Contradictions, mathematical science and incompleteness

Abhishek Majhi

"The very possibility of mathematical science seems an insoluble contradiction."

H. Poincare, *Science and Hypothesis* (1905)

I. MY SCHOOL DAYS AND ARITHMETIC

During my school days, my father used to tell me that, "Arithmetic is the hardest of all disciplines of mathematics." I did not understand at that time the depth of those words. Arithmetic gradually became a lesson of pure mathematics only. Algebra, geometry and especially calculus became important for learning physics. Geometric explanation of calculus, e.g. the explanation of "instantaneous velocity" by the drawing of a tangent to a curve with a pencil on a paper, made so much sense in solving problems in physics that I never felt the lack of arithmetical accuracy.

Nonetheless, I suppressed a disturbing 'childish nuisance' in my mind that "the chord becoming a tangent only depends on the sharpness of my pencil". But, I had no other option than drawing with some pencil on a paper to express a geometric thought. Also, I wondered – if the accuracy of geometry depends on the sharpness of the drawing object, then why does such information not appear in the corresponding equations? Acceptance of such incompleteness in my understanding, only seemed practical at that time because it allowed me to qualify my exams. The psychological discomfort due to my 'childish nuisance' decayed to passivity as I advanced to college education and so on.

However, now I find no shame in confessing that I have never really understood the definition of derivative in calculus [2], although I applied such tools, quite comfortably, in physics so as to build a career in a discipline of science, based on such acceptance and belief. So, was my act of belief and acceptance just an unacceptable contradiction, since I was pursuing science where each and every step needed to be understood by questioning and through reasoning? Or, was my act legitimate, in spite of being contradictory, because of its practicality? I believe, some contradictions are useful because these set the premise of some action and need not be resolved. However, some contradictions are fatal and need to be resolved.

According to the general standpoint all contradictions are unacceptable and need to be resolved. However, then I wonder, how has it gone unnoticed by so many of my respected teachers and examiners that I have built my career on contradictions? The only answer that I can think of is that, I and my respected teachers and examiners must have shared some common premise that is founded on contradictions. Hence, they were unable to judge me otherwise. Interestingly, this common premise is the collection of *languages* that I use to express my thoughts on *mathematical science*, namely, English grammar, pure mathematics and some non-mathematical symbols of expression (called units). Now, I realize that what I write down by the name "mathematical science", is founded on *contradictions* and resolving the ones fatal for theoretical exposition can solve some hard problems like singularity resolution of gravity, leading to an "asymptotically safe" theory. The problem lies in the misunderstanding of "unit"(quantity) and "unity"(number) and in the belief that "scientific truth is unassailable"[27] or infallible or *exact*.

II. UNITS, MEASUREMENTS AND INEXACTNESS

Measurement is the founding stone of scientific beliefs. At least, I feel confident of my knowledge of mathematical science, when I find that what I can say about what can be measured or observed, tallies (albeit approximately, always) with the explanations provided by the *calculations* performed with pure mathematics along with the *physical interpretations* given in terms of *units*. And, it is due to this confidence that I call these 'scientific beliefs' as 'scientific facts' instead.

It appears to me that the only difference between physics and pure mathematics are the different kinds of units which are used to provide physical interpretations. Calculations belong to pure mathematics and physics is explained in terms of units corresponding to physical dimensions like length, time, etc. For example, when I write, "1 metre + 1 metre = 2 metre", I certainly perform the calculation "1+1=2". Length dimension implied by the expression "metre" (unit) does not play an active role in the process. Also, when I say that "1 metre = 100 centimetre", I express the comparison of two length units in terms of numbers. Therefore, what I understand from this observation is that a unit, while compared to another unit, corresponds to some number giving the comparative smallness, largeness or equality. When the comparison yields the equality, the notion of unity or the number "one" (e.g. Arabic-Hindu numeral "1") is used to express it; while the unit is an expression of physicality (symbolically expressed as metre, second, etc.), the number represents the expression of comparison between two units (e.g. 1 km = 1000 metre). Thus, "unit" by itself is not the number "one" or "unity" [3].

When *exact* equality is *impossible*, which is always the case in actual practice, I need access to smaller units, until my practical purpose is fulfilled [5]. For example, if I am a tailor and making measurement to make a trouser, I am only concerned about the accuracy up to centimetre at most; but, when I am doing the double slit experiment in lab, accuracy up to millimetre is the least requirement. Nevertheless, the thickness, of the marks on measuring tape or of the cross wire of the eye piece, is absolutely necessary for measurement to be possible. But the thickness itself is not measurable in the process i.e. measurement performed on a *base* (premise) which itself is not measurable. Thus, the base on which measurement is made, acts as an *immeasurable*[1], making the outcome inexact. Hence, I consider it as my inability to perform measurement without an error.

So, after choosing a convenient unit for measurement, say λ_0, I express the measurement of an extension (length, breadth, etc.) as $e = (N_e + \delta_e)\lambda_0$ such that $0 < \delta_e < 1$ and $N_e = 0, 1, 2, \cdots$. $e = n_e\ \lambda_0$, where $n_e := N_e + \delta_e$. Thus, $N_e \geq 1 \Rightarrow e > \lambda_0$, $N_e = 0 \Rightarrow e < \lambda_0$ and the ideal case of equality is not achievable because $\delta_e \neq 0$. Ideal or exact numbers occur in ideal thoughts of pure mathematics and not in the practical world. I call n_e as practical number which reveals comparison between two units e and λ_0 of same physical dimension. If $n_e \ll 1$ or $n_e \gg 1$, then I conclude that $e \ll \lambda_0$ or $e \gg \lambda_0$ respectively; statements like "$n_e \to 0$", "$n_e \to \infty$" make no sense and hence, illogical.

