COULD INFINITY SOLVE THE ANALOG-DIGITAL DILEMMA?

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Abstract. No significant aspect of reality seems to be infinitely divisible, except perhaps space and time. Two entities usually considered as continuums modeled by the densely ordered set of the real numbers. The formal consistency of the analog model of spacetime depends, therefore, on the consistency of the densely ordered sets, which in turn depends on the consistency of the actual infinity hypothesis. Under the assumption that reality is itself consistent, that dependence makes it possible to test the consistency of the analog model: to prove the inconsistency of the actual infinity. This paper presents five short arguments suggesting that notion could be, in fact, inconsistent. In consonance with that possibility, the paper introduces a new way of discussing on some elementary aspects of spacetime and of relativity in cell-automaton like models. It also suggests an experimental way to test a simplified digital model of spacetime.

Contents

1. Introduction

We have always presumed that anything that can be divided can be divided infinitely many times. It is an old pre-Socratic idea, perhaps the most primitive way of conceiving the divisibility of things.[1] Although the things we have been able to divide never divide in that way. Ordinary matter, elementary particles, electromagnetic energy, electric and non electric charges, are all of them of a discrete nature, with an indivisible minimum. Only space and time remain as possible infinitely divisible entities. And the prevailing idea is that they are infinitely divisible. The result is the so called continuum spacetime: between any two points infinitely many other different points do exist, which, as Wittgenstein said, makes one feel dizzy.

Although the discrete nature of spacetime have already been proposed in different areas of physics,[2] it surprises the little attention we usually pay to the fact that the assumed continuity of spacetime is founded on the actual infinity hypothesis. This foundation makes the consistency of any continuum depends on the consistency of the actual infinity hypothesis: if the actual infinity were inconsistent all continuums would also be inconsistent. It is in this sense that the actual infinity hypothesis could be the key to solve the dilemma on the digital or analog nature of reality.

The actual infinity does not get along with reality. When infinity appears in physical equations, physicists are forced to remove it from them because of the unsolvable problems it invariably leads to. A removal that usually requires a lot of hard work,[3] which suggests that physicists never question the consistency of the actual infinity,[4] as if that consistency had been demonstrated. Nothing further from the truth: the actual infinity is only a hypothesis subsumed by the Axiom of Infinity. Authors of the intellectual stature of Brouwer, Poincaré or Wittgenstein, among others, rejected that hypothesis.

Let us recall what is assumed when one assumes the Axiom of Infinity. Consider the list of natural numbers in its natural order of precedence: 1, 2, 3, ... According to the hypothesis of the actual infinity that list exists as a *complete totality*, i.e as a totality that contains, all at once, all natural numbers. The ordered list of natural numbers does exist as a complete totality in spite of the fact that no last number completes the list. Modern infinitists defend that *all* natural numbers could be counted, even in a finite time.[5] The alternative is the hypothesis of the potential infinity, that rejects the existence of *complete* infinite totalities and then the possibility of counting all natural numbers. From this perspective, natural numbers result from the endless process of counting: it is always possible to count a number greater than any given number. But it is impossible to complete the process of counting, so that the complete list of all natural numbers makes no sense.

Nothing in the history of science can be compared with the fecundity of the actual infinity as source of paradoxes. As we will immediately see, at the end of the XIX century one of those paradoxes served to lay the foundations of modern transfinite mathematics. Ever since, infinitism maintains an hegemonic position, almost free of criticism, in contemporary mathematics. But, as could not be otherwise, the actual infinity can also be questioned. The next section resumes five short arguments[6] that put into question the formal consistency of the actual infinity hypothesis, and then the consistency of any continuum, of any analog model.

2. Testing the hypothesis of the actual infinity

Most of the paradoxes related to infinity result from the violation of the euclidian Axiom of the Whole and the Part,[7] among them, the so called paradoxes of reflexivity in which a whole is paired off with one of its proper parts.[8] Galileo's paradox[9] is a well known example of reflexive paradox. Authors as Proclus, J. Filopón, Thabit ibn Qurra al-Harani, R. Grosseteste, G. de Rimini, W. of Ockham etc. found many others examples.[10] The strategy of pairing off the elements of two sets is not exactly a modern invention. Aristotle used it to refute Zeno's Dichotomies.[11] And since then, it has been used by many authors with different discursive purposes, although never (including the case of Bolzano[12]) as an instrument to consummate the violation of the old euclidian axiom. Things began to change with Dedekind, who stated the definition of the infinite sets just on the basis of that violation: a set is infinite if it can be put into a one to one correspondence with one of its proper subsets.[13] Dedekind and Cantor inaugurated the so called paradise of the actual infinity, where exhaustive injections (bijections or one to one correspondences) play a capital role.

