The Ultimate Physics of Number Counting

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"The sequence of natural numbers turns out to be unexpectedly more than a mere stringing together of identical units: it contains the whole of mathematics and everything yet to be discovered in this field."

— C.G. Jung

This essay puts forward the idea that the elementary physical process in the universe is the counting procedure of natural numbers. If true, it would imply that the ultimate possibility in physics is the discovery of this archetypal and fundamental numerical order in nature. In pursuing this astounding idea with methods from modal logic and set theory, it is argued that the number counting process may indeed be sufficient for a complete quantum description of the evolving universe. Completeness means that the universe is understood as a quantum system without an external classical world and without any outside observers. As a consequence, the observable world including its cognizant observers become *emergent* phenomena that arise from inside the system. Emergence is known from systems with a sufficient number of physical constituents and with a sufficiently complex evolution of the latter. It manifests itself in global physical behavior that cannot be understood properly by looking only at the local constituents.

Imagining the universe as a closed and discrete quantum system is not a very new thought. Already in 1982 Feynman [10] outlined some implications of the assumption that the universe is a *quantum computer* representable as a tensor product of many finite Hilbert spaces of low dimension, such as qubits. Considerations like this look at the cosmological evolution of the universe as a run of a quantum automaton. This happens in a similar manner to classical cellular automata, like for example Conway's *Life* [11], which do evolve in a sequence of discrete steps. A quantum automaton is thus a finite system in some predefined initial state together with some rules that govern its stepwise evolution. The rules themselves have to be consistent with the laws of quantum physics; classical cellular automata can therefore only be a limiting case of quantum automata. But even in simple classical cellular automata rich varieties of complex patterns emerge [21], it is therefore at least not implausible to think of the physical universe as the output of a quantum automaton.

The quantum automaton paradigm and its problems

The hypothesis that natural numbers are *all-that-is* and *all-that-can-be* has a relationship to the *quantum stage paradigm* of the universe (see, for example, the works of Eakins and Jaroszkiewicz [7, 8, 9]). In this setting the universe is represented as a quantum automaton in a Hilbert space $\mathcal{H}$ of very large but finite and fixed dimensionality N, where a pure state vector $\Psi_\alpha \in \mathcal{H}$ represents the current stage of the universe. This state is an element of an orthonormal basis given through the family of non-degenerate eigenstates of a Hermitian operator Σ^α acting on $\mathcal{H}$; the family of eigenstates of Σ^α is the *preferred basis* while the operator itself is referred to as the *self-test* of the universe. Also, the state of the universe is subject to change which is governed by some rules, yet unknown, that map Ψ_α onto its successor $\Psi_{\alpha+1}$. Thus the index α labels the successive stages of the universe; it is called *exotime*. These rules guarantee further that $\Psi_{\alpha+1}$ also is an eigenvector but this time of a different Hermitian operator $\Sigma^{\alpha+1}$. The argument goes that at each stage α the Hilbert space $\mathcal{H}$ factors in a tensor product

$$\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$$

of n Hilbert spaces each having a prime dimensionality. States in $\mathcal{H}$ can be total factor states, they can be totally entangled, or they may contain factors of entangled states. Thus Ψ_α admits the general form

$$\Psi_\alpha = \Psi_\alpha^1 \otimes \ldots \otimes \Psi_\alpha^{f_\alpha}$$

where $f_\alpha \leq n$. Since it is believed that any self-test has the capacity to change the factor structure of a given state Ψ_α when going from stage α to the next stage $\alpha + 1$, the corresponding transition amplitude calculated with Born's rule may or may not factorize. This observation allows to look at groups of factors that become entangled in the successor stage, or at entangled states from subregisters that become factorized within the next stage. When followed over several successive stages, the transition amplitudes between states resemble the structure of causal sets (for details, see [8]); it is in this manner that the building blocks of Einstein locality seem to be accessible. Moreover, Eakins and Jaroszkiewicz speculate about further implications of their approach, such as the possibility that highly factorized states should correspond to a quantum system with emergent classical behavior.