III. CONTRADICTION IN EXACT MATHEMATICAL SCIENCE

Unfortunately, physics is plagued with such illogical statements e.g. one writes for Newton's law of gravitation

$$\lim_{r \to 0} \frac{Gm_1 m_2}{r^2} = \infty. \tag{1}$$

The question here is not that whether there is a technique that can solve the problem of a divergent physical quantity. Rather, the issue is regarding the logical validity of the statement "$r \to 0$". Certainly, r represents the length, measured with a chosen length unit λ_0, of a geometric line drawn between two dots on the paper to represent the distance between two "point masses" m_1 and m_2. r has the form $r = n_r\lambda_0$ where n_r is a real positive number. The reason for writing "$r \to 0$" is to enquire what the theory indicates when the two dots overlap and looks like one dot i.e. drawing of a geometric line becomes impossible.

An attempt to make the statement "$r \to 0$" look logical can possibly be "$r = 0\lambda_0$ as $n_r \to 0$" and also one may provide a justification that what this means is simply what one calls "a point" by putting a dot of the pencil on a paper. And then, of course, it is also supported by the statement that "a point has zero length dimension". Now, such a statement is trivially deceptive because I need to be able to see the dot on the paper so as to claim that there is "a point". However, this implies that the dot certainly has an extension by virtue of which I can say that it has some "negligible" length dimension. Thus, if I can see a dot and yet I say that it has "zero" length dimension, then it appears to me as a blatant lie or a fatal *contradiction*. If I stick to the statement of "zero length dimension", which implies "no extension", then I can not put a dot on the paper to represent "a point". And, if there is no dot on the paper, geometric explanation of physics and any physical interpretation based on it is impossible.

I would rather humbly accept my inability to do exact mathematical science and allow practical inexactness to consider "a point" as having a negligible length dimension and write $r > s \Rightarrow n_r >$

[1] It is the irremovable error that makes measurement possible.

$\epsilon := s/\lambda_0$ such that $0 < \epsilon \ll 1$, where s represents the minimum extension needed to express further i.e. to draw. Starting from such "a point", when I draw a line on the paper that represent some number n_r, it is erroneous up to ϵ that represents the extension of the drawing object (viz. tip of the pencil) compared to the chosen length unit for measurement.

One considers an object 'as a whole' while describing its motion. Then $\epsilon \ll 1 \Rightarrow s \ll \lambda_0$ where s represents some intrinsic length unit corresponding to the existence of the object irrespective of its shape, size, etc. This is why putting $\epsilon = 0$ is equivalent to saying that "the object does not exist" and hence, no observation. In that way, the characteristic extension of the tip of the pencil, signifying the existence of the tip of the pencil, is the representative of the intrinsic length unit of the object, signifying the existence of the object. Newton did put $\epsilon = 0$ and built science on a fatal contradiction. That is why he needed "mass" to express existence of an object – a concept that has always been the source of conceptual difficulties [25, 27]. Such fatal contradictions still remain unresolved and not even realized. I can safely call such a belief system as *Newton's legacy.*

IV. CONTRADICTIONS AND INCOMPLETE STATEMENTS IN CALCULUS

Calculus, as applied to physics, is fraught with such contradictions owing to its reliance on the geometric foundations and also, based on incomplete statements. Let me start by quoting Cauchy (see page 11 of ref.[2]) to note how he defined, what is known today as, "derivative of a function" – "$\cdots$ *function* $y = f(x) \cdots$ *variable* $x \cdots$ *an infinitely small increment attributed to the variable produces an infinitely small increment of the function itself.* $\cdots$ *set* $\Delta x = i$, *the two terms of the ratio of differences*

$$\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}. \tag{2}$$

will be infinitely small quantities. $\cdots$ *these two terms indefinitely and simultaneously will approach the limit of zero, the ratio itself may be able to converge toward another limit,...*"

The mathematical statements corresponding to Cauchy's "infinitely small quantities", in modern notation, look like "$\Delta x \to 0, \Delta y \to 0$" in the above quoted example. Since Δx is a quantity (or unit), but "0" is just a number (not a quantity or unit), I can not understand how to make sense of such statements.

Nevertheless, it is crucial to note that both "$\Delta y \to 0$" and "$\Delta x \to 0$" are necessary to define the derivative. Therefore, expression (2) *should* look like

$$\frac{dy}{dx} := \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i} \qquad \ni y = f(x), \Delta x = i. \tag{3}$$

The symbol "$\ni$" stands for "such that" and ":=" stands for "defined as". Then, considering Δx as the distance traversed by an object during the time elapse Δt, the "instantaneous velocity" should be defined as

$$v \equiv \frac{dx}{dt} := \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\Delta x}{\Delta t} \ \ni \Delta x := x(t + \Delta t) - x(t) \tag{4}$$

where the symbol "$\equiv$" stands for "equivalence".

Certainly, as I have explained before, both the statements "$\Delta x \to 0$", "$\Delta t \to 0$" are contradictions if I try to represent them geometrically by drawing diagrams. More importantly, Cauchy's quoted statements are trivially *incomplete* because of the phrase "infinitely small quantities". As I understand, the word "quantity" is used for an expression of "physicality" and not any number. For example, "some quantity of rice" by itself is not any number, but it is expressed as a number while compared to another quantity, say, "kilogram", "gram", etc. So, "quantity" is just "unit" and two units of same physical dimension yield some number on comparison. Therefore, "infinitely small quantities" is a meaningless and *incomplete* phrase that acquires meaning only through a *completion* such as "infinitely small quantities *compared to* another quantity" [9].

