain an empirical decay curve.

Figure 2

decay to take place, e.g., hits at the Coulomb barrier for α decay or oscillations of an electromagnetic multipole for γ decay. The iterations were followed until they escaped into a final interval, $[0.53, 0.54]$, representing the final state. The results are shown in Fig. 3, where the number of survivors is plotted against the number of iterations. A well-defined exponential "decay curve" results from the procedure.

The same basic results can be obtained by iterating any unimodal map, another example of "universality" in chaos. Naturally, an analogy of this sort is far from a proof — something actually impossible to attain for any quantum mechanical phenomenon. Nevertheless, it illustrates that minute differences in initial conditions can lead to significant differences in trajectories in phase space, and consequently in behavior. Although this kind of argument provides no insight
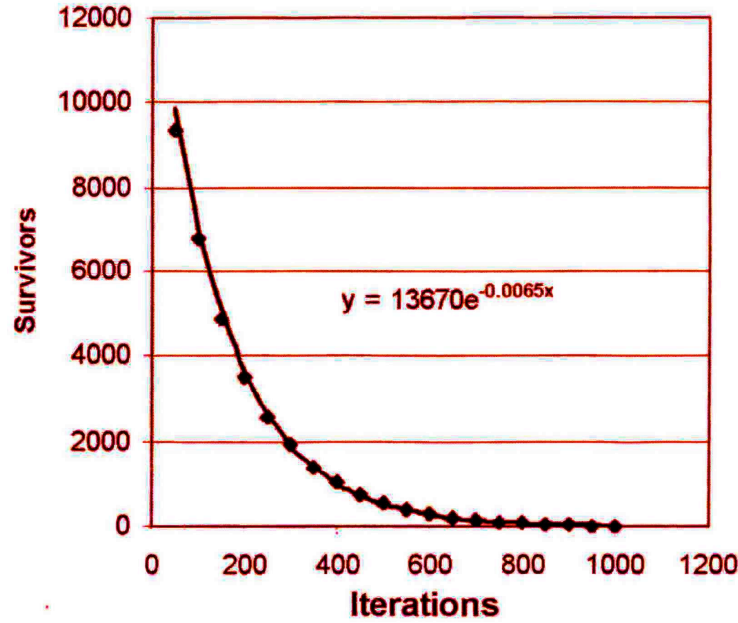
Exponential decay curve produced by iterating the logistic map with $A = 4$. To represent the initial state, 10,000 evenly-spaced points were chosen from the interval, $[0.2, 0.2 + 10^{-11}]$, and the resulting trajectories were followed until they escaped into the interval, $[0.53, 0.54]$, representing the final state. This plot of the number of survivors vs the number of iterations produces an empirical exponential decay curve having a half-life of about 107 iterations

Figure 3

into mechanisms, it does provide a parallel with first-order quantum transitions, whether they be radioactive decay or atomic or molecular de-excitations.

3.3 Bell-Type Inequalities and Conditional Statistics. Bell's theorem [19] and Bell-type inequalities lie at the heart of renewed interest in the foundations of quantum mechanics. Bell-type inequalities place limits on the statistical correlations between "entangled" pairs of particles generated in common (e.g., a singlet [spins antiparallel] pair of electrons) but whose properties (e.g., spins or polarizations) are measured "at a distance." These limits are derived for "classical" systems; quantum mechanics allows these limits to be exceeded under certain conditions. During the last several decades dozens of experiments have been performed, and they vindicate quantum mechanics. Conclusions drawn from these results usually involve statements declaring that "local reality" does not exist — Einstein's "spooky action-at-a-distance" does exist! (Signals travel faster than the speed of light.) For an excellent overview of the history, experiments, and philosophical interpretations resulting from the flurry of activity on Bell-type systems, read some of the essays in *Quantum [Un]Speakables* [20], particularly Chapter 6 by Clauser, which details the difficulties encountered by those who dared question the orthodox Copenhagen interpretation of quantum mechanics.

Let us examine the "classical" derivation of a simple, experimentally-friendly Bell-type inequality, the CHSH inequality [21]. A pair of particles is prepared, and one particle is sent to each

of the prototypical information-theory experimentalists, Alice and Bob, who are separated far apart and are effectively incommunicado. Alice is equipped to make measurements Q and R on her particles, each of which could result in an outcome of $+1$ or -1 — e.g., Q could be a measurement of spin or polarization with respect to a vertical axis, R with respect to a skewed axis. Similarly, Bob can make measurements S or T. Alice and Bob each choose which measurement to make at random, and they can decide on which measurement to make even after the particles have left their point of origin. After accumulating measurements on enough pairs to be statistically meaningful, Alice and Bob get together to compare results. They decide to compare the quantity,

$$QS + RS + RT - QT = (Q+R)S + (R-Q)T.$$

Note the minus sign on each side. Because Q and R independently can be $+1$ or -1, one or the other terms on the right side must be 0. Either way,

$$QS + RS + RT - QT = \pm 2.$$

When experimental efficiencies are included and speaking in terms of probabilities,

$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2,$$

where $E(QS)$ is the mean probability obtained for QS, with corresponding terms for the other pairs. This "CHSH inequality" puts an upper limit on the statistical correlations on a particular combination of products obtained by presumably independent measurements made at effectively infinite separation.