These preliminary results and thoughts surely motivate for further work in this direction, but our immediate goal is to take a step back and to recapitulate the common assumptions and prerequisites that form the basis of this quantum stage approach to the universe. In doing so we list a group of questions that are at the source of all arguments presented in this work.

1. If the universe admits a representation by means of a Hilbert space $\mathcal{H}$ of fixed finite dimension N, what causes the choice of the number N? For now, there does not seem to be an immediate physical reason behind the choice of N. We know that at present time this number must be gigantic but has this been the case throughout the history of the universe? In other words, is it necessary that the Hilbert space is static with a fixed number of dimensions? Additionally, N must not be prime since otherwise no non-trivial tensor product of subregisters is available. Is there a physical reason behind this?

2. We know that Hermitian operators represent observables in quantum physics, but why should the self-test of the universe Σ^α be Hermitian and non-degenerate. At least such an extrapolation from local physical experience to the universe as a whole is quite bold. Must we simply accept it as a matter of fact or can we possibly find a reason that explains these properties of Σ^α?

3. How does the preferred basis, i.e. the family of orthonormal eigenvectors of Σ^α, emerge at each stage of the universe? This question—also known as the problem of pointer states—has been a central issue in various approaches to the measurement process in quantum physics (e.g., in the decoherence framework and in the many-worlds approach).

4. How does state reduction or, more appropriately phrased, state *selection* occur at each stage of the universe? In [8] it has been plausibly argued that Ψ_α always is a pure state, but how does the universe make a choice between the available elements of the preferred basis? This question addresses the second central and undecided issue of the quantum measurement process: is von Neumann's formal characterization of the measurement procedure, that is the distinction between processes of type II (unitary dynamics) and type I (reduction), the final word or can we do better in characterizing a (non-deterministic) process responsible for state selection?

5. What is the mathematical structure of the Hilbert

space $\mathcal{H}$? Since any quantum theory of the universe should propose a construction of the physical Hilbert space, we ask whether we can identify the preferred basis elements of $\mathcal{H}$. This is the step of going from an abstract to a concrete Hilbert space where physical states are explicitly given.

All five questions are about the ultimate physical nature of observers, about the observable world, and about the process of measurement. Our goal is to introduce two basic principles and to find their proper mathematical representations in order to gain further insights into these problem statements and, eventually, into the nature of the quantum universe. To the knowledge of the author, these principles as well as the mathematical methods related to them have not yet been widely used in this fundamental problem domain.

An imperfect quantum world

The first principle concerns the ability to perceive nature through experiments (every measurement we call an experiment). By experiments we do not only mean an experimental physical set-up and its conduction in the usual sense but also the ultimate class of experiments that we carry out on ourselves in order to become aware of any experiment whatsoever: sensory perception. The *imperfection principle* says that every experiment in nature has to be blurred in some sense. This means that there should always be a set of several measurement outcomes such that each member of this set must not be perceptually separated from any other member of the same set. Experiments of this kind we call *imperfect experiments*, and hence the principle demands that any experiment in nature must be imperfect. This makes sense in many cases because empirically we know that experimental data has limited precision. But there are types of experiments where it is apparently more difficult to recognize the validity of the imperfection principle.