Albeit needlessly I note in the passing that the foundations of differential geometry laid down by Riemann in ref. [23] inherits Cauchy-like incomplete statements.

V. CALCULUS WITH COMPLETE STATEMENTS – CONTRADICTIONS RETAINED

I complete the incomplete statements as follows. Let me choose the units for length and time measurements as λ_0 and T_0 respectively. I write $\Delta x = \Delta n_x \lambda_0$, $\Delta t = \Delta n_t T_0$. Then

$$\text{“}\Delta x \text{ is infinitesimally small compared to } \lambda_0\text{”} \equiv \text{“}\Delta x \ll \lambda_0\text{”} \equiv \text{“}\Delta n_x \ll 1\text{”}, \tag{5}$$

$$\text{“}\Delta t \text{ is infinitesimally small compared to } T_0\text{”} \equiv \text{“}\Delta t \ll T_0\text{”} \equiv \text{“}\Delta n_t \ll 1\text{”}. \tag{6}$$

Then,

$$\text{condition } (5) \Rightarrow 0 < \epsilon < \Delta n_x \ll 1 \tag{7}$$

$$\& \quad \text{condition } (6) \Rightarrow 0 < \tilde{\epsilon} < \Delta n_t \ll 1. \tag{8}$$

Such considerations would resolve the contradictions, but it would render the definition of derivative inapplicable to write down any theory of physics because exact “zero” (i.e. $\epsilon = 0 = \tilde{\epsilon}$) is a necessity for continuity [2]. Therefore, at the moment I retain the contradictions and deliberately make the following *illogical* replacements:

$$\text{“}\Delta n_x \ll 1\text{”} \xrightarrow{r.b.} \text{“}\Delta n_x \to 0\text{”}, \tag{9}$$

$$\text{“}\Delta n_t \ll 1\text{”} \xrightarrow{r.b.} \text{“}\Delta n_t \to 0\text{”}. \tag{10}$$

The symbol “$\xrightarrow{r.b.}$” denotes “replaced by”. In a nutshell, now I use the following *illogical equivalence* (denoted by the symbol “$\stackrel{?}{\equiv}$”):

$$\text{“}\Delta n_x \to 0\text{”} \stackrel{?}{\equiv} \text{“}\Delta x \ll \lambda_0\text{”}, \tag{11}$$

$$\text{“}\Delta n_t \to 0\text{”} \stackrel{?}{\equiv} \text{“}\Delta t \ll T_0\text{”}. \tag{12}$$

If the above conditions hold, along with the illogical notations, “instantaneous” velocity can be defined using Cauchy’s definition of derivative as follows

$$\boxed{v \equiv \frac{dx}{dt} := \left.\frac{\Delta x}{\Delta t}\right|_{\substack{\Delta x \ll \lambda_0 \\ \Delta t \ll T_0}} = \left(\lim_{\substack{\Delta n_x \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_x}{\Delta n_t}\right) \frac{\lambda_0}{T_0} = n_v v_0 \ \ni n_v := \lim_{\substack{\Delta n_x \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_x}{\Delta n_t} \ \& \ v_0 := \frac{\lambda_0}{T_0}.} \tag{13}$$

I am considering only positive numbers here for the sake of simplicity i.e. $\Delta n_x, \Delta n_t, n_v$ are real positive numbers.

Now, I consider $\Delta v = \Delta n_v v_0$. Then, “instantaneous” acceleration (a) can only be defined when

$$\Delta v \ll v_0, \tag{14}$$

$$\Delta t \ll T_0. \tag{15}$$

Then, using illogically equivalent statements “$\Delta n_v \to 0, \Delta t \to 0$”, I define “instantaneous” acceleration, according to Cauchy’s prescription, as follows:

$$\boxed{a \equiv \frac{dv}{dt} := \left.\frac{\Delta v}{\Delta t}\right|_{\substack{\Delta v \ll v_0 \\ \Delta t \ll T_0}} = \left(\lim_{\substack{\Delta n_v \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_v}{\Delta n_t}\right) \frac{\lambda_0}{T_0} = n_a a_0 \ \ni n_a := \lim_{\substack{\Delta n_v \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_v}{\Delta n_t} \ \& \ a_0 := \frac{v_0}{T_0}.} \tag{16}$$

Condition (14) is necessarily satisfied if and only if $v < v_0$. However, there can be many choices of v_0 for which such condition can be valid. So, I need to postulate some $c_0 \ni v_0 < c_0$ for any chosen v_0, which is certainly a statement about my ability to choose measuring units. In a nutshell, the necessary conditions for “instantaneous” acceleration to be defined are

$$\boxed{\Delta v \ll v_0, \quad v < v_0 \quad \ni v_0 < c_0.} \tag{17}$$

Now, to define “force (f)”, I need to define “instantaneous” rate of change of momentum (dp/dt) so as to be able to write $f \propto dp/dt$ following Newton. I consider momentum ($p = n_p p_0$) to be a quantity in its own right, where p_0 is the chosen unit for momentum measurement. I denote a change in p as $\Delta p = \Delta n_p p_0$, where Δn_p is a real positive number and p_0 is the chosen unit for measurement of momentum. Then, following necessary (illogical) arguments, as the prior cases, I write

$$\boxed{\frac{dp}{dt} := \frac{\Delta p}{\Delta t}\bigg|_{\substack{\Delta p \ll p_0 \\ \Delta t \ll T_0}} = \left(\lim_{\substack{\Delta n_p \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_p}{\Delta n_t}\right) \frac{p_0}{T_0} = n_f f_0 \ \ni n_f := \lim_{\substack{\Delta n_p \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_p}{\Delta n_t} \ \& \ f_0 := \frac{p_0}{T_0}.} \tag{18}$$

Then, following similar arguments as prior cases inclusive of a postulate of some $P_0 \ \ni p_0 < P_0$, the conditions for the derivative of momentum to be definable are the following:

$$\boxed{\Delta p \ll p_0, \quad p < p_0 \quad \ni p_0 < P_0.} \tag{19}$$

If I want to define the derivative of force, then there needs to be a postulate of some $F_0 \ \ni f_0 < F_0$. More implications can be found in ref.[30].