2.1. **Reinterpreting the paradoxes of reflexivity.** As is well known, an exhaustive injection between two sets A and B is a correspondence between the elements of both sets in which each element of A is paired with a different element of B, and all elements of A and of B result paired. When at least one element of the set B results unpaired the injection is said non-exhaustive. Exhaustive and non-exhaustive injections can be used as instruments to compare the cardinality of finite sets. But if the sets are infinite only exhaustive injections can be used. It seems reasonable to assume that if after pairing every element of a set A with a different element of a set B all elements of the set B result paired, then A and B have the same number of elements. But it seems also reasonable, and for exactly the same elementary reasons, that if after pairing every element of a set A with a different element of a set B, one or more elements of the set B remain unpaired, then A and B have not the same number of elements. Notice that exhaustive and non-exhaustive injections make use of the same basic method of pairing the elements of two sets, without carrying out any arithmetic operation.

Exhaustive and non-exhaustive injections should have the same validity as instruments to compare the cardinality of infinite sets just because they use *exactly the same comparison method*. However, only exhaustive injections do have that privilege. We should pay attention, however, to the fact that the existence of both exhaustive and non-exhaustive injections between two sets could be indicating the existence of a contradiction (that both sets have and have not the same cardinality), in whose case the distinction in favor of exhaustive injections would be the distinction of a term of a contradiction to the detriment of the other. Note the existence of transfinite cardinals as $\aleph_0$, for which it holds $\aleph_0 = \aleph_0 + \aleph_0$ and the like, is derived from assuming the existence of sets whose elements can be paired off with the elements of some of its proper subsets,[14] but not vice versa. So, under penalty of circular reasoning, we cannot infer from the derived existence of transfinite cardinals with arithmetical peculiarities, the existence of the sets from which those cardinals are derived. We are simply discussing if the method of pairing the elements of two sets is appropriate to compare their cardinality; and if it is, why non-exhaustive injections are arbitrarily rejected, because that rejection could be concealing a fundamental contradiction.

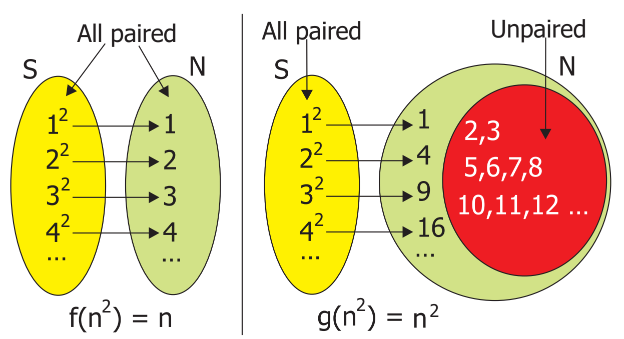


Figure 1. The suspicious power of the ellipsis: sets S and N have (left) and have not (right) the same cardinality.

Assume for a moment that exhaustive and non-exhaustive injections are both of them valid instruments to compare the cardinality of infinite sets. In these conditions, let B be an infinite set. By definition, there exists a proper subset A of B and an exhaustive injection f from A to B proving both sets have the same number of elements. Consider now the injection g from A to B defined by $g(x) = x, \quad \forall x \in A$, which evidently is non-exhaustive (the elements of the nonempty set B-A are not paired). Injections f and g would be proving respectively that A and B have and have not the same number of elements.