For example, consider a Stern-Gerlach experiment with a detector screen placed behind the magnetic field. The spin value in z-direction, along which the measurement is performed, of a spin $1/2$ particle shall be determined. Imagine the measurement outcome now is a dot at the upper half of the screen signalizing that the measured particle has a value of $+1/2$ in z-direction. As it seems, there is no fuzziness in the measurement outcome since the particle spin in z-direction has been uniquely derived by measurement. But is this really the case? In this situation the experiment outcome consists of the physical object 'screen' together with a physical object 'dot' on it. If we now come closer to the screen we may observe a chemical reaction, blurred across an area on the screen, which gives rise to the visible dot. The dot, being a cloud of chemically interacting parts (these parts can be groups of molecules, for example), has many physical degrees of freedom and these degrees of freedom are correlated with the measured particle because they *materialize* the experimental result. In quantum physics it is only through the experiment result that a quantum entity becomes a physical object with a measured physical attribute. Hence, it makes little sense to say that the observed particle has only one degree of freedom ($+1/2$ or $-1/2$ in z-direction) because what we actually observe as a measurement result (by means of the chemically interacting cloud on the screen) is a physical system that has many. Parts can therefore be viewed as the material constituents of the physical object 'particle with spin $+1/2$ in z-direction'. Many of these parts can be visually resolved and separated but others become indiscernible in our visual field no matter how close we observe the cloud because every time we *zoom-in* a new family of parts may emerge. In this sense every experiment result can be partitioned such that the imperfection principle holds.

The imperfection principle is intimately related to our senses (not only to our visual sense but to all our senses that interact with the outer world) in that every physical experiment ultimately is an experiment carried out through our sensory apparatus. From a mathematically point of view any family of parts forms a complete ortholattice realized through a non-transitive binary relation called the proximity relation. From a physical point of view we can recognize parts as a realization of Poincaré's *physical continuum*.

A physical continuum is a conceptual alternative to the mathematical continuum of the real numbers: it allows a finite set of experimental outcomes, or sensations to appear continuous in the sensory field of the observer. Even though it has never become part of the canon in modern physics, this concept has an interesting history. Poincaré, for instance, made a clear distinction between the physical continuum and the mathematical continuum of the reals, noting that both alternatives are admissible [17].

The physical continuum can be mathematically introduced as follows. Let X denote a set of finite size N representing mutually exclusive events that in our context represent the set of all possible outcomes of a physical experiment, and let $\mathcal{P}(X)$ be its power set. A *proximity relation* P is a binary relation between the elements of X that is reflexive and symmetric, but not necessarily transitive, and the pair (X, P) is the proximity space [4, 19, 3]. For each $x \in X$ the set

$$Q_x = \{y \in X : xPy\}$$

is called a *quantum associated to* $x \in X$. A quantum is the smallest recognizable subset of X, and any subset of X that is a union of of some quanta is called a *quantum set* [19] or a *part* [3, 4]. We denote the set of all quantum sets as $\mathcal{Q}_P$.

Fundamentally, proximity relations represent indistinguishable outcomes in perceptive fields. They express an inherent and irreducible limitation in our ability to receive information from nature through any perceptive apparatus—no matter how intricate the latter may be physiologically or technically.[1]

As anticipated, quantum sets can be used to construct models of quantum logic. The set $\mathcal{Q}_P$ is recognized as a complete ortholattice, that is a tuple $\mathcal{L}_P = (\mathcal{Q}_P, \cap_P, \cup_P, ^\perp)$, if we equip $\mathcal{Q}_P$ with a join operation $\cup_P$ taken as the usual set-theoretic union, with a meet operation $\cap_P$ of two quantum sets as the union of all quanta in their set-theoretical intersection, and with an unary relation $^\perp$ with

$$^\perp Q = \{y \in X | \, (\exists x \notin Q)(xPy)\}$$

for any $Q \in \mathcal{Q}_P$. A complete ortholattice is known to be a proper model for quantum logic in the sense of von Neumann and Birkhoff [5]. However, it has not been introduced here as a lattice of closed subspaces of a Hilbert space but rather as a lattice of quantum sets (or, parts) for a given proximity space. In this manner proximity relations can be viewed as an alternative entry to the quantum realm [3].

Not all sets in $\mathcal{P}(X)$ are quantum sets; nevertheless a given proximity relation offers a mathematical classification of any two sets in $\mathcal{P}(X)$ as follows: Given $A, B \in \mathcal{P}(X)$, we say A and B are *separated* if $A \cap B = \emptyset$ and if for all $x \in A$ it is $Q_x \cap B = \emptyset$; and due to the symmetry of the proximity relation the same holds for the elements of B. Generally, for any two sets A and B which are not separated one distinguishes two cases: superposition and incompatibility [19, 3, 4]. Both cases resemble situations in Hilbert spaces of quantum systems where two states may arise in a linear superposition, and where two observables may be incompatible, respectively.