Remark: The reader may be tempted to identify the postulate of c_0 with that of Einstein's postulate of a velocity upper bound c, called "velocity of light in vacuum", that led to relativity[11]. In that case, I would like to remind that the theory of electromagnetic fields that describes "light" and any mathematical explanation of the word "vacuum", are based on calculus founded upon Cauchy's incomplete statements, which look quite ugly if one argues that science is a *logical* exposition of observed phenomena. Here, the postulate of c_0 is necessary to define acceleration using calculus with complete and meaningful statements. It is rooted to the understanding of the difference between quantity (unit) and number (unity). To understand how c_0 is related to Einstein's c, a restructuring of mathematical science is indispensable. However, in doing so one should resolve the fatal contradictions which I have still retained. Unfortunately, such resolution makes the definition of derivative inapplicable in physics.

VI. RESOLVING CONTRADICTION: TOWARDS NON-SINGULAR GRAVITY

I have already pointed out that retaining the ϵ-s in the conditions (7) and (8), would resolve the fatal contradictions and also render the illogical replacements (9) and (10) unnecessary and definition of derivative inapplicable to physics. Also, I have mentioned why Newton needed the concept of "mass" as a consequence of putting $\epsilon = 0$ in the relation $s = \epsilon\lambda_0$ in the concluding paragraph of section (III). Nevertheless, to draw the connection with existing literature and conventional wisdom, I shall retain the concept of "mass" and manifest its relation with the intrinsic length units for the existence of objects. In what follows, I set the prelude to an inexact mathematical science that is non-singular, logical and practical at its core, owing to such ϵ-s. I believe the subtle essence of arithmetic over geometry should be manifest.

Newton's law of gravitation is a statement about the mutual force between two objects, having a separation or distance. The velocity unit c_0 lets me express that statement as follows

$$F = F_0 \frac{s_1 s_2}{d^2} \quad \ni F_0 := c_0^4/G, \ s_1 := Gm_1/c_0^2, \ s_2 := Gm_2/c_0^2, \tag{20}$$

where d represents the thought of distance that needs to be given a description by drawing a line or, as Newton would write, "*Geometry does not teach us to draw these lines, but requires them to be drawn*"[2]. But, I can only draw a line by joining two dots if and only if they do not overlap. For overlapping dots, geometric description is impossible, but it is not a catastrophe. This is what I show next, followed by the geometric nature of Newton's law of gravity [14].

The crucial issue is how I can express d. The length units available are s_1 and s_2 which are intrinsic to the theory. Also, there is the unit chosen for measurement or making observation i.e. λ_0. Now, the involvement of two length units s_1 and s_2 gives rise to a problem because when I compare them with d, the two different units give rise to two unities ("one"-s) i.e. $d = (N_1 + \delta_1)s_1 \ni N_1 = 1_1, 2_1, 3_1, \cdots ; 0 < \delta_1 < 1_1$ and $d = (N_2 + \delta_2)s_2 \ni N_2 = 1_2, 2_2, 3_2, \cdots ; 0 < \delta_2 < 1_2$. Therefore,

[2] See the *Preface to the First Edition* on page no. - xvii of ref.[13].

no arithmetic operations are possible because there is no meaning of writing "$1_1 + 1_2$" [12]. This necessitates the proposition of an *undecidable* length unit (ℓ_u) by virtue of which any notion of extension can be expressed as some real positive number i.e. $s_1 = \alpha_1 \ell_u, s_2 = \alpha_2 \ell_u, d = \alpha_d \ell_u, \lambda_0 = \alpha_{\lambda_0} \ell_u$ such that $\alpha \gg 1$ for all the α-s[3]. This lets me write down

$$d > s_i \equiv \alpha_d > \alpha_i \Rightarrow \frac{s_i}{d} = \frac{\alpha_i}{\alpha_d} = \frac{1}{(X_i + 1)} \quad \ni X_i = (N_i - 1) + \delta_i \quad \forall i \in [1, 2]. \tag{21}$$

$$\therefore \quad \frac{s_i}{d} \equiv \frac{\alpha_i}{\alpha_d} = \sum_{n=0}^{\infty} (-X_i)^n \ni s_i < d < 2s_i \equiv \alpha_i < \alpha_d < 2\alpha_i \quad \textbf{[Small Distance Expansion (SDE)]}, \tag{22}$$

$$\& \quad \frac{s_i}{d} \equiv \frac{\alpha_s}{\alpha_d} = \frac{1}{X_i} \sum_{n=0}^{\infty} (-X_i^{-1})^n \ni d > 2s_i \equiv \alpha_d > 2\alpha_i \quad \textbf{[Large Distance Expansion (LDE)]}, \tag{23}$$

where the words "large" and "small" carry a clear and obvious comparative sense. Now, an object considered "as a whole" is represented by putting a dot on the paper and called "point mass"[15]. The depiction of any geometric curve with that dot represents the motion of the object from the observer's perspective and this is known as "point (or classical) mechanics"[15]. I represent this dot, or "object as a whole", by the following condition:

$$s_i \ll \lambda_0 \Rightarrow s_i = \epsilon_i^{\lambda_0} \lambda_0 \equiv \alpha_i = \epsilon_i^{\lambda_0} \alpha_{\lambda_0} \ni \epsilon_i^{\lambda_0} \ll 1 \quad \forall i \in [1, 2]. \tag{24}$$

