

The Distinct Nature of Physics and Cosmos

Thomas J. McFarlane
thomasmc@stanfordalumni.org

February 8, 2011

Abstract

The question of whether reality is necessarily continuous or discrete (i.e., analog or digital) is investigated by examining the nature of physics. It is argued that the view of physics as describing substance—common since ancient Greece—is today obsolete, and that modern physics is better understood as a way of describing reality as mathematical order. The question of whether reality is discrete or continuous is then reframed as a question of the nature of theories and the mathematics that they use. Because both measurement and theory are fundamentally grounded in discrete mathematical concepts based on distinctions, it is concluded that any description of reality by physics is necessarily discrete at its foundations. This conclusion points to a more fundamental insight into the nature of reality beyond the scope of physics.

Reconsidering the Nature of Physics

Among the most fundamental questions about reality are questions related to basic dichotomies. Indeed, the pre-socratic philosophers began rational speculation about reality by considering whether reality is fundamentally constant (Parmenides) or changing (Heraclitus), whether reality is ultimately reducible to one substance (Parmenides) or many (Democritus), and whether space and time are continuous or discrete (Zeno's paradoxes) [1]. Similar dichotomies also appear throughout the history of modern physics, such as whether light is composed of waves (Huygens) or particles (Newton) and whether space is absolute (Newton) or relative (Leibniz). The resolution of such basic questions in physics has always required a reconsideration of the fundamentals of physics itself. As Werner Heisenberg wrote,

The existing scientific concepts cover always only a very limited part of reality, and the other part that has not yet been understood is infinite. Whenever we proceed from the known into the unknown we may hope to understand, but we may have to learn at the same time a new meaning of the word "understanding." [2]

So, let us begin by examining the development of physics as a way of understanding reality. Modern physics traces its roots to the speculative theories of ancient Greek philosophers. Among these philosophers we can isolate two basic approaches. On the one hand, there were those such as Aristotle who developed theories about the fundamental substances of reality and their causal relations [3]. This was the approach that dominated the Western worldview until modern times. On the other hand, there were others such as Pythagoras and Plato who described reality as essentially mathematical. Rather than seeking a substance, they were looking for mathematical form.

Pythagoras, for example, famously declared that everything is number [1], while Plato described the fundamental nature of material elements in terms of geometrical forms [4].

It was not until the 17th century that modern physics as an empirical science began to develop, distinguishing itself from the speculative theories of the ancient Greeks through the systematic coupling of theory with experiment. The key to this development was the rejection of the dominant Aristotelian conception of physics as the study of substance and a shift in emphasis toward the mathematical approach of Plato: It was only by mathematically formulating a theory that it could be precisely connected with quantitative empirical measurements, thereby providing an effective and rigorous means to constrain speculative theories to those that conformed to such empirical measurements. As Heisenberg wrote,

Galileo turned away from the traditional science of his time, which was based on Aristotle, and took up the philosophical ideas of Plato. He replaced the descriptive science of Aristotle by the structural science of Plato. When he argued for experience he meant experience illuminated by mathematical constructs. Galileo, as well as Copernicus, had understood that...by idealizing experience, we may discover mathematical structures in the phenomena, and thereby gain a new simplicity as a basis for a new understanding. [5]

Today, mathematics is so essential to modern physics that a theory of physics without mathematics would be inconceivable. Ever since Galileo wrote that the book of nature is written in the language of mathematics [6], it has been axiomatic that any theory of physics must be formulated in mathematical terms. In fact, our current theories of physics do not describe independently existing substances but patterns of mathematical order. For example, the standard model of particle physics does not describe elementary particles as eternal atomic substances as Democritus did. Instead, it provides us with certain irreducible representations of symmetry groups that correspond to deep invariant patterns discerned from measurements. We are free to speculatively posit the independent existence of substances corresponding to these mathematical invariants, but such a philosophical overlay has no empirical content. Thus, modern physics describes mathematical form rather than substance, and it is more akin to Plato than Democritus [7, 8, 9]. At the basis of the modern approach to physics is not the search for an understanding of physical substance but the search for mathematical order subject to the constraints of empirical measurement. To again quote Heisenberg,