There is a physical difference, however. To see it we can choose $\mathcal{L}_\mathcal{H}$, a complete ortholattice of closed subspaces in a Hilbert space $\mathcal{H}$, but at the same time we may obtain another complete ortholattice $\mathcal{L}_P$ through any Hermitian operator admitting an orthonormal basis $b \subset \mathcal{H}$ of eigenvectors. Each eigenvector $x \in b$ corresponds to a measurement outcome documented with the associated eigenvalue $\lambda_x \in \mathbb{R}$, and a proximity relation P is given as

$$(xPy) \quad \text{exactly if} \quad (\lambda_x \text{ is indistinguishable from } \lambda_y).$$

Now use P to define $\mathcal{L}_P$ and there is no necessity to imply that $\mathcal{L}_\mathcal{H}$ and $\mathcal{L}_P$ are isomorphic ortholattices in any obvious sense.

[1] In [4] Bell uses this classification to demonstrate that the human visual field resembles quantum behavior through superposition. More recently, Planat [16] gave an interpretation of the perception of time on the grounds of Poincaré's ideas.

Remarkably, the proximity space approach to quantum logic is general. This follows because *all* complete ortholattices $\mathcal{L}_{\mathcal{H}}$ representing closed subspaces of a separable Hilbert space $\mathcal{H}$ are isomorphic (as ortholattices) to proximity spaces based on the proximity relation

$$(sPt) \quad \text{exactly if} \quad (s, t) \neq 0\,,$$

for all $s, t \in \mathcal{H}\backslash\{0\}$, and where $(.,.)$ is the inner product on $\mathcal{H}$ [3]. It is in this sense that proximity relations give a general approach to the mathematical foundations of quantum physics. We can now make this thought to a guiding principle.

Imperfection Principle. *Every measurement within the universe is imperfect, that is, it gives rise to a nontrivial proximity relation.*

With this basic statement at hand, a direct connection can be made to modal logic, i.e. a non-classical logic that allows for modalities of propositions such as possibility and necessity. This connection follows because every proximity relation naturally defines a model of modal logic, referred to as Kripke model (or, Kripke structure), where the proximity relation P becomes the *accessibility relation* between *possible worlds* [19, 12]. However, such a link to modal logic presumes that a proximity relation is already given. Thus, to be more meaningful, can we point to a truly elementary accessibility relation? Arguably the most basic relation in mathematics is the membership relation '$\in$' between members and their sets, and in the next paragraphs we lay out how modern set theory may create a large variety of *possible worlds*. Let's begin this task by introducing the second guiding principle.

Counting, simulating, and being in the quantum universe

The *simulation principle* says that the physical universe is a discrete quantum automaton which executes an unlimited and elementary process known from set theory. The set theory in question is the Structural Theory of Sets (STS) [1] which is a universal, nonwellfounded set theory based on infinitary modal logic. Non-wellfounded set theories are more general than conventional set theories, such as the classical Zermelo-Fraenkel-Axiom of Choice (ZFC) set theory: they do not have an Axiom of Foundation. In the former *exotic sets* like

$$a = \{b, a\} \quad \text{or} \quad a = \ldots\{\{\{b\}\}\}\ldots$$

may appear which are undefined or which can lead to paradoxes in conventional ZFC set theory. These unusual sets are termed *hypersets*: they represent *self-referential* structures and situations because, in a seemingly strange twist, a non-wellfounded set may become a member of its own member [2]. Indeed, paradoxes that plague classical set theory can be resolved in hyperset theory [1]. Structural set theory is a universal set theory for two reasons. First, its model $\mathcal{M}$ can be seen as the largest extension of a model of ZFC set theory that still preserves the property of modal characterization; and second, it circularly contains its own model in $\mathcal{M}$.