Thus, from the observer's perspective, I can write

$$\mathbf{O}_{\lambda_0} \left[\frac{s}{d} \right] := \frac{\epsilon_i^{\lambda_0} \alpha_{\lambda_0}}{(\alpha_d/\alpha_i) \epsilon_i^{\lambda_0} \alpha_{\lambda_0}} = \frac{\epsilon_i^{\lambda_0}}{(X_i + 1) \epsilon_i^{\lambda_0}} = \frac{\epsilon_i^{\lambda_0}}{x_i^{\lambda_0} + \epsilon_i^{\lambda_0}} \quad \ni x_i^{\lambda_0} := X_i \epsilon_i^{\lambda_0}. \tag{25}$$

where "$\mathbf{O}$" stands for "observed". So, from the observer's perspective, the SDE and the LDE can be written as

$$\textbf{Observer's SDE:} \quad \mathbf{O}_{\lambda_0} \left[\frac{s_i}{d} \right] = \frac{\epsilon_i^{\lambda_0}}{x_i^{\lambda_0} + \epsilon_i^{\lambda_0}} = \sum_{n=0}^{\infty} \left(-\frac{x_i^{\lambda_0}}{\epsilon_i^{\lambda_0}} \right)^{-n} \ni x_i^{\lambda_0} < \epsilon_i^{\lambda_0} \iff d < 2s_i, \tag{26}$$

$$\textbf{Observer's LDE:} \quad \mathbf{O}_{\lambda_0} \left[\frac{s_i}{d} \right] = \frac{\epsilon_i^{\lambda_0}}{x_i^{\lambda_0} + \epsilon_i^{\lambda_0}} = \frac{\epsilon_i^{\lambda_0}}{x_i^{\lambda_0}} \sum_{n=0}^{\infty} \left(-\frac{\epsilon_i^{\lambda_0}}{x_i^{\lambda_0}} \right)^{-n} \ni x_i^{\lambda_0} > \epsilon_i^{\lambda_0} \iff d > 2s_i. \tag{27}$$

Here the symbol " $\iff$ " stands for "if and only if". Now, I write eq.(20) from the observer's perspective

$$\frac{F}{F_0} = \mathbf{O}_{\lambda_0} \left[\frac{s_1}{d} \right] \cdot \mathbf{O}_{\lambda_0} \left[\frac{s_2}{d} \right] = \frac{\epsilon_1^{\lambda_0}}{x_1^{\lambda_0} + \epsilon_1^{\lambda_0}} \cdot \frac{\epsilon_2^{\lambda_0}}{x_2^{\lambda_0} + \epsilon_2^{\lambda_0}}. \tag{28}$$

Two cases follow immediately through some elementary calculations. Henceforth, I drop the λ_0-superscript for convenience of representation.

A. Two overlapping dots

Let me consider the following conditions:

$$0 < x_1 < \epsilon_1, 0 < x_2 < \epsilon_2 \iff s_1, s_2 < d < 2s_1, 2s_2 \Rightarrow \frac{1}{2} < \frac{d}{s_1 + s_2} < 1 \equiv \frac{1}{2} < \frac{\alpha_d}{\alpha_1 + \alpha_2} < 1. \tag{29}$$

If follows from eq.(28) that

$$\left. \frac{F}{F_0} \right|_{ASDE} = 1 - \left(\frac{r_1}{s_1} + \frac{r_2}{s_2} \right) + \left(\frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} + \frac{r_1 r_2}{s_1 s_2} \right) - \cdots \tag{30}$$

$$= 1 - \frac{c_0^2}{G} \left(\frac{r_1}{m_1} + \frac{r_2}{m_2} \right) + \frac{c_0^4}{G^2} \left(\frac{r_1^2}{m_1^2} + \frac{r_2^2}{m_2^2} + \frac{r_1 r_2}{m_1 m_2} \right) - \cdots, \tag{31}$$

---

[3] A bit more elaboration on ℓ_u is available in End Note.

where

$$r_i := x_i\lambda_0,\ Gm_i/c_0^2 = s_i = \epsilon_i\lambda_0 \quad \forall i \in [1,2]. \tag{32}$$

Here, "ASDE" stands for Absolute SDE. Eq.(30) or eq.(31) has no geometric interpretation. The situation describes the case of two overlapping dots on the paper. That is, no separation is realized between the two dots so that a line can be drawn to give a geometric interpretation with some "coordinate distance r", which is indeed a necessity even before one talks about "coordinate invariance"[7]. I *wonder* whether, in course of addressing my 'childish nuisance', I have shown that gravity is "asymptotically safe"[16], albeit with some ridiculously simple calculations that, I believe, can be taught to an undergraduate student.

B. Universal free fall: an artefact of geometric triviality

For the following condition:

$$\epsilon_1 < x_1, \epsilon_2 < x_2 \iff 2s_1, 2s_2 < d \quad \equiv \quad 2\alpha_1, 2\alpha_2 < \alpha_d, \tag{33}$$

it follows from eq.(28) that

$$\left.\frac{F}{F_0}\right|_{ALDE} = \frac{s_1 s_2}{r_1 r_2}\left[1 - \left(\frac{s_1}{r_1} + \frac{s_2}{r_2}\right) + \left(\frac{s_1^2}{r_1^2} + \frac{s_2^2}{r_2^2} + \frac{s_1 s_2}{r_1 r_2}\right) - \cdots\right]. \tag{34}$$

Here, "ALDE" stands for Absolute LDE.