We must therefore decide if exhaustive and non-exhaustive injections do have the same validity as instruments to compare the cardinality of all types of sets. If they have, then all infinite sets are inconsistent. If they don't, at least one reason should be given to explain why they don't. And, if no reason can be given, then the arbitrary distinction in favor of the exhaustive injections should be arbitrarily declared in an appropriate ad hoc axiom. Until then, the foundation of (transfinite) set theory rests on the basis of one of the terms of a contradiction.[15] As could not be otherwise in a theory of such foundations, inconsistencies appeared from the very beginning: the set of all ordinals and the set of all cardinals were proved to be inconsistent by Burali-Forti[16] and Cantor respectively. According to Cantor those sets are inconsistent because of their excessive infinitude.[17] One can be infinite but only within certain limits. By the appropriate ad hoc axioms, it was finally stated that some infinite totalities, as the totality of cardinals or the totality of ordinals, do not exist because they lead to contradictions. It can easily be proved, however, that in a naive (not limited by axiomatic restrictions) set theory, each set of cardinal $C > 1$ originates nothing less than 2^C inconsistent totalities.[18]

2.2. **A rational version of Cantor's 1874 argument.**[19] Since the set $\mathbb{Q}$ of rational numbers is denumerable we can consider a one to one correspondence f between this set and the set $\mathbb{N}$ of natural numbers. Let $\langle q_i \rangle_{i\in\mathbb{N}}$ be the ω-ordered sequence[20] of rationals defined by $q_i = f(i)$, $\forall i \in \mathbb{N}$. Obviously $\langle q_i \rangle_{i\in\mathbb{N}}$ contain all rational numbers. Let now x be a rational variable whose initial value is b, the right endpoint of any rational interval (a, b) and consider the following sequence of definitions:

$$\begin{cases} i = 1, 2, 3, \dots \\ q_i \in (a, b) \wedge q_i < x \Rightarrow x = q_i \end{cases} \tag{1}$$

that compares x with the successive elements of $\langle q_i \rangle_{i\in\mathbb{N}}$ that belong to (a, b), and defines x as the compared element each time the compared element is less than the current value of x. Assume also the following restriction: each successive definition (x-definition from now on) is carried out if, and only if, x results defined as a rational number. Assume finally, that while the successive x-definitions can be carried out, they are carried out. The value of x once performed all possible x-definitions,[21] whatsoever be the finite or infinite number of times it has been defined, will be a rational number within the interval $(a, b]$ because *it was always defined as a rational number within* $(a, b]$*, and only as a rational number within* $(a, b]$. Consider then the rational interval (a, x) and any element η within (a, x), as for instance $1/2(a + x)$. It evidently holds $\eta \in (a, b)$ and $\eta < x$. But η does not belong to $\langle q_i \rangle_{i\in\mathbb{N}}$ for if that were the case there would exist a q_v such that $\eta = q_v$, and then we would have $q_v < x$, which, according to (1), means the number of performed x-definitions is less than v. But this is impossible because for each $n \leq v$, $f(n)$ is a well defined rational number, being therefore possible to decide if it is within (a, b) and if it is less than the current value of x, which in turn makes it possible to perform the firsts v x-definitons. So, at least the firsts v x-definitions have been carried out and then it must hold: $x \leq q_v$. The rational η proves, therefore, the existence of rational numbers within (a, b) which are not in $\langle q_i \rangle_{i\in\mathbb{N}}$, and then the falseness of the initial assumption on the countable nature of $\mathbb{Q}$. According to Cantor's proof on the denumerability of $\mathbb{Q}$, we can conclude this set is and is not denumerable. The sequence of definitions (1) leads to some other contradictory results the reader could easily find.[22]

2.3. **Jordan curves of infinite length.** Let $f(x)$ be a real valued function whose graph is a Jordan Curve[23] $\mathbf{J}$ in the euclidian plane $\mathbb{R}^2$. Let a, b be the endpoints of the arc $\widetilde{ab}$ in $\mathbf{J}$. We will write $L(a, b)$ to denote the length of $\widetilde{ab}$:

$$L(a, b) = \int_a^b \sqrt{1 + (f(x)')^2} dx \tag{2}$$

Assume that $\mathbf{J}$ has an infinite length. In these conditions let r be any real number greater than 0 and assume $\mathbf{J}$ is partitioned clockwise from a point x_1 into a certain number of adjacent parts $\widetilde{x_1 x_2}$, $\widetilde{x_2 x_3}$, $\widetilde{x_3 x_4}$ ... so that each part $\widetilde{x_i x_{i+1}}$ has a *finite length* equal or greater than r:

$$L(x_i, x_{i+1}) \geq r, \ \forall i \in I \tag{3}$$

where I is the set of the partition indexes. This partition, in symbols $\langle x_i \rangle_{i\in I}$, must be infinite otherwise, and being finite the length of the parts, $\mathbf{J}$ would have a finite length. In addition, and according to Cantor, $\langle x_i \rangle_{i\in I}$ cannot be uncountably infinite.[24] The partition $\langle x_i \rangle_{i\in I}$ must therefore be countably infinite. Accordingly, the sequence $\langle x_i \rangle_{i\in I}$ must be β-ordered, being β a transfinite ordinal, i.e. an ordinal of the second class according to Cantor's terminology.[25]