Structural set theory operates concurrently on two sides. On one side it is formulated in infinitary modal logic, i.e. a modal logic which includes infinite logical conjunctions, while on the other side it represents all those sets that satisfy modal sentences. These modal sentences are *analytical experiments* which means they are possible statements about a set, and the result is the set that satisfies a statement. In this manner STS is an *analytical* set theory where sets are discovered as opposed to *synthetic* set theories, such as ZFC, where sets are built by means of the usual iterative concept of set.

There is a natural process that comes with a recursive formulation of modal sentences about a set that is the object of structural analysis. This process is represented as a counting sequence ordered by instances of the ordinals. Each ordinal α gives rise to a stage of *structural unfolding* of a set. Thus *a priori* an arbitrary set is completely unknown; instead it reveals its structure only step-wise through the successive stages of unfolding. The higher the stage ordinal α the better

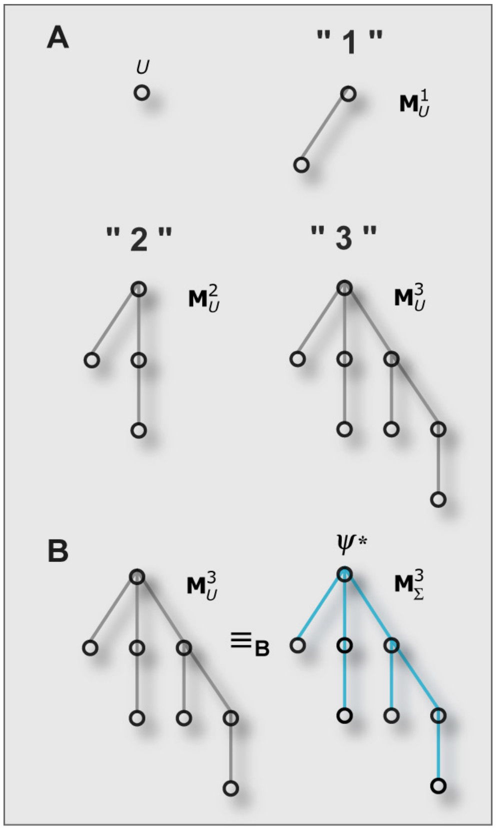

Figure 1: (**A**) Universal structural unfolding of the set U for the first three stages ("1", "2", "3", ...). The evolving tree structure represents the succession of natural numbers, represented through ordinals, as the elementary process in the physical universe. (**B**) Illustration of the Bisimulation Principle realized at the third stage: $\alpha = 3$. Objects in $\mathbf{M}_U^3$ are sets in non-wellfounded set theory and links represent the membership relation '$\in$'. Bisimilar objects in $\mathbf{M}_\Sigma^3$ are possible experimental outcomes of the universal quantum system (the elements of the preferred basis b^3). Links depict the proximity relation between them, and the root ψ^* is the selected state of the current stage. The universe as a quantum system thus becomes a simulation of the structural unfolding process.

is our information about the analyzed set. For finite α the structural unfolding process becomes the recursive formulation of natural numbers through von Neumann ordinals, as shown in Figure 1 for the first three stages, where each von Neumann ordinal is the well-ordered set of all smaller ordinals.

But how to construct a quantum computer out of this unfolding sequence of natural numbers? It turns out that the Bisimulation Principle is entirely sufficient for this task. Structural unfolding generates a α-sequence $\{\mathbf{M}_U^1, \mathbf{M}_U^2, \ldots\}$ of Kripke models. At each stage of unfolding a Kripke model forms a rooted tree with edges that define the accessibility relation through the elementary membership relation $\in$ between sets and their members (which in general are sets again). This tree of structual unfolding then becomes the simulation object on the universal quantum automaton, defined through the Kripke structure $\mathbf{M}_\Sigma^\alpha$ with a proximity relation P_Σ^α. More rigorously, a simulation is established by the mathematical concept of *bisimulation*: two Kripke structures are bisimilar if they are satisfied by the same collection of modal sentences, and bisimulation is an equivalence relation $\equiv_B$ between Kripke structures.