In general, then s_2 and s_1 can have a non-trivial relationship depending on d. However, here I wish to show under what strict assumptions Newton's law of gravitation follows. So, I *assume*

$$s_2 = n_0 s_1 \ \equiv \ \alpha_2 = n_0\alpha_1 \ni n_0 > 1 \Rightarrow d > 2s_2 > 2s_1. \tag{35}$$

Then, instead of eq.(34), I obtain:

$$\left.\frac{F}{F_0}\right|_{ALDE} = \frac{s_2 s_1}{r_1^2}\left[1 - \frac{2s_1}{r_1} + \frac{3s_1^2}{r_1^2} - \cdots\right] \tag{36}$$

$$\Rightarrow \quad F|_{ALDE} = \frac{Gm_2 m_1}{r_1^2}\left[1 - \frac{G}{c_0^2}\frac{2m_1}{r_1} + \frac{G^2}{c_0^4}\frac{3m_1^2}{r_1^2} - \cdots\right] \tag{37}$$

where obviously $m_2 = n_0 m_1$. Now, it is interesting to note that the sub-leading terms of eq.(37) become more and more suppressed as $d \gg 2s_1$. Therefore, the condition for only the leading term to become practically relevant is $d > 2s_2 \gg 2s_1$. Then, I can neglect the sub-leading terms of eq.(36) and write

$$F|_{ALDE} \approx F_0\frac{s_1 s_2}{r^2} = \frac{Gm_1 m_2}{r^2} \quad \ni s_2 \gg s_1 \equiv m_2 \gg m_1 \tag{38}$$

where I have deliberately written "r" in place of "r_1" to emphasize the *triviality of geometry*. In other words, this leading term does not carry the information regarding the test mass (m_1 in this case), like geometry does not carry the *non-trivial* information of the drawing object.

Also, this is the essence of universal gravitational free fall i.e. all test objects fall with same acceleration. It is quite easily understandable, just by looking at eq.(37), that an attempt to define the gravitational field due to the source object (m_2) as $F|_{ALDE}/m_1$ gives an expression that still depends on m_1. The dependent terms account for the "back-reaction" of the test object. However, with the restrictions from which eq.(38) follow $F|_{ALDE}/m_1$ yields the usual Newtonian gravitational field that does not depend on the test object.

Therefore, Newton's theory of universal gravitational free fall [14], that founds the basis of Einstein's equivalence principle[7], is an artefact of the triviality of geometry and the corresponding experimental verifications have been made in such a regime that the higher order corrections are unobservable. It remains a question though, whether such terms play significant roles in the galactic scenarios where neither Newtonian gravity nor Einsteinian general relativity works e.g. the unexplained anomalous rotation curves in galaxies. Such observations led to the "missing mass" problem which can be comfortably named as one of the most acute crises in modern physics [17].

VII. CONCLUDING REMARKS AND MY INCOMPLETENESS – "I"

Whether it is the symbolic expression "$s = \epsilon\lambda_0 \ \ni 0 < \epsilon \ll 1$" corresponding to the statement "an object as a whole" (represented by the dot of the pencil on the paper) for the resolution of a fatal contradiction or the completion of Cauchy's incomplete statements to justify the applicability of calculus in physics, it depends on me that how I describe the observed phenomena by choosing the appropriate units like λ_0, T_0, etc. restricted by my ability to make such choices. Therefore, it appears to me that the description of reality in mathematical science is just a manifestation of human choices, which is apparently against Einstein's philosophy of "impersonal" truth[6] as he wrote on page no. 2 of ref.[6], *"We are accustomed to regard as real those sense perceptions which are common to different individuals, and which therefore are, in a measure, impersonal. The natural sciences, and in particular, the most fundamental of them, physics, deal with such sense perceptions. The conception of physical bodies, in particular of rigid bodies, is a relatively constant complex of such sense perceptions."* Einstein's principle of general covariance is the mathematical manifestation of his philosophy about truth and reality that should be "impersonal". However, Einstein could not express in mathematical terms his concern about the "measure" dependence because he developed on the science that Newton wrote based on fatal contradictions and Cauchy's calculus founded upon *incomplete* statements (inherited by Riemann too [23]). At the moment, excluding this essay, physics is based on Einstein's philosophy of "impersonal" (universal, fundamental) truth and reality and yet plagued with the monumental problems like singularity, missing mass, etc. Considering this essay included, the singularity problem has been resolved and the necessity of concept of "mass" has been questioned along with an explanation of why Newton needed to invoke it in the first place due to a misunderstanding of a fatal contradiction. It appears to me the truth in neither impersonal nor personal but a subtle admixture of both that needs to be understood and described according to situation i.e. how the observer relates to the observed. When I make the choices of the measuring units, I certainly make these choices of making observations and describe the reality of an object accordingly. For example, I can tune λ_0, if I am able, such that ϵ becomes greater than 1 and then certainly I am now asking a different question that what the object is made up of. So, I may call the reality described in this way as relational and practical.

Nevertheless, contradictions can be both useful and fatal. The fatal ones need to be resolved and the useful ones need to be used, as I have done. Let me explain and conclude. I began with writing about my father's arithmetic nitpicking. Now, let me state that it was my mother who pushed me to go through the education system by accepting the contradictions and ignoring my 'childish nuisance'. Can you *decide* which one of the two views is more "fundamental"? I can not decide, because I have used both the views to write this essay and to become able to write this essay. Therefore, I believe in the *practical* and not in the *fundamental* reasoning. And what I find to be practical, is founded on contradictions and on undecidable premises, of which the most basic is the following – when I write "I", I *express the inexpressible* and thus making a contradiction to begin with – an useful one – without which I can not express. And, what can not be expressed, can neither be computed nor tested (proved or disproved). "I" is *the* undecidable premise that lets me *do*.