   Derivation of the quantum mechanical version is similar, except the particles are prepared in the explicit Bell singlet state,

$$|\Psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

where $|01\rangle$ means the first and second particle show spin down and up, respectively, and $|10\rangle$, vice versa. The follow-through is a bit more intricate than the classical derivation [21,22], but since I shall not be questioning it, here is the end result:

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2}$$

Here $\langle\cdots\rangle$ are the quantum mechanical expectation values. Quantum mechanics allows greater correlations than does "classical" mechanics, and the experiments have observed such increased correlations, vindicating quantum mechanics and squelching classical mechanics.

   Or do they? The quantum particles were clearly prepared in the "entangled" singlet state, and presumably the classical particles were also prepared in a correlated state; yet there is noth-

ing in the "classical" derivation above that prevents the classical state from being factored — and entanglement requires that a global state cannot be factored. Perhaps we are not comparing classical mechanics against quantum mechanics but independent probabilities against conditional probabilities.

Classical mechanics is rife with unexpected correlations. Many of these, such as correlations in wind velocities in tornadoes and hurricanes or correlations in cosmic ray distributions, have been studied in the context of nonextensive (Tsallis) entropy, as noted under point 8) in §2.2 above. Different subsystems of these classical systems do show correlations with other subsystems, and their entropies are not additive but contain an interference term, which can make the overall entropy greater (or less) than the sum of the individual entropies — in other words, the whole can be greater (or less) than the sum of its parts. This casts doubts on inequalities such as the CHSH inequality above, for it means that in *nonlinear* classical systems the upper limit on statistical correlations is too restrictive. And if the classical limit can increase into the region predicted by quantum mechanics, the use of such inequalities to rule out classical mechanics becomes ineffective or moot.

4. Conclusion: The Physics of Limits and Unexpected Correlations

Nonlinear dynamics can be every bit as counterintuitive as quantum mechanics, or for that matter, relativity. However, like relativity, once one examines nonlinear dynamics in detail, it becomes quite reasonable. And the existence of peculiarities generated by nonlinear dynamics and chaos that are peculiarly similar to so-called paradoxes produced by the Copenhagen interpretation of quantum mechanics — these should at the very least raise our eyebrows and make us question whether or not quantum mechanics is strictly linear.

Our brains are not innately wired to thinking naturally about infinite or infinitesimal limits, nor are we completely comfortable with conditional probabilities. We have progressed far since Zeno's paradoxes, yet nonlinear dynamics and chaos theory, although permeating almost every field of science, still has not hit the mainstream of physics or its textbooks. Perhaps we need a paradigm shift in our way of thinking.

To appreciate modern, deterministic chaos theory fully one must think in terms of taking limits. That is what the "butterfly effect," extreme exponential dependence (and divergence of trajectories in phase space) on initial conditions is all about. (In fact, one can even envision philosophical ramifications about questions such as free will: If chaos is deterministic, yet in the infinitesimal limit unpredictable — is this a sort of predestination that actually becomes free will?!)

We also tend to ignore unexpected correlations in classical systems — splitting a problem up into independent, linear parts makes computation so much simpler and neater. After all, nonlinear dynamics is inherently cumbersome and messy — almost no calculations can be accomplished in closed form. Yet nature is inherently nonlinear, as the biologists have led the way in

demonstrating. Physicists should be more wary of drawing far-reaching conclusions from strictly linear arguments.

A humorous yet serious example of the difficulties we have in thinking about probabilities is the Monty Hall problem. A contestant (on the old TV show, "Let's Make a Deal") has a choice of which of three doors to open. Behind one door is an automobile, and behind the other two are goats. When the contestant picks a particular door, which remains unopened, the host of the show opens one of the other doors to show a goat behind it. Now the contestant has the choice of remaining with his/her original choice or switching to the remaining unopened door, What should the contestant do to maximize the chances of winning the car? Almost everyone's first take is that it doesn't matter whether or not he/she switches — the chances remain at 1/3 for each door. To the contrary, the contestant doubles chances of winning by switching. Conditional probabilities are involved: the contestant's first choice limits the host's actions. Think about it, or even play a game about it by tossing coins — or go on-line to read about it [23]. The important point is that similar conditional probabilities come into play when considering the ramifications of Bell-type experiments.

Nonlinear dynamics and chaos may well change the way we think about nature. Ignoring nonlinear dynamics places undue limits on how we do science. Indeed, the limits of physics are intimately involved with the physics of limits and unanticipated correlations.

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