The ultimate root of appearances is therefore not matter but mathematical law, symmetry, mathematical form. ...Plato's notion makes a renewed entry into science, to the effect that a mathematical law, a mathematical symmetry, ultimately underlies the atomic structure of matter. [10]

Problems with Physics of Substance

Viewing physics as a description of the nature of substance has several additional problems. In fact, such a view of physics may well turn out to be impossible. One problem, for example, is that experimental violations of Bell's inequality [11, 12] have falsified any conception of physical substance that satisfies both 1) the principle of locality and 2) realism (i.e., the view that physical properties exist independent of measurement). The Kochen-Specker theorem also creates problems for realism [13]. Thus, while the mathematical formulation of quantum theory continues to

agree perfectly with empirical measurements, interpreting the mathematical formalism in terms of an underlying physical substance is problematic at best.

Even supposing it were possible to interpret the quantum mechanical formalism in terms of an underlying substance, there is still the problem of deciding which interpretation is correct. Any given mathematical formalism may be compatible with several different interpretations, and because empirical measurements only connect directly with the mathematical formalism, experiments do not provide a means to select between interpretations. Thus, the various interpretations of the quantum mechanical formalism have no basis in empirical measurements and are not strictly speaking part of the empirically-grounded formalism.

Another problem with viewing physics as a description of substance is that natural laws can be formulated either in terms of differential equations or in terms of stationary action integrals, but these formulations have very different interpretations. As Hermann Weyl describes,

The differential formulation corresponds to the causal conception according to which the state at one instant determines the change of state during an infinitesimal interval of time; the second, the integral formulation, savors of teleology. However, both laws are mathematically equivalent. Thus natural law is completely indifferent to causality and finality; this difference does not concern scientific knowledge, but metaphysical interpretation. [14]

Thus, the physics itself is contained in the mathematical formalism of the theory, not in some speculative philosophical interpretation of the formalism in terms of a substance and its causal relations. Such a philosophical interpretation of the formalism of physics is essentially non-empirical. It is an unnecessary appendix, a metaphysical residue of prescientific thought. Physics does not take us beyond the phenomenal or empirical and give access to the noumenal or the Kantian thing-in-itself [15]. Rather, it describes the mathematical order that is revealed by a particular method of viewing reality, namely, the method of empirical science. As Heisenberg wrote,

The measuring device has been constructed by the observer, and [so] we have to remember that what we observe is not nature in itself but nature exposed to our method of questioning. Our scientific work in physics consists in asking questions about nature in the language we possess and trying to get an answer from experiment by the means that are at our disposal. [16]

In view of the above, consider again the question of whether reality is fundamentally discrete or continuous. We are of course free to speculate, as the ancient Greeks did, as to whether the fundamental substance of nature is discrete or continuous, but that is a question for speculative philosophy rather than for physics. If we are to ask this kind of question in the context of modern physics, we should more properly ask whether the mathematics used in physics to investigate nature is necessarily discrete or continuous. This includes both the mathematics used to formulate physical theories as well as the mathematics used to make empirical measurements.

Mathematics and Physical Theories

Of course, the most fundamental physical theory is not known today, and perhaps it never will be. But even supposing such a theory were known, deciding whether it is truly discrete or continuous is complicated by the fact that the theory would not likely have a single unique mathematical formalism. Classical mechanics, for example, can be formulated in terms of Newton's laws of motion,

the Lagrange equations, Hamilton's equations, or the principle of stationary action. All of these are mathematically equivalent formulations of the same laws of classical mechanics. Similarly, quantum mechanics can be equivalently formulated in terms of Heisenberg's matrix mechanics, Schrödinger's wave equation, or Feynman's path integral. General relativity also can be formulated either in terms of Einstein's field equations or using a stationary principle. Consequently, there is no reason to expect that a fundamental theory of physics will have a unique mathematical formulation of its laws.

More generally, most mathematical structures do not have a single unique definition, but several different definitions that are mathematically equivalent. For example, the natural numbers have various equivalent but quite distinct definitions, including the Peano axioms [17], the von Neumann construction in set theory [18], and the classical set theoretic definition of Russell and Frege [19]. This type of equivalence of mathematical structure is made formal by category theory [20]. Because the nature of such equivalences is to preserve the essential structure of the formalism, we might expect that a discrete mathematical structure would not be equivalent to a continuous one. Indeed, Cantor established the nontrivial result that an uncountable infinite (e.g., the continuum of real numbers) is 'larger' than a countable infinite (e.g., the natural numbers) [21]. Thus, it seems that, at least in this respect, discrete mathematical structures are fundamentally distinct from continuous structures. Nevertheless, this type of distinction does not seem to provide a means to determine whether a theory of physics is essentially continuous or discrete.