Is it science that I just wrote? I believe it is beyond science because I discussed the premise on which science is founded upon. Then, why did I write these in an essay regarding science? It is because I am humble and open minded enough to admit my incompleteness in pursuit of science. Are "you"?

** END NOTE **

The much acclaimed rigour that Cauchy provided to calculus [18] could not eliminate the problems associated with Newton's "evanescent" quantities [13], which were met with sharp criticisms from Berkeley who, quite sarcastically, dubbed those as "the ghosts of departed quantities"[19]. Robinson pointed out this issue in ref. [20] while discussing non-standard analysis (NSA) [21]. However, Robinson founded NSA on formal mathematical logic and got trapped in the same basic dilemma regarding the discrimination of pure mathematics and physics, i.e. misunderstanding the difference between "unit" and "unity" or "number" and "quantity". The reason is that formal mathematical logic has the same shortcoming. For example, the revolutionary results, of Goedel on incompleteness and undecidability in arithmetic (page no. 592 of ref. [29]), or of Tarski on undefinability in formal systems of language [22], are devoid of a clear cut prescription of how one can express the thought of physicality or existence, in terms of units and numbers. Although Frege did minutely dissect the issues regarding expressions of thought about an object's existence, relation with bodily sense-perceptions, meaning of numbers, functions, variables, etc. and the essence of language to write down scientific thoughts, but surprisingly he never applied (or was unable to apply) such clarity of reasoning for practical purpose like to solve some problem in physics [26]. This is why logically founded set theory serves no purpose in solving any open problem in physics with immediate effect and instead fraught with inherent inconsistencies or paradoxes [24].

According to me, the crux of the problem lies in the thought about the existence of an object and its corresponding expression by virtue of which one calls an object physical or real. It goes right back to Newton's philosophy of science where he expressed his thought about the existence of an object through "mass" dimension. That the notion of "mass" is problematic becomes apparent from the writings of Poincare [27] and especially Mach [25]. However, physics can not be written without "mass".

Interestingly, inexact mathematical science suggests otherwise i.e. the concept of "mass" is redundant. It is immediately understandable from my calculations that "mass" does not play any independent role in the calculations. Rather the intrinsic length units indicate the existence of objects. Newton needed to invoke "mass" because he considered intrinsic length units (extensions) to vanish and plagued science with incomplete statements and contradictions. Such philosophy has become so hard coded in the thoughts of several successive generations that nobody has any problem in writing "$r \to 0$" in spite of knowing that "r" is not a pure number.

I have discussed my calculations in terms of "mass" only to make this essay not too outrageous to believe in. Otherwise, it is certainly possible to do this analysis (and more) with only length units. All the interpretations need to change. The question definitely arises that where to begin with then. The answer is simple and it is to express, what one represents on the paper with a pencil, in terms of units, numbers, etc. An object (represented by the dot - an extension), associated distance (environment i.e. not object – blank page surrounding the dot – an extension associated with non-existence), an observer (measuring length unit – also an extension) and the undecidable length element ℓ_u that bears the burden of "length dimension" and lets the thought of extensions to be written in terms of numbers. ℓ_u is undecidable because it is one of the premises on which expressions are made and therefore, can not be proven or disproven by the theory. Since inexact mathematical science takes into account the role of the observer explicitly within the theory, ℓ_u is experimentally undecidable or irrefutable [28]. I must emphasize that one should not confuse ℓ_u with some "fundamental length unit" hocus-pocus that prevails in modern physics because ℓ_u is not some fixed or exact length unit. Any notion of extension (intrinsic unit (s), distance (d), measuring unit (λ_0), etc.) can be expressed as $e = N_e\ell_u + i_e = (N_e + \delta_e)\ell_u \ni N_e \geq 2,\ 0 < \delta_e := i_e/\ell_u < 1\ \forall\ e \in [s, d, \lambda_0]$, where i_e is the unknown part of the corresponding extension even after written in terms of ℓ_u. In fact, one would not be mistaken to think that I should have written (ℓ_u, i_e) as *undecidable element*. However, that would have needed a philosophical discussion that is beyond the scope of this essay.

Nonetheless, I may point out that the dot (existence) and the blank page (non-existence), together ensure the thought of an object. It is a perception of thought that is neither based on pure "existence" nor pure "non-existence". Rather it is a *middle way* that leads to the totality of the perception. Further, consideration of "the object as a whole", represented by "$s = \epsilon\lambda_0 \ni 0 < \epsilon < 1$", certainly depends on the observer's choice of λ_0 and therefore, can be violated at will i.e. I can bring a magnifying glass and look at the dot that now looks like a patch. If the magnification is high enough the patch may fill my vision and the blank page not visible any more. Therefore, what I call "existent" or "not existent" depends on how I observe or relate to the observed. This is the essence of relational existence ("sunyata"). The *choice* of the measuring unit depends on my will, but restricted by the ability of my body – that is why I needed a magnifying glass to zoom into the dot. Therefore, "I" makes the choice restricted by the ability of the associated body. So, "I" is the basic undecidable premise whose expression is a contradiction. Taking all these into account, I find my attitude very similar to certain aspects of Indian philosophy [31–33].

