Integrating Mathematics and Science

Abstract

The author addresses the question: Is it possible to integrate math and science? The metaphor of divorce is used to distinguish mathematics' reliance on quantitative methods from science's requirements for tools to encode structural and quantitative relationships. The reader's attention is engaged by inviting them to experience varying degrees of cognitive dissonance as they struggle to understand quantitatively problems that require the integration of both. The reader is guided, through the introduction of structurally based axioms, into the development of a mathematics where the symbols have quantitative and structural referents.

"A truly realistic mathematics should be conceived, in line with physics, as a branch of the theoretical construction of the one real world, and should adopt the same sober and cautious attitude toward hypothetic extensions of its foundations as is exhibited by physics." - Hermann Weyl [1]

Is it possible to integrate math and science? An interesting question, as there is a widespread belief that the integration of math and science has a long history of success. The success was considered so great that physicist Eugene Wigner [2] wrote a paper: "The Unreasonable Effectiveness of Mathematics in the Natural Sciences." The reason for raising this question is that many agree with Freeman Dyson [3] when he states: "I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce." Dyson characterized the divorce in terms of lost opportunities for advancement of both math and science. David Hestenes [4] ascribes the lack of an understanding of science in our high school mathematics teachers to this divorce.

The divorce can also be characterized by the fact that mathematics no longer has an intuitive base such as underlies Euclidian geometry. Hermann Weyl [5] characterizes this shift: "The set-theoretical approach is the stage of naïve realism which is unaware of the transition from the given to the transcendent." He further characterizes it as an attempt to "jump over its own shadow," to "leave behind the stuff of the given." What has essentially happened is that mathematics has shifted over to a complete reliance on axioms and the employment of symbols, which in themselves signify nothing. "Nothing" in the sense that the contentual meaning of early arithmetic has as been replaced by ideal elements. Ernst Cassirer [6] expressed this idea thusly: "The claim to grasp the substance of things in number has gradually been withdrawn: but at the same time the insight has been deepened and clarified that in number is rooted the substance of rational thought." Hilbert [7] put it this way: "The numerals and letters in algebra [can be considered] as signs that in themselves mean nothing but are merely building blocks for ideal propositions."

Implications of the Divorce

To understand the full implications of the divorce, to some degree, requires an understanding of the work of Immanuel Kant. Kant was very much interested in the operations of the mind, and in particular how the mind could be absolutely certain of something without having any experience of it. When delving into the mind, he found that it utilizes four different categories for constructing concepts: quantity, quality, relation and modality. Our systems of ideas or interpretive frameworks for constructing models of the world require these elements. The one of interest here is quantity, which he determined was composed of three components: totality, plurality and unity [8]. These components when symbolized, may capture his meaning thus: Totality (T) = plurality (n) × unity (μ), or $T = n \times \mu$. Plurality here is essentially the concept of number, which is the ratio of totality to unity ($n = T/\mu$). Let this interpretation be termed the *science idea of number* as it was expressed by Newton [9]: "By Number we understand not so much a Multitude of Unities, as the abstracted Ratio of any Quantity, to another Quantity of the same kind, which we take for Unity." The vinculum (/) in mathematics specifies a divide operation, and scientists often express it this way as well. However in terms of science, it could be used to specify the existence of a perceived structural connection between quantities rather than the construction of a new quantity.

Now lets shift gears and consider Kant's idea of quantity from a different perspective. The totality could be thought of as either unity or plurality, where the plurality is just the cardinality or measure of the whole. This is a mathematics idea of number as – in Euclid's words - "a multitude composed of units." [10] Symbolically this can be expressed as: $T = 1$ or $T= n$. In other words, quantities are expressed by symbols, or numbers are abstractions of quantities. Thus any quantity can be expressed as 1 or n, depending upon whether one considers it a unity or a plurality. Gottlob Frege [11] illustrates this shift by stating: "One pair of boots may be the same visible and tangible as two boots. Here we have a difference in number to which no physical difference corresponds …". In other words, the source of the meaning of numbers is not the referent itself. This way of thinking about numbers is often called *reification*. This is in essence Weyl's idea of transcendence, or alternatively as the shift from number as adjective to number as noun. Numerals used in this way are able to mathematize relative magnitudes, although structural information may be used in constructing the symbol.