Even though discrete and continuous mathematical structures have some fundamental differences, a discrete mathematical formalism in physics can be regarded as an approximation to a continuous one, and vice versa. For example, while the continuous equations of fluid dynamics are used to describe macroscopic behavior of fluids, a gas is more fundamentally described as discrete atoms [22]. The continuous formulation is used as an approximation to a deeper discrete structure. Conversely, a discrete structure can emerge as a description of what may have a deeper continuous structure. For example, ray optics, which can be considered a description in terms of trajectories of discrete particles, is an approximation to an underlying continuous wave optics [23]. So, underlying a discrete description in physics may be a deeper continuous description.

From considerations of a theory's mathematical formalism alone, it thus appears difficult to settle the question of whether a continuous or discrete representation is truly fundamental. Some additional considerations, however, may help bring some insight.

The Nature of Measurement

Together with mathematical theory, the second cornerstone of modern physics is empirical measurement. It is relevant, therefore, to ask whether our measurements are fundamentally continuous or discrete.

Often it is taken for granted that observable quantities in physics correspond to the continuum of real numbers. For example, the classical variables for position and time are normally considered to be real valued, and the eigenvalues of quantum mechanical observables are real valued. Empirical measurements, however, never directly produce irrational quantities since the measurement of an irrational quantity would require infinite precision. This fundamental limit of finite precision constrains any measurement result to a finite number of significant digits, i.e., to a rational number. For example, a meter stick with marks every millimeter can not produce a measurement result with more than three significant digits. Thus, the measurement of a classical

variable such as length always results in a rational number. Similarly, in the context of quantum theory, the measurement of an observable with a continuous spectrum will also have a result with finite precision.

Moreover, in quantum theory, any measurement operator can be decomposed into projection operators that represent yes-no propositions. The result of such an irreducible measurement of a system produces one bit of information [24]. Thus, the most basic measurement of a system is discrete. More generally, insofar as a measurement procedure can produce a definite result at all, it produces a finite amount of information which has a discrete representation.

In addition, because measurement procedures always involve one or more types of error, a measurement result is never an exact rational number but a rational number together with a measurement uncertainty (i.e., an estimate of the dispersion in the results of repeated measurements). For instance, a simple length measurement using a meter stick with marks every millimeter might produce a result such as 0.531 m with an uncertainty of 0.001 m. More generally, measurements result in a rational value together with a distribution representing the uncertainty. Thus, due to measurement uncertainty, it is impossible to measure a single exact value, whether irrational or rational.

Because the initial conditions, boundary conditions, or universal constants always involve errors which are propagated, all predicted measurements will have errors as well. To take a simple example, consider a square whose sides have a length of 1 m. Using the Pythagorean theorem, we would then predict the diagonal has an irrational length of $\sqrt{2}$ m. It is a non-physical idealization, however, to assume that the length of the sides is measured to be exactly 1 m. In fact, the two sides are measured with some standard deviation error, say 1 mm, giving the sides an actual measured length of $1.000 \text{ m} \pm 0.001 \text{ m}$. Propagating the errors, it follows that the predicted measurement of the length of the diagonal in meters is not exactly $\sqrt{2}$ but rather a distribution of rational numbers $\{x \in \mathbb{Q} : \sqrt{2} - 0.001\frac{\sqrt{2}}{2} \leq x \leq \sqrt{2} + 0.001\frac{\sqrt{2}}{2}\}$.

Since we never directly measure irrational quantities, or even single rational values, the real number continuum does not correspond to actual empirical measurements. Rather, the real numbers are used as a convenient calculating tool, similar to the use of complex numbers in the physics of AC circuits. Just as the complex representation of a sinusoidal signal in an AC circuit is very useful for calculations but is not considered physically real, the representation of physical quantities using real numbers is useful in physics for intermediate calculation but does not reflect the discrete physical results of empirical measurements.