Coming back to the strict mechanical aspects of my views, let me point out the following. Since I have suggested that intrinsic length unit can replace the concept of "mass", one may wonder what happens to G, which is considered as "fundamental constant". To answer this I need to explain the role of "time unit", which I could not do within the limited scope of this essay. The only hint that I can give here is that the smallness of the numerical value (with respect to 1) of G is a manifestation of extremely slow perception of relative motion of the associated objects of observation and extremely small intrinsic length units of the objects (compared to the measuring length unit).

I thank Gopal Sardar for repeated and thorough readings of earlier versions of this manuscript and consequently offering critical and valuable comments on this essay which helped me to get it to the present final form.

[1] R. E. Bradley, C. E. Sandifer, *Cauchy's Cours d'analyse: An Annotated Translation* , Springer-Verlag New York (2009).
[2] D. M. Cates, *Cauchy's Calcul Infinitesimal*, Springer International Publishing (2019). 1, 3, 4
[3] I find a similar view expressed by Frege: *"A distinction must be drawn between one and unit."*; see page no. 58 of ref.[4]. 2
[4] G. Frege, *The Foundations of Arithmetic - A logico-mathematical enquiry into the concept of number*, Northwestern University Press (1980), translated into English by J. L. Austin. 10
[5] My view clashes with that of Einstein regarding what he wrote, in the first footnote in page 4 of ref.[7], except that a measurement with a chosen unit only yield whole number. Einstein's belief in *exact* measurement led only to a development in science that could not evade the problem of singularity which was already present in Newtonian physics. Such science has only been made to appear more sophisticated by later believers of exact science e.g. see ref.[8]. 2
[6] A. Einstein, *The Meaning of Relativity*, Routledge Classics (2003). 8
[7] A. Einstein, *Relativity: The Special and the General Theory*, Forgotten Books (2010). 7, 10
[8] S. W. Hawking, R. Penrose, *Proc. Roy. Soc. Lond. A.* 314, 529-548 (1970). 10
[9] The sense of incompleteness that I have discussed here is related to the understanding of units and numbers, which I believe, can be understood by a school student. This is much more elementary than the "operational" incompleteness of symbols like "d/dx", etc. as has been discussed in ref.[10]. 3
[10] A. N. Whitehead, B. Russell, *Principia Mathematica*, Vol. 1, Cambridge University Press (1927). 10
[11] A. Einstein, *Annalen der Physik*, 17, 891-921 (1905). 5
[12] Frege pointed out this issue on page no. 57 of ref.[4]: "*But the mere .... difference is .... utterly incompatible with .... arithmetic*". 6
[13] I. Newton, *Principia, Volume I - The Motion of Bodies*, University of California Press (1966); English translation by A. Motte. 5, 8
[14] I. Newton, *Principia, Volume II - The System of the World*, University of California Press (1966); English translation by A. Motte. 5, 7
[15] H. Goldstein, C. P. Poole, J. L. Safko, *Classical Mechanics*, Addison Wesley (2001). 6
[16] S. Weinberg, *Critical Phenomena for Field Theorists* in *Understanding the Fundamental Constituents of Matter (The Subnuclear Series 14)*, edited by A. Zichichi, Springer US (1978). 7
[17] R. H. Sanders, *The Dark Matter Problem: A Historical Perspective* (2010). 7
[18] J. V. Grabiner, *The Origins of Cauchy's Rigorous Calculus*, Dover Publications (2005). 8
[19] G. Berkeley, *The Analyst* (1734) [edited by D. R. Wilkins], online link. 8
[20] A. Robinson, *The Metaphysics of Calculus*, Studies in Logic and the Foundations of Mathematics, Volume 47, Pages 28-46 (1967). 8
[21] A. Robinson, *Non-standard analysis*, North Holland Publishing Company, Amsterdam (1966). 8
[22] A. Tarski, *The Concept of Truth in Formalized Languages* in *Logics, Semantics, Metamathematics - Papers from 1923 to 1938*, translated by J. H. Woodger, Second Edition, Hackett (1983). 9
[23] B. Riemann, *The hypotheses on which geometry is based*, on pg. 257, section XIII, of *Bernhard Riemann Collected Papers*, Kendrick Press (2004). 3, 8
[24] A. A. Fraenkel, Y. Bar Hillel, A. Levy, *Foundations of Set Theory - Studies in Logic and The Foundations of Mathematics, Volume 67*, Elsevier (1973). 9
[25] E. Mach, *The Science Of Mechanics - A Critical and Historical Account of its Development*, 4th Edition (2007). 3, 9
[26] G. Frege, *Collected papers on mathematics, logic, and philosophy*, Blackwell (1984). 9
[27] H. Poincare, *Science and Hypothesis*, The Walter Scott Publishing Co. Ltd., New York (1905). 1, 3, 9
[28] What I have called "experimentally undecidable" is perhaps what Popper would call "irrefutable": K. Popper, *Conjectures and Refutations: The Growth of Scientific Knowledge*, Routledge (2002). 9
[29] J. van Heijenoort, *From Frege to Goedel - A Source Book in Mathematical Logic, 1879-1931*, Harvard University Press (2002). 8
[30] A. Majhi, *Comments on application of calculus in physics*, online link. 5
[31] G.C. Nayak, *Madhyamika Sunyata: A Reappraisal*, Indian Council of Philosophical Research (2001). 9
[32] J. L. Garfield, *The Fundamental Wisdom of the Middle Way: Nagarjuna's Mulamadhyamakakarika*, Oxford University Press, USA (1995).
[33] M. Siderits, S. Katsura, *Nagarjuna's Middle way – the Mulamadhyamakakarika (Classics of Indian Buddhism)*, Wisdom Publications (2013). 9