It seems certain that this conflict in meaning could and may have created a significant amount of confusion about the meaning of number. An example may help in explaining how this confusion arises. Consider the simple math problem: $2 + 2 + 2 = 6$. The twos and six are members of the set of whole numbers. However, this is often written as $3(2) = 6$. Is this 3 a member of the set of numbers to which 2 and 6 belong? Would $3 + 2$ equal 5? The three is a count on two's but the two and six are counts on ones. Another perspective is that the 2 & 6 are cardinal numbers, whereas the 3 is a *relationship number* that scales 2 to 6.

Let's restate this: in math $T = n$ (cardinal concept of number) and in science $T = n \times \mu$ (relationship concept of number). These formulations highlight the differences between numbers as nouns versus adjectives. The difficulty encoding science concepts mathematically arises because numerical symbols by themselves are incapable of encoding both quantitative and structural information. It is the structure versus quantitative interpretation of number and the consequences of the difference that make mixing math and science like mixing oil and water.

Relationship Numbers

The implications of the divorce can be further illustrated by an examination of pi (π) from both a mathematics and science perspective. The meaning given pi from a mathematics perspective is that it is a member of the families of irrational, real and transcendental numbers but not a member of families of rational or algebraic numbers. In other words, pi is just a symbol whose referent is itself in a class of elements. However as Weyl also points out, the elements in these classes have no meaning other than as an ordered relation to one another.

The meaning of pi from a science perspective is very different. Pi refers to the relationship that the circumference of a circle has to its diameter, which is encoded as a ratio. In general the "n" in $T = n \times \mu$, is what is sometimes called a pure or dimensionless number, in the same sense that pi ($\pi = C/d$) could be called pure or dimensionless. Perhaps a better classification for "n" is relationship number, as its meaning does not come from the symbol but rather the referent, which does contain units.

A deeper understanding of relationship numbers can easily be illustrated by justifying the meaning of pi. Consider a circumference of a circle (C) and its diameter (d). If the diameter changes then so does the circumference, which is a law-like behavior. Starting with this idea, meaning gets developed in the following way:

1)	$C = C$	law of identity (laws of thought)
2)	$d/d = 1$	the relationship that d has with d is symbolized by 1
3)	$C = 1\,C$	axiom $a = 1a = a1$
4)	$C = (d/d)\,C$	substitution d/d in place of 1
5)	$C = (C/d)\,d$	axiom $a\,(b/b) = (a/b)\,b$
6)	$C = \pi\,d$	π has a referent of (C/d)

The axiom $a\,(b/b) = (a/b)\,b$ is only true under special circumstances, as there has to be something defining a connection between a and b. In this particular case the law mentioned above is what is defining the connection. These axioms are similar to those in mathematics but are deeply grounded in phenomenology, psychology and philosophy.

Experimentally the relationship C/d is found to be pi (π) = 3.1415… . Thus from a science perspective pi is just a symbol for the connection or relationship that C has to d. Stated in another way, pi is the symbol and the referent of the symbol is the context, which provides the meaning. J.S. Mill [13] makes a similar symbol/ referent argument when he states: "Three pebbles do not make the same impression on our senses as when they lie before us in two separate piles as when they are collected into a single pile." Frege takes the opposite position regarding this point when discussing boots; mathematically they are different, as the meaning does not come from the referent but rather from the set of numbers. Frege's point is that the whole may be the same but the part is different.

The science or relationship concept of number bridges the gap between rational, real, and imaginary numbers. Rational numbers are relationships between discrete quantities, real numbers are relationships between continuous quantities (as demonstrated by Clifford [14]) and imaginary numbers are relationships between complex quantities: e.g. vectors. This concept bridges the gap in the sense that, from a science perspective, these number types encode structure in the phenomenological referent.

The Circle/Square Problem

Stated simply, the reason that there is difficulty in integrating math and science is that a number in mathematics get its meaning from its relation to other numbers in an ordered set, which can be used to reason quantitatively. In science the meaning comes from the referent, which can be used to reason structurally. How these differences play out can be illustrated with what is called the circle/square problem [15].