Thus the decisive thought is to realize that two Kripke models, $\mathbf{M}_U^\alpha$ and $\mathbf{M}_\Sigma^\alpha$, are equivalent in the sense of bisimulation even though both represent different mathematical structures: $\mathbf{M}_U^\alpha$ refers to an abstract membership structure in set theory revealed through universal unfolding, while $\mathbf{M}_\Sigma^\alpha$ refers to the physical space of indiscernible experimental outcomes. In this manner the preliminary simulation principle turns into its proper mathematical form, the Bisimulation Principle, which says:

Bisimulation Principle. *The physical universe is a quantum simulation of the structural unfolding process of an absolutely unknown set U; thus for all ordinal stages α of structural unfolding of the universe U the Kripke models $\mathbf{M}_U^\alpha$ and $\mathbf{M}_\Sigma^\alpha$ are bisimilar, viz.* $\mathbf{M}_U^\alpha \equiv_B \mathbf{M}_\Sigma^\alpha$.

That this is not an empty statement warrants the next result, Proposition 1, which shows that the physical space comes with the desired Hilbert space properties of quantum systems. This positive result [12] directly addresses the five questions about the quantum stage paradigm of the universe and it becomes a cornerstone of the Bisimulation Principle.

Proposition 1 *The Bisimulation Principle valid at every finite ordinal α, $\mathbf{M}_U^\alpha \equiv_B \mathbf{M}_\Sigma^\alpha$, implies the following statements:*

(i) There is a Hilbert space $\mathcal{H}^\alpha \supset \mathcal{H}^{\alpha-1}$ with even dimensionality $\dim \mathcal{H}^\alpha = N = 2^\alpha$.

(ii) There is a preferred basis $b^\alpha \subset \mathcal{H}^\alpha$, i.e. the basis elements form a family of orthonormal elements in $\mathcal{H}^\alpha$ such that each basis element is a non-degenerate eigenvector of a Hermitian operator Σ^α acting on $\mathcal{H}^\alpha$.

(iii) There is a unique basis element $\psi^* \in b^\alpha$ which is the selected quantum state of the universe at stage α bisimilar to the root of the tree $\mathbf{M}_U^\alpha$. This basis element encodes the maximum information about the current stage.

A basic test for any quantum mechanical description of the universe is the necessity for an explanation of an apparently smooth three-dimensional manifold structure that on many length scales does not exhibit any quantum character whatsoever. A smooth three-dimensional space is one of the basic pillars of our external experience. Surely, there are further levels of difficulties related to this issue; for instance, the problem of how a quantum model of the universe may plausibly emerge into a unified description of space *and* time resulting in a four-dimensional manifold structure being locally isomorphic to Minkowski space. And finally, the question of how a general representation of space, time and matter could ever be accomplished to incorporate full general relativity. With regard to any solution of these problems we may have just gained first insights into the possible building blocks of locality in physics; for example, the factor structure of the selected state of the universe may be responsible for a classical Einstein universe, where the interplay of factorized and entangled states may give rise to causal sets [8, 9].

The present work permits for a slightly broader view on the problem of the basic building blocks of the universe, which are the elusive fundamental degrees of freedom in physics. Today there are at least two roads leading to this problem. The first is followed in those attempts which surmise that the fundamental degrees of freedom of the universe should be closely related to geometrical points in general relativity. All attempts that try to construct a canonical quantization of general relativity can be found here. But there is a second path where *en route* it is not presumed that such a relationship to general relativity exists. Physical approaches of the second type look for other fundamental aspects of nature that are not directly associated with relativistic space-time structure. Thus even though a consistency proof with general relativity remains to be done, other aspects of nature may be relevant to the problem. Can we begin to explore these aspects and identify their degrees of freedom?