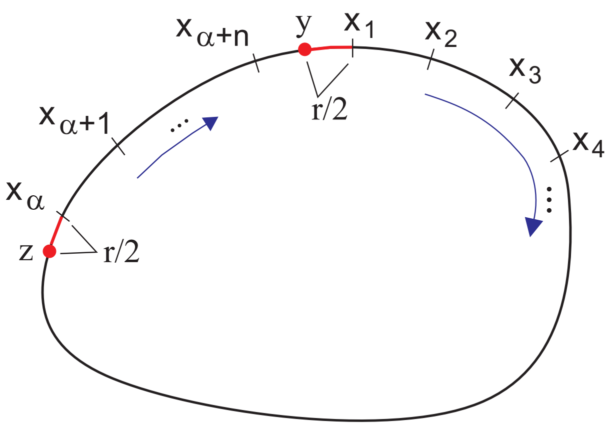


Figure 2. z can only belong to the impossible immediate predecessor of $\widetilde{x_\alpha x}_{\alpha+1}$.

Now consider a point y anticlockwise from x_1 and such that $L(y, x_1) = r/2$. From (3) we infer that y can only belong to the last element of $\langle x_i \rangle_{i \in I}$. Accordingly this sequence must have a last element and then it must holds: $\beta = \alpha + n$, being α a transfinite ordinal of the second class second kind,[26] and n a finite ordinal. The partition $\langle x_i \rangle_{i \in I}$ is then (α+n)-ordered. If must therefore exist an α-th second class second kind ordinal in I so that $\widetilde{x_\alpha x}_{\alpha+1}$ has not immediate predecessor.[27] Now then, according again to (3), the point z anticlockwise from x_α and such that $L(z, x_\alpha) = r/2$ can only belong to a part immediately preceding $\widetilde{x_\alpha x}_{\alpha+1}$, which is impossible. This proves that $\langle x_i \rangle_{i \in I}$ must be, but cannot be, $(\alpha + n)$-ordered, a contradiction we have derived from our initial assumption on the infinite length of **J**. We must therefore conclude that Jordan Curves of infinite length are inconsistent objects, and so must be any metrical space compatible with them.[28]

2.4. **Hilbert's machine.** The following argument makes use of a theoretical device, Hilbert's machine, inspired by the emblematic Hilbert Hotel. It is composed of the following elements (see Figure 3):

(1) An infinite wire divided into two infinite parts, the left and the right side:
   (a) The right side is divided into an ω-ordered sequence of adjacent sections $\langle S_i \rangle_{i \in \mathbb{N}}$ which are indexed from left to right as S_1, S_2, S_3,.... They will be referred to as right sections.
   (b) The left side is also divided into an ω-ordered sequence of adjacent sections $\langle S'_i \rangle_{i \in \mathbb{N}}$ indexed now from right to left as ..., S'_3, S'_2, S'_1; being S'_1 adjacent to S_1. They will be referred to as left sections.
(2) An ω-ordered sequence $\langle B_i \rangle_{i \in \mathbb{N}}$ of sliding beads, physically connected, which are inserted in the wire as the beads of an abacus, being each bead B_i initially placed on the right section S_i.
(3) A mechanism inside the wire that slides simultaneously each bead exactly one section to the left if, and only if, no bead is removed from the wire as a consequence of the sliding.

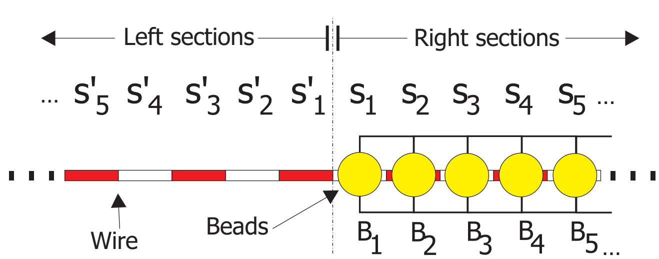


Figure 3. Hilbert's machine ready to begin the sequence of L-slidings.