Consider the circle and square, which have been partitioned into four equal sections and joined together to form a new object.

Figure 1

Mathematically each shaded area can be symbolized quantitatively by the fraction one quarter (¼), which is the reification of a relationship number. However from an examination of the figures it would intuitively appear that:

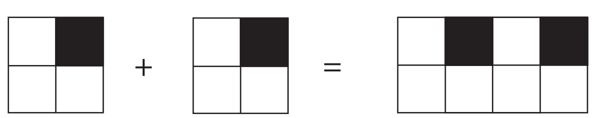
7) $$\frac{1}{4} + \frac{1}{4} = \frac{1}{4}$$

However, this is not the answer one might expect from a purely mathematical quantitative perspective, which is:

8) $$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

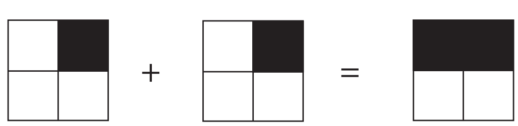
One rationale for the solution to this difficulty is that only like things can be added, since from a quantitative perspective the units are not equivalent. However as shown below the problem persists, because the process of joining like things should be no different than joining unlike things.

Figure 2



Mathematics circumvents this difficulty by utilizing the relationship concept to construct the symbol, then reifying the structural symbol into a quantitative symbol, which allows it to be added. This reification shifts the meaning of the symbol (¼) from structural to quantitative. The shift is essential, as relationships do not have the quantitative addition property. The reified image is shown below; the unshaded areas are kept to maintain the naming convention.

Figure 3



Do the same difficulties arise utilizing the mathematics of science? Let the shaded section of the square (x) be equal to the plurality ¼ times the unity (S) and similarly for the circle (C): $x = 1/4S$ and $y = 1/4C$. Note how the ¼ encodes the structural information and the product of ¼ with C or S provides the quantitative information.

9) $\frac{1}{4} S + \frac{1}{4} C = \frac{1}{4} (S + C)$

10) $\frac{1}{4} S + \frac{1}{4} S = \frac{1}{4} (S + S)$

Figures 1 & 2 are certainly intuitively obvious both structurally and quantitatively. Each symbol in equations 9 & 10 encodes the structural and quantitative information derived directly from the objects themselves. The algebraic operations of science match perfectly with the structural and quantitative information contained in each diagram. Note that this consistency between structural and quantitative reasoning provides a sense of validity to the operations. This will be further developed below.

So why is mathematics unable to symbolize the joining of the two objects as depicted in Figures 1 & 2? Once reified, the numeral ¼ encodes quantitative information. However the joining operation requires both structural and quantitative information to symbolize the result.

Is (1= 1) True In Both Mathematics and Science?

Weyl [16] poses a challenge when he states: "The demonstration of the consistency of classical mathematics was then to be achieved by showing, within the constraints of strict finitistic evidence insisted on by Hilbert, that the formal metamathematical counterpart of a classical proof in that system can never lead to an assertion evidently false, such as $\propto(1= 1)$."

Is it the case that $\propto(1= 1)$ necessarily has to be false in science? In other words is $1 = 1$ a universally accepted principle across both math and science?

To answer this question requires an examination of the axiom expressed above: $C = 1C$, since C is not a number but rather a quantity. $C = C$ can be accepted as a given as it is the law of identity. Suppose however that there are a number of circumferences that are the same in that they have the same magnitude. Let C' be one such circumference.

11)	$C = C$	law of identity
12)	$C' / C' = 1$	the relationship that a has with a is symbolized by 1
13)	$C = 1 C$	axiom $a = 1a = a1$
14)	$C = (C' / C') C$	substitution C'/C' in place of 1
15)	$C = (C/ C') C'$	axiom $a (b/b) = (a/b) b$
16)	$C = 1 C'$	

This result could be interpreted as stating that "this" is equivalent to one of "those". This is just a restatement of Kant's idea that we can't know the noumenon but only the phenomenon. In other words the best that can be done is to be able to describe “this” in terms of “that” where the “that” is not the thing as it is in itself. This is a variation on Alfred North Whitehead’s idea of the “fallacy of misplace concreteness”: “There is an error; but it is merely the accidental error of mistaking the abstract for the concrete.” [17]