There is a notion of continuity and distance for proximity spaces. Given a proximity space (X, P) we say (X, P) is *P-continuous* if for any $x, y \in X$ there is a set $\{x, z_1, \ldots, z_m, y\}$, with integer $m \geq 0$, such that the set $\{xPz_1, \ldots, z_mPy\}$ exists. Thus even though X is discrete, a proper notion of perceptual continuity can be defined because within each sequential pair of points in $\{x, z_1, \ldots, z_m, y\}$ one point is indiscernible from the other. We call the set $\{x, z_1, \ldots, z_n, y\}$ *open path* from x to y and concurrently assume that an open path does not contain closed paths, i.e. each element appears exactly once along the way. Define the length of an open path as $l(x, y) := m$, and set the trivial case $l(x, x) = 0$ for all $x, y \in X$. It follows that for any ordinal stage of the quantum universe the Kripke structure $\mathbf{M}_\Sigma^\alpha$ is P-continuous because it is a proximity space. Since $\mathbf{M}_\Sigma^\alpha$ is a rooted tree the path is unique and we can define the *tree metric* d_T^α on $\mathbf{M}_\Sigma^\alpha$ as

$$d_T^\alpha : b^\alpha \times b^\alpha \to \mathbb{N}_0 \quad \text{with} \quad d_T(\psi, \psi') := l(\psi, \psi').$$

We see that the Bisimulation Principle invokes a tree metric structure on all possible worlds in the preferred basis. But what does this metric mean physically? To better answer this question, let us first briefly reconcile the general character of metrics in physics, and here especially the role of distance in space.

In classical physics, the elementary similarity relation between objects is their distance in three-dimensional space. It is given by a value of a function conventionally understood as a metric on a three-dimensional Riemannian manifold. Distance in space has been the fundamental mathematical relation in physics because space itself has been understood as the stage where all physical action happens. Before general relativity space had the role of a completely rigid and passive structure unable to expose any interaction or feedback with physical objects. Space (and time) served solely as a *block universe*—not more than a convenient labeling method for physical objects in coordinates of three-space and in time. General relativity gave space and time a dynamical role and therefore a true physical meaning. However, general relativity still shares the point of view that (local) three-dimensional space and time ought to be fundamental elements of physical experience. This heritage is a remainder from times when the universe was regarded as a rigid bock and it finds its expression in the fact that Einstein's field equations determine a metric tensor of a four-manifold as a solution. But, in a broader perspective, quantum theory showed us that physical systems may in general have degrees of freedom that do not require a representation in three-dimensional space.[2]

The quest for a theory of quantum gravity is the search for a theory of the fundamental degrees of freedom in physics; therefore difficulties can be expected early in any attempt to construct a quantum version of general relativity, simply because the latter initially narrows the view to points three-space and time as the main candidates while the former allows for a broader view where the elementary degrees of freedom might be altogether different. What, then, can be said at least about the fundamental degrees of freedom? A reasonable minimum assumption would be the possibility of pairwise comparison because in any physical theory there should be observationally accessible and distinguishable degrees of freedom; further, there should be a degree of similarity for already distinguished degrees of freedom.

In the present approach such a similarity relation naturally comes with a tree metric. So, with the previous thoughts in mind, we want to extend our view on similarity relations in physics and ask: are there physical objects that are comparable by means of a tree metric? There are such objects and to find them one has to remember that physical objects are carriers of information. This means that in general a physical object's identity is not fully confined to its geometric points in classical space-time, but that an essential part of the object's identity in the universe can be found only within the information that it carries.