As can be immediately inferred from the above definition of Hilbert's machine, the simultaneous sliding of each bead one section to the left (L-sliding from now on) will be possible if, and only if, there is an empty section on the left of B_1, making it possible that each bead can simultaneously L-slide one section to the left: the first one to the next empty section on its left, and each $B_{n,n>1}$ to the section previously occupied by B_{n-1}. Since the sections $\langle S'_i \rangle_{i \in \mathbb{N}}$ of the left side are ω-ordered, each section S'_n has an immediate successor S'_{n+1} placed just on its left. According to the actual infinity hypothesis, the infinitely many left sections exist as a complete totality in spite of the fact that there is no last section completing the sequence.

Assume now that while the successive L-slidings can be carried out, they are carried out. Once performed all possible L-slidings we will have the following two contradictory results.

- Once performed all possible L-slidings, at least one bead remains inserted in the wire. Assume, on the contrary, that once performed all possible L-slidings no bead remains inserted in the wire. This would imply:
  (1) The existence of a last left section in the wire through which all beads can be removed from the wire.
  (2) Infinitely many violations of Hilbert's machine law of functioning: an L-sliding is performed if, and only if, all beads remains inserted in the wire.
  And both consequences are impossible: the first because the left sections $\langle S'_i \rangle_{i \in \mathbb{N}}$ of the wire are ω-ordered and there is no last element in an ω-ordered sequence. The second because the same arbitrary violation could be expected from any definition or procedure involving infinitely many steps, which, obviously, would invalidate all transfinite mathematics.

- Once performed all possible L-slidings no beads remains inserted in the wire. Consider any bead B_v and assume that once performed all possible L-slidings it remains inserted in the right section S_k. Since B_v was initially inserted in S_v only a finite number $v - k$ of L-slidings would have been performed. As a consequence of these $v - k$ L-slidings the first bead B_1 will be placed in the left section S'_{v-k} and since this section has an immediate successor on its left, the section S'_{v-k+1}, a new L-sliding would be possible. Consequently not all possible L-slidings would have been performed. A similar reasoning can be applied if B_v were finally placed on a left section S'_n, being now the number of performed L-slidings exactly $n + k - 1$. Thus, if all possible L-slidings are performed no bead remain inserted in the wire.

2.5. **A rational inconsistency.** The set $\mathbb{Q}$ of rational numbers, in its natural order of precedence, is densely ordered: between any two rationals infinitely many different rationals do exist. But, being denumerable,[29] it is also ω-ordered: between any two successive rationals no other rational exists. The argument that follows takes advantage of this astonishing *singularity*. For the sake of simplicity, we will only consider the set $\mathbb{Q}^+$ of positive rationals, which is also denumerable and densely ordered. Let then f be a one to one correspondence between the set $\mathbb{N}$ of natural numbers and $\mathbb{Q}^+$. Evidently, f makes it possible to ω-order $\mathbb{Q}^+$:

$$\mathbb{Q}^+ = \{q_1, q_2, q_3, \dots\}, \text{ where } q_i = f(i), \ \forall i \in \mathbb{N} \tag{4}$$

Let now x be a rational variable whose initial value is 1 and consider the following ω-ordered sequence of definitions:

$$\left.\begin{array}{r} d_i = |q_{i+1} - q_1| \\ d_i < x \Rightarrow x = d_i \end{array}\right\} \ i = 1, 2, 3, \dots \tag{5}$$

where $|q_{i+1} - q_1|$ is the absolute value of $q_{i+1} - q_1$, and '<' stands for the usual dense ordering of $\mathbb{Q}$; i.e $d_i < x$ means $d_i - x < 0$. Definition (5) defines the sequence $\langle d_i \rangle_{i \in \mathbb{I}}$ and redefines the variable x each time the defined term d_i is less than x.

It is immediate to prove that each one of the successive definitions (5) leaves x defined as a positive rational number. Evidently, d_1 is a well defined positive rational number and then x results defined as d_1 if $d_1 < 1$, or it retains its original value if it is not. So the first definition leaves x defined as a positive rational number. Assume the n-th definition also leaves x defined as a positive rational number. Being the field of rational numbers closed under the operation of subtraction, it is quite clair that d_{n+1} is a well defined positive rational number. Thus, the (n+1)-th definition defines

x as d_{n+1} if $d_{n+1} < x$, or x retains its previous well defined positive rational value if $d_{n+1} \geq x$. In consequence all infinitely many definitions (5) leave x defined as a positive rational number, whatsoever it be.