A few additional comments about the axiom n = 1n are necessary at this juncture. In mathematics, the number 1 can be considered as an identity operator that maps “n” onto itself, however the number 1 is not a member of the set n. In science, 1 encodes the structural relationship C′ has to C and thus allows “this” to be described in terms of “that”. It also supplies a symbolic means for encoding the philosophical claim that “this” and “that” are the same in some sense, as technically they are not the same. They could be located in different places or times, or be different in some other sense. The claim is philosophical as it requires decision making as to which distinctions to drop.

It is now possible to fully answer the question as to whether 1 = 1 is a universal truth. From a mathematics perspective, 1 = 1 is a universal statement of symbolic identity, however, if applied quantitatively 1 = 1 is only true if the suppressed units are identical. From a science perspective, 1 = 1 only expresses equivalence between relationships if a/a can be substituted for b/b. Thus in science 1 = 1 is not an axiom.

Mathematical Operations Encode Relationships

The effectiveness of mathematics in science seems undeniable and yet, as has been shown, there is a huge chasm between them. Numbers, as shown above, can be used to encode whole-to-part structural relationships between quantities. However, mathematical operations can encode quantitative relationships as well. Let “a” be of magnitude different from “b” and let (-b) be the *opposite* of “b”.

17)	$a = a$	law of identity
18)	$b + (-b) = 0$	relationship b has with (-b) is symbolized by 0
19)	$a = a + 0$	axioms $a = a + 0$ & $b + (-b) = 0$
20)	$a = a + (b + (-b))$	substitution $b + (-b)$ in place of 0
21)	$a = (a + (-b)) + b$	axiom $b + (-b) = (-b) + b$
22)	$a = \Delta a + b$	substitution $\Delta a = a - b$

The axiom $a = a + 0$ is necessary to establish a quantitative relationship with another quantity. The axiom $b + (-b) = 0$ expresses the quantitative relationship that b has to its opposite (-b). Thus Δa encodes the quantitative relationship that a has to (-b), a *difference*, and is computed using the subtraction operation. The axiomatic shift $a + (b + (-b))$ to $(a + (-b)) + b$ is only possible if there is a law-like quantitative relationship defined between “a” and “b”.

In science, differences can have structural relationships to other differences. Let Δs be a difference in position and Δt a difference in time. Starting with a basic identity:

23)	$\Delta s = \Delta s$	law of identity
24)	$\Delta s = 1(\Delta s)$	axiom $a = 1a = a1$

25)	$\Delta s = (\Delta t/\Delta t)\ \Delta s$	substitution $\Delta t/\Delta t$ in place of 1
26)	$\Delta s = (\Delta s/\Delta t)\ \Delta t$	axiom $a\ (b/b) = (a/b)\ b$
27)	$\Delta s = v\ \Delta t$	substitution velocity (v) in place of $\Delta s/\Delta t$

The law that justifies the axiomatic shift producing $\Delta s/\Delta t$ is that *a change in position (Δs) requires a change in time (Δt)* or equivalently an object can't be in two places at the same time. It is not possible to divide distance by time, which is sometimes considered to be the case. The vinculum (/) here is used as a means of encoding a structural relationship defined by a law.

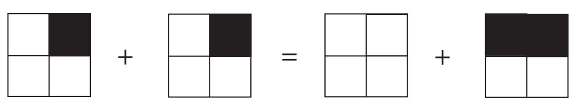
This derivation requires an underlying postulate that every object defines a relationship between changes in location and changes in time regardless of whether it is considered to be moving in a reference frame. The value of $(\Delta s/\Delta t)$ has to be determined by experiment and relies upon the reference frame chosen. Note that this derivation implies that velocity is not relative but rather is indeterminate. Only the difference between "this and that", the velocities of the object and the reference frame, can be determined.