Three examples may better illustrate this idea. A printed book, for instance, could be correctly described by means of a large amount of individual physical particles altogether forming a certain solid state. Such a description would involve a vast collection of equations representing the fabric of the paper, while other equations would describe the behavior of ink particles, and so on. However, such a representation would make it practically impossible to decipher the book's story and an immanent part of the book's identity would be lost. Another example are black holes. Black holes can be interpreted as classical solutions to Einstein's field equations but this is probably a minor part of the whole story. Theoretical evidence holds that black holes are carriers of information placed on their surface, the event horizon, and this information will likely be accessible through a full quantum description of black holes. A classical black hole solution in general relativity merely becomes a description of the physical carrier but it is probably not a suitable representation of the information the black hole carries. Our third example is the biological macromolecule DNA. Here too we may give a reductionist description giving rise to a large collection of interacting atoms in three spatial dimensions, yet again such a representation would hardly reveal the genetic code and its biological meaning.

[2] For example, consider a quantum spin system.

Hence, it is not hard to find physical objects which are *primely* carriers of information and which play a secondary role as extended objects in space, and any theory which claims to present the universal physical degrees of freedom should address this issue. Think again of DNA: the essential similarity relation between DNA strands is not their distance in three-dimensional space but their *phylogenetic* distance in the Tree of Life of all phyla. And, indeed, tree metrics and related structures turn out to be very useful when it comes to compare the *information encoded* on physical objects [15].

We can now begin to study the discrete tree metric structure that follows naturally from the Bisimulation Principle of structural unfolding. Any tree metric can be isometrically embedded through code words into the Banach spaces L_1 and l_1. This characterization determines an embedding of the tree metric into the Euclidean space l_2. The construction of an Euclidean distance matrix out of the embedded vectors in the Hilbert space l_2 represents the self-test Σ^α and its eigenvectors form a non-degenerate basis of $\mathcal{H}^\alpha$, which constitute the possible worlds of the Kripke structure. These are—in brief summary—the essential steps to arrive, through induction over the ordinals [12], at the main Proposition 1.

With regard to an emergent classical manifold structure Euclidean distance matrices for large N have remarkable properties [6]; for example, when going from small negative to large negative eigenvalues the corresponding eigenvectors undergo a localization-delocalization transition (Euclidean distance matrices have one positive eigenvalue and all other eigenvalues are negative.) Large negative eigenvalues admit a continuous representation of the Euclidean distance matrix leading to an integral equation locally similar to a Laplace equation.

The follow-up problem of a three-dimensional space can be approached by reverting to J. A. Wheeler's old "bucket of dust" pregeometry model from 1964 [20] which, many years later and after Wheeler himself dismissed it, was picked up again and generalized by Nagels [14]. The model input is kept at a minimum and assumes only a set of abstract nodes ("things") which can be freely linked into pairs. Given (a) the probability p of two arbitrarily chosen nodes to be adjacent, i.e. to be a pair in the proximity relation, is uniform and small, and (b) that the total number of nodes is large, it can be shown that the most likely distribution of nodes has striking similarities with a closed three-dimensional space of constant positive curvature. Both prerequisites are met in our model as for higher stages of structural unfolding, $\alpha \gg 1$, this probability p becomes inversely proportional to the (truly gigantic) number of elements in the preferred basis, $N = 2^\alpha$.

Another interesting observation arises from the reflexivity property of the proximity relation: $\psi P^\alpha_\Sigma \psi$ for all preferred basis elements $\psi \in b^\alpha$. This is true mathematically because $x \in Q_x$ for any quantum Q_x, which simply means that any measured result is indistinguishable from itself. As a consequence, through bisimulation, self-referentiality is imposed onto all possible worlds in $\mathbf{M}^\alpha_U$ leading to non-wellfounded sets. To this end, reflexivity of an accessibility relation has been assigned to self-awareness of an agent represented by the underlying Kripke structure [18], and the self-referential structure of hypersets in structural set theory can be directly linked to representational self-awareness [13]. Hence, with the Bisimulation Principle, deeper implications about the non-wellfounded quantum universe are within reach.

It then appears not unreasonable to conclude that natural numbers may indeed become both the primary archetypes of the physically possible, and the tangible connection between the spheres of matter and mind.

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