The completion of the sequence of definitions (5) defines x as a positive rational number a certain finite or infinite number of times. Consider then the following two alternatives for the resulting value of x:

(1) The completion of definition (5) leaves x defined as a positive rational number.
(2) The completion of definition (5) does not leave x defined as a positive rational number, despite of the fact that every performed definition left x defined at a positive rational number.

Since the completion of any *finite* sequence of definitions leaves the object defined as in its last definition, the second alternative, which implies the violation of (5), could only be an unexpected consequence of completing an *infinite* sequence of definitions in which no last definition completes the sequence. But, if that were the case, the same violation could be expected from any other definition or procedure involving infinitely many steps without a last completing step, in whose case transfinite mathematics would lose all of its meaning. We must reject the second alternative and conclude that the completion of definition (5) leaves x defined as a positive rational. It is now immediate to prove the following two contradictory results.

In the usual dense order of $\mathbb{Q}$, and once completed (5), the rational $f(1)+x$ is less than any rational greater than $f(1)$. Assume it is not. There would be a rational $f(k)$ greater than $f(1)$ and less than $f(1)+x$:

$$f(1) < f(k) < f(1) + x \tag{6}$$

and then:

$$0 < f(k) - f(1) < x \tag{7}$$

which is impossible because $x \leq |f(k) - f(1)|$ at least from the definition of d_{k-1}.

On the other hand, in the usual dense order of $\mathbb{Q}$, and once completed (5), the rational $f(1)+x$ is not less than any rational greater than $f(1)$. In fact, the rational $f(1) + 0.1 \times x$, for instance, is greater than $f(1)$ and less then $f(1) + x$.

3. Spacetime and relativity in Cell Automata Like Models

If the hypothesis of the actual infinity were inconsistent all continuums derived from it, as the spacetime continuum, would also be inconsistent. In these conditions, if reality is itself consistent, space and time could only be of a digital nature. As noted above, the hypothesis of a digital (discrete or discontinuous) spacetime has already been proposed in different areas of physics, although the corresponding theories are invariably developed within the same continuous-mathematics framework. The model of a new science based on the digital notion of cellular automata has also been proposed.[30] As could not be otherwise, our objective here is not to propose a new digital model of reality but simply to suggest a new way of discussing on geometry and motion in a simplified digital spacetime, which in turn suggests a new understanding of relativity. And, although we will not deal with it here, the problem of change, that remains unsolved from Parmenides time,[31] could also found a solution within the framework of a digital reality.

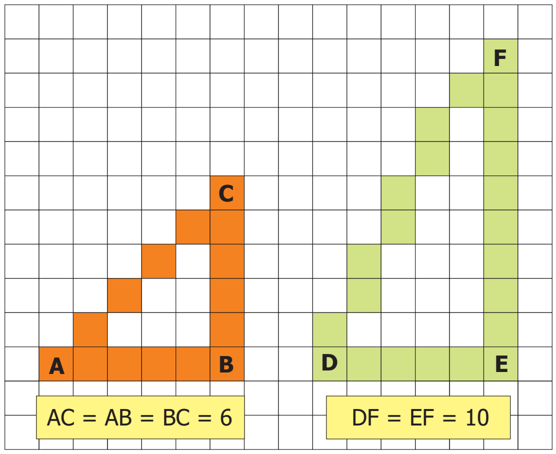
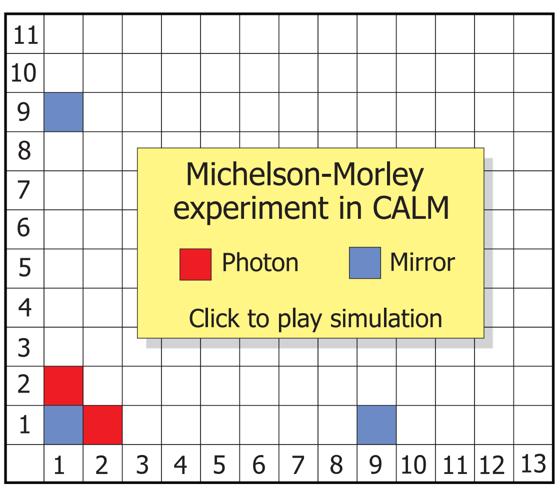


Figure 4. Pythagoras d-theorem: the hypothenuse has the same number of sits (space units) as the greater of the legs.