Reasoning Processes

One of the major strengths of mathematics is the power of algebraic or axiom-based reasoning. Equations can be transformed in a vast variety of ways by purely mechanical means. Consider an extension of Equation 10 above, mechanically applying the rules of algebra:

28) $\frac{1}{4} S + \frac{1}{4} S = \frac{1}{2} (S)$

Figure 4



However as can be observed in Figure 4, the accepted answer requires an underlying law of science: *objects being invariant under regrouping*. It is this law that provides the logic and reasoning necessary for validating the final answer.

Consider a simple unit conversion problem to illustrate transitive reasoning: $(a \rightarrow b)$ & $(b \rightarrow c)$ then $(a \rightarrow c)$. Consider a length (L) of six feet, (encoded mathematically as $L = 6$) which is to be expressed in terms of inches. Since each foot is defined to be 12 inches, it is possible to calculate $L = 6\ (12) = 72$. In science it is possible to justify the answer by reasoning follows:

29)	$L = L$	law of identity
30)	$L = L\ (ft/ft)$	axiom $a = 1a = a1$
31)	$L = (L/ft)\ ft$	axiom $(b/b)\ a = (a/b)\ b$
32)	$L = (L/ft)\ ft\ (in/in)$	axiom $a = 1a$
33)	$L = (L/ft)\ (ft/in)\ in$	axiom $(b/b)\ a = (a/b)\ b$
34)	$L = (6ft/1\ ft)\ (12\ in/1\ in)\ in$	
35)	$L = 72\ in$	axiom $a = 1a = a1$

Examining this closely shows that arriving at the answer of 72 is nothing other than transitive reasoning. If six feet has a relation to one foot and one foot has a relation to one inch then six feet has a relation to one inch: (6 ft → ft) & (ft → in) then (6ft → in). In other words, *a product of relationships encodes transitive reasoning*, a reasoning process which has become invisible in mathematics.

Combining Laws/Axioms and Structure/Quantity

The simple examples above have been chosen to clearly illustrate the benefits of combining laws and axioms to encode structure and quantity. However, using these tools, complex problems are also rendered with equal simplicity.

Vector algebra, of whatever flavor chosen, currently requires a steep learning curve. It can be done much more easily by encoding both structure and quantity. Consider two vectors "**a**" and "**b**", constrained by a law that *their structure is invariant under translations and rotations*. This invariance allows structural reasoning to be done independently of reference frame.

36)	**a** = **a**	law of identity
37)	bb/bb = 1	1 is real part of a complex number
38)	**a** = **a** (bb/bb)	axiom a = 1a = a1
39)	**a** = (ab/bb) **b**	axiomatic shift
40)	**a/b** = (ab/bb)	

This is essentially Hamilton's [18] idea, as he understood that this ratio had to encode both the relative magnitude and orientation, in other words the structural information. Here **a/b** can be interpreted as the relationship that **a** has to **b**, and ab/bb, which can be thought of as the quotient of vector division, produces an entity that behaves like a complex number or quaternion. The intent was the same, but the formulation is different. The format ab/bb is necessary so that the product of relationships "**a**" to "**b**" and "**b**" to "**a**" produces a relationship that can be symbolized by 1.

The structural information once retained can be utilized in the following way. Let **a** be a velocity vector and **b** the initial velocity. The complex number (ab/bb) references all the physics that determines the motion of the object. In other words, the physics provides a rationale for constructing a conversion factor that allows "this" vector to be described in terms of "that" vector. The importance of this approach is that each symbol can have a direct referent, as illustrated in the circle/square problem. This changes the understanding of mathematics by illuminating the meaning of mathematical operations and how they can be applied in contentual settings. In science it clarifies the meaning of symbols and operations contained in formulas. Showing that complex numbers, contrary to the opinion expressed by Wigner [19], encode structural information, suggest the possibility that other mathematical methods may have referents.

By allowing any vector to be written as the vector sum of two other vectors, this system opens up many elegant, even trivial proofs in geometry including the sine law, cosine law and the

Pythagorean theorem. Its development requires the constraint that **a** + **b** and **c** are quantitatively the same but structurally not the same, an application of the *law of the included middle*.

This simple beginning, whose only difference from those above is that the *fancy form of one* b/b is replaced by bb/bb, can be used to generate the Geometric Algebra developed by David Hestenes [20].