Moreover, repeated measurements or sequences of measurements form a discrete set. We may, of course, conveniently represent this discrete data in continuous terms. For example, according to Shannon's sampling theorem, a sequence of discrete measurement samples can be represented completely and equivalently as a continuous bandwidth-limited signal [25]. Thus, although a discrete set of measurement results may be represented continuously to simplify calculation, such a continuous representation is not itself measured. Rather, the continuity is a convenient abstraction from the discrete empirical measurements.

Thus, the nature of measurement in physics is fundamentally discrete.

The Discrete Foundations of the Mathematical Continuum

Even supposing that the real number continuum were required by both measurement and theory in physics, the mathematics of the continuum itself is based on discrete concepts. The real number

continuum is traditionally constructed using countable Cauchy sequences of rational numbers or using Dedekind cuts of the countable set of rational numbers. In either case, the uncountable continuum is ultimately described in terms of countable (hence discrete) sets of rational numbers. Furthermore, the rational numbers are themselves based on pairs of natural numbers, and natural numbers in turn are formally constructed from a discrete successor operation, as in the Peano axioms [17], or from discrete sets, as in the von Neumann construction [18]. In the von Neumann construction of natural numbers, for example, the numbers 0, 1, 2 are defined in terms of pure sets as

$$\begin{aligned}0 &= \{\} \\1 &= \{\{\}\} \\2 &= \{\{\}, \{\{\}\}\}.\end{aligned}$$

Furthermore, a set is defined by specifying what is contained in the set, i.e., distinguishing the contents of the set from everything else. Thus, the continuum of the real numbers is ultimately defined in terms of the most basic discrete structure: making a distinction. So, because the mathematics of the continuum is ultimately rooted in the discrete concept of distinction, even if a fundamental physical theory required formulation in terms of the real number continuum, it would still be discrete at its foundation, as would be the measurements that connect the theory to empirical observation.

Reality and Physics

By definition, the measurements and mathematical formulations of physics describe only that aspect of reality that is capable of being characterized in terms of distinction and order. Thus, the Greek word for order, *cosmos*, can be used to refer to this aspect of reality that is revealed as order. Physics may then be defined as a way of viewing reality as order, as a cosmos.

According to this view of the nature of physics, traditional notions of independently existing substance are obsolete and unnecessary. The nature of modern physics reveals by its very definition a cosmos, but it does not thereby compel any philosophical interpretation of this invariant order in terms of independently existing substance. Insofar as the distinctions we use to describe order are free imaginative constructs, they are not so much properties inherent in reality itself, but the basic elements that make it possible to characterize and describe a cosmos at all. We may then redefine objectivity in purely mathematical terms, without any implication of an independently existing substance. As formulated by Hermann Weyl,

Objectivity means invariance with respect to the group of automorphisms. [26]

With objectivity defined as mathematical structures invariant to certain group transformations, the meaning of an *objective cosmos* can be freed from philosophical assumptions of an independent metaphysical substance. Instead, an objective cosmos is simply defined as that aspect of reality that is revealed when seeking order in the form of mathematical invariants in empirical measurements. With these definitions, the unreasonable effectiveness of mathematics in the physical sciences is no longer a mystery [27]. It is only unreasonable if one posits the existence of a substance independent of our theories and modes of measurement.

Physics may thus be viewed as a kind of theoretical and empirical lens that is ultimately built from discrete distinctions, and reveals by its very nature discrete order. Because physics is based

on discrete mathematical concepts, in both its theory and measurement, it is fundamentally discrete in its description of reality. We may thus conclude that the description of reality through the lens of physics will necessarily reveal structures that are fundamentally discrete in nature. The cosmos, as the mathematically invariant order that is imaged by this lens, is thus also necessarily discrete.

The cosmos described by physics, however, is a characterization of only that aspect of reality which is revealed when we look through the lens of discrete mathematical concepts which are all traced back to the primordial act of making a distinction. There is still—lest we forget—that aspect of reality that is *not* revealed as order. This aspect may be called the *complement of the cosmos*. Because the cosmos is discrete, this suggests that its complement is a continuum—not the mathematical continuum which has definite structure, but an indefinite continuum, a formless void (i.e., the original meaning of the Greek word *chaos*) that lacks any order and is thus beyond comprehension in terms of concepts or distinction.