Without going into further details, consider a two-dimension cell-automaton like model (CALM) and assume that each of its cells represents an indivisible space unit (sit: space unit) of an isotropic length[32] σ; consider also the duration of each automaton state represents an indivisible unit of time (tit: time unit) of duration τ. The content (state) of a sit is the simplest object that can move throughout the automaton and the motion can only be to an adjacent sit. A trajectory would be any succession of adjacent sits. Let us now define the distance between two sits A and B as the less number of adjacent sits that connect A with B. The sped in such a model would be the number of tits elapsed to move to the next adjacent sit of a trajectory. In this model there would be a maximum insurmountable sped: one tit per sit. The second principle of relativity would not be necessary in a CALM physics.

A point of note in CALM geometry is the digital version of Pythagoras theorem (Pythagoras d-theorem from now on): the hypotenuse of a right triangle has the same number of sits as the greater of the legs. Thus, if h is the number of sits of the hypotenuse of a right triangle whose legs have x and y sits respectively, being $x \leq y$, se will have $h = y$. If $h'\sigma$, $x\sigma$ and $y\sigma$ are the lengths of the hypothenuse and legs of the same triangle in the continuum model, we will have:

$$h'\sigma/h\sigma = h'/y = h'/\sqrt{h'^2 - x^2} = 1/\sqrt{1 - x^2/h'^2} \tag{8}$$



which has the same form as the relativistic factor γ. Consider now a Michelson-Morley-like interferometer moving with uniform linear motion of sped v in a CALM. Along the arm parallel to v, light traverses a distance $(y+x)$ sits when it moves in the same direction as the arm, and $(y-x)$ sits when it moves in the opposite direction, being y the number of sits of the arm and x the number of sits the apparatus moves while light travels between the two mirrors placed in the direction of v. This ray then moves a total of $y + x + y - x = 2y$ sits. According to Pythagoras d-theorem the ray moving perpendicular to v also moves $2y$ sits. The experimental conclusion would be, therefore, the same as in the continuum model. But the explanation of those results would be quite different. In fact, while in the continuum model a length contraction in the direction of v is needed, in the case of a CALM Pythagoras d-theorem suffices. Fitzgerald-Lorentz's contraction could be interpreted as a transformation between the digital and the analog model of reality. In fact, in accordance with (8) and taking into account that $x = vt$ and $h' = ct$, being c the speed of light, we can write:

$$h'/y = h'/\sqrt{h'^2 - x^2} = 1/\sqrt{1 - x^2/h'^2} = 1/\sqrt{1 - v^2/c^2} = \gamma \tag{9}$$

This short discussion on spacetime and motion in CALM seems to indicate the convenience of discussing the meaning of the theory of relativity in a digital reality: could that theory be interpreted as the result of describing a digital reality from an analog perspective?. Or in other words, could it be interpreted as a continuous-mathematics description of a digital reality?

4. A simple experimental test for Pythagoras digital theorem

Pythagoras d-theorem provides an experimental way to prove its own validity. Consider the interferometer depicted in Figure 5. The semi-silvered mirror m1 divides the laser beam L1 into L2, that moves towards the mirror m2, and L3 that moves towards the mirror m3. L2 is reflected by mirror m2 and then moves to m3 where it is reflected in the same direction as L3, with which it interferes. The interference is then observed in the appropriate detector.