This system, which could perhaps be called *structural algebra*, maintains structural as well as quantitative information thus enabling both structural and quantitative reasoning.

Conclusion

Freeman Dyson makes the claim that the divorce between math and science resulted in missed opportunities. I would claim that it goes much deeper as the divorce has created a chasm between structure and quantity, laws and axioms and ultimately theory and experiment.

It is these chasms that are the source of cognitive dissonance that the reader may have experienced in making sense of the circle/square problem, the meaning of vector division, the axioms that support it and the law of invariance under translation and rotation, as they involve mental shifts between quantitative and structural information.

The confusion that is caused by mathematics' ability to shift the meaning of symbols without distinguishing the shift has a long history. It lies at the heart of the long-standing debate on whether 5 + 7 = 12 is synthetic or analytic, the resolution to which is that if perceived quantitatively, it is synthetic, perceived structurally, it is analytic.

Only by shifting to a more encompassing mathematics, where each symbol has a referent, can metaphorical, analogical and transitive reasoning processes, essential for understanding science be transparent to the user. It is this tight connection between theory and experiment that can guide the modification of laws to more closely decode experimental results.

Today the seemingly "unreasonable effectiveness" of mathematics is that it provides operations necessary for encoding various structural types; science is "unreasonably effective" in its use of mathematics to algebraically encode structure and quantity. It all works since the axioms of mathematics, to some extent, mirror those of science.

[1] Weyl, Hermann (2009)[1949]. *Philosophy of Mathematics and Natural Science*, p.235, Translation O. Helmer, Princeton: Princeton University Press

[2] Wigner, Eugene (1960). "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," *Communications in Pure and Applied Mathematics* 13(1), New York: John Wiley & Sons.

[3] Dyson, Freenam J. (1972). "Missed Opportunities", *Bulletin of the American Mathematical Society* 78(5), p.635.

[4] Hestenes, David (2008). "Notes for a Modeling Theory of Science, Cognition and Instruction", In E. van den Berg, A. Ellermeijer & O. Slooten (eds.) *Modeling in Physics and Physics Education*, University of Amsterdam.

[5] Weyl, *op.cit.,* pp.65-66.

[6] Cassirier, Ersnt (1923). *Substance and Function and Einstein's Theory of Relativity*, p.27, W. C. Swabey and M. C. Swabey, translators, Chicago: Open Court.

[7] Hilbert, David (1996) [1927]. "The Foundations of Mathematics", in *The Emergence of Logical Empiricism* publ., Garland Publishing Inc.

[8] Kant, Immanuel (1965) [1787]. *Critique of Pure Reason*, p.113, N.K. Smith translator, New York: St. Martin's Press.

[9] Newton, Isaac (1720) [1707]. *Universal arithmetick: or, A treatise of arithmetical composition and resolution,* p.2, J. Raphson, translator, London: J. Sennix.

[10] Euclid (1956) [c.300 BC]. *The Thirteen Books of Euclid's Elements,* Vol. II, Book VII, p.277, T.L. Heath, translator, New York: Dover.

[11] Frege, Gottlob (1980) [1950]. *The Foundations of Arithmetic*, p.33, J.L.Austin, Translator, Evanston: Northwestern.

[12] Weyl, *op. cit.*

[13] Mill, John Stuart (1882). *A System of Logic, Ratiocinative and Inductive,* Eighth Ed., p.318, New York: Harper.

[14] Clifford, William K. (1955). *The Common Sense of the Exact Sciences*, New York: Dover.

[15] MacDuff, Rob (2014). "What is One Quarter Plus One Quarter?" *Cognitive Instruction in Modeling Math and Physics,* https://trueddotorg.wordpress.com/2014/03/07/one-quarter/

[16] Weyl, *op.cit.,* pp.65-66.

[17] Whitehead, Alfred North (1997) [1925]. *Science and the Modern World*, p.51, New York: Free Press (Simon & Schuster).

[18] Hamilton, William R. (1886). *Elements of Quaternions*, London: Longmans, Green.

[19] Wigner, *op. cit.*

[20] Hestenes, David (1993). *New Foundations for Classical Mechanics* (Second Ed.), Dordrechr: Kluwer Academic.