Reality in its totality, then, encompasses both the cosmos (order) and its complement (chaos). But, more fundamentally, it is prior to the even distinction between cosmos and chaos, form and formlessness, discrete and continuous. Its ultimate nature is therefore ineffable, beyond the scope of mathematics, physics, and even thought itself, which depends on making distinctions. Insofar as it can be known at all, it must be known through other means.

References

- [1] Guthrie, W. D. C. *A History of Greek Philosophy: Volume 1, The Earlier Presocratics and the Pythagoreans*, (Cambridge University Press, 1962).
- [2] Heisenberg, Werner. *Physics and Philosophy: The Revolution in Modern Science*. (New York: Harper & Row Publishers, 1962). p. 201.
- [3] Aristotle, *Physics*. tr. R. P. Hardie and R. K. Gaye. (Oxford: The Clarendon Press, 1930).
- [4] Cornford, F. M. *Plato's Cosmology: the Timaeus of Plato, Translated with a Running Commentary*. (Indianapolis: Hackett Publishing Company, 1997).
- [5] Heisenberg, Werner. *Encounters with Einstein*. (Princeton: Princeton University Press, 1989). p. 8.
- [6] Galileo Galilei, "The Assayer." Published in *Discoveries and Opinions of Galileo*. (New York: Doubleday & Company, 1957).
- [7] Heisenberg, Werner, *Across the Frontiers* (Woodbridge, Connecticut: Ox Bow Press, 1990). p. 110.
- [8] Heisenberg, Werner, *Encounters with Einstein*. (Princeton: Princeton University Press, 1989). p. 17.
- [9] Tegmark, M. (2008) "The Mathematical Universe." *Foundations of Physics* 38: 101150.
- [10] Heisenberg, Werner. *Across the Frontiers*. p. 10-11.
- [11] J.S. Bell, *Physics* 1, 195-200 (1964).
- [12] Aspect et al., *Phys. Rev. Lett.* 49, 1804 (1982).
- [13] S. Kochen and E.P. Specker, "The problem of hidden variables in quantum mechanics", *Journal of Mathematics and Mechanics* 17, 5987 (1967).
- [14] Weyl, Hermann. *The Open World*. (Woodbridge, Connecticut: Ox Bow, 1989). p. 37.
- [15] Kant, Immanuel. *The Critique of Pure Reason*. tr. Norman Kemp Smith. (London: Macmillan, 1929).
- [16] Heisenberg, Werner. *Physics and Philosophy*. p. 58.
- [17] Giuseppe Peano, *Arithmetices principia, nova methodo exposita* (Bocca, Turin, 1889).
- [18] von Neumann, "On the Introduction of Transfinite Numbers, published in *From Frege to Gödel: A source book in mathematical logic, 1879-1931*. ed. Jean Van Heijenoort (Harvard University Press, 1967).
- [19] Russell, Bertrand, *Principles of Mathematics* (W. W. Norton & Company, 1903).
- [20] Mac Lane, Saunders, *Mathematics, Form and Function* (Springer-Verlag, 1986).
- [21] Cantor, Georg (1874), "On a Property of the Collection of All Real Algebraic Numbers", published in Ewald, William B., ed., *From Immanuel Kant to David Hilbert: A Source Book in the Foundations of Mathematics* (Oxford University Press, 1996).
- [22] Landau, L. D.; Lifschitz, E. M., *Fluid Mechanics. Course of Theoretical Physics, 2nd ed.* (Pergamon Press, 1987).
- [23] Sommerfeld, A., Runge, I. (1911). Anwendung der Vektorrechnung auf die Grundlagen der geometrischen Optik. *Annalen der Physik* 35, 277-299.
- [24] Zeilinger, A., "A Foundational Principle for Quantum Mechanics", *Foundations of Physics*, vol. 29, p. 631 (April 1999).
- [25] Shannon, C. E. "Communication in the presence of noise", *Proc. Institute of Radio Engineers*, vol. 37, no.1, pp. 10-21, Jan. 1949.
- [26] Weyl, Hermann. *Symmetry*. (Princeton, N.J.: Princeton University Press, 1952). p. 132.
- [27] Eugene Wigner (1960). "The Unreasonable Effectiveness of Mathematics in the Natural Sciences". *Communications on Pure and Applied Mathematics* 13 (1): 114.