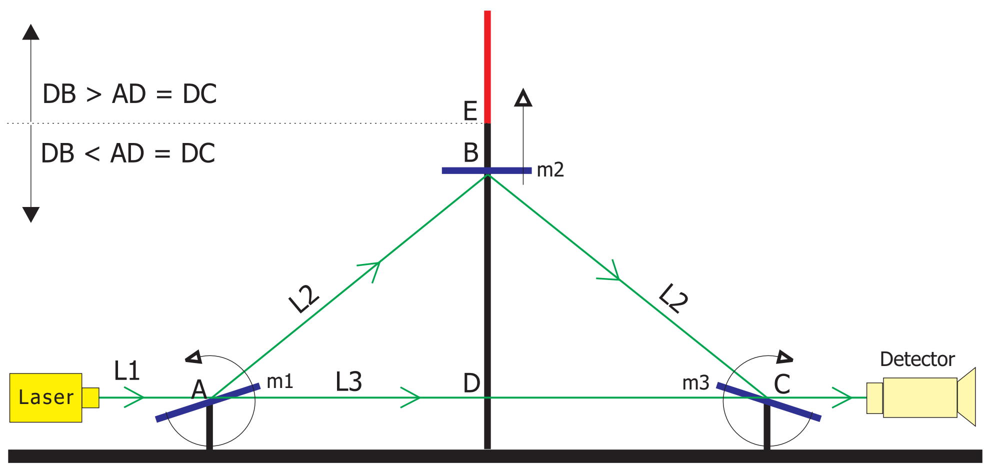


Figure 5. Schematic depiction of an interferometer to test the digital nature of spacetime.

Mirrors m1 and m3 can synchronically rotate in opposite directions while mirror m2 can move up and down so that L2 meets L3 in the same place of m3. The legs AD and DC of the right triangles ADB and DBC are always of the same length, but the common leg DB changes as the mirror m2 moves up and down. According to Pythagoras d-theorem, while B is below E it will hold: AB = AD = BC = DC. Thus, we could move m3 up and down and observe the same interference pattern whenever B remains below E. In these conditions, in fact, L2 and L3 move through the same number of sits. On the contrary, if B moves above E it will hold AB = DB > AD; BC = DB > DC and a different pattern of interference should be observed because in this case the number of sits L2 traverses increases as DB increases, while the number of sits traversed by L3 remains constant. Pythagoras d-theorem would be experimentally proven if two interference patterns were observed: the one when m2 moves below E; the other when it does above E. Evidently, and due to the extreme simplicity of the discussed model, a negative result would not invalidate the digital model of spacetime.

Notes

[1]Perhaps because our sensorial perception is continuous. Although neurologically built with discrete entities: the atoms of knowledge [25], [26]. ←

[2][21], [19], [34], [14], [31], [3], [32], [33], [4], [23], [24]. ←

[3][15], [20], [18], [19]. ←

[4]Things are not different with mathematicians. ←

[5]By performing the following supertask [28]: count each number n at the precise instant t_n, the n-th element of an ω-ordered sequence $\langle t_i \rangle_{i\in\mathbb{N}}$ of instants within the finite interval (t_a, t_b) whose limit is just t_b. ←

[6]For the extended versions of these and other arguments take a glance at http://www.interciencia.es. ←

[7]The whole is greater than the part is one of the common notions of Euclid's Elements [13]. ←

[8][30], [12]. ←

[9]The set of natural numbers can be paired with one of its proper subsets: the set of its squares [16]. ←

[10][30]. ←

[11]Aristotle finally rejected his pairing method and proposed the distinction between the actual and the potential infinity [2], [1]. ←

[12][5]. ←

[13][11]. ←

[14][11], [9]. ←

[15]Unbelievable as it may seems, the axiomatic foundation of set theory has always ignored this problem. ←

[16][6]. ←

[17]Letter to Dedekind quoted in [10, pag. 245], [17], [22]. ←

[18] Since the elements of a (naive) set can be sets, sets of sets, sets of sets of sets and so on, consider the following relation $\mathbf{R}$ between two sets: two sets A and B are $\mathbf{R}$-related, written $A\,\mathbf{R}\,B$, if B contains at least one element which forms part of the definition of at least one element of A. For instance, if:

$$A = \{\ \{\{a,\ \{b\}\}\},\ \{c\},\ d,\ \{\{\{\{e\}\}\}\}\dots\ \} \tag{10}$$

$$B = \{1, 2, b\} \tag{11}$$

then A is $\mathbf{R}$-related to B because the element b of B forms part of the definition of the element $\{\{\ a,\ \{b\}\ \}\}$ of A. In these conditions let X be any set such that $|X| > 1$, and let Y be any non empty subset of X. From Y we define the set T_Y according to:

$$T_Y = \{Z \mid Z \cap Y = \emptyset\ \wedge \neg\exists V(V \cap Y \neq \emptyset \wedge Z\,\mathbf{R}\,V)\} \tag{12}$$

T_Y is, therefore, the set of all sets that do not contain elements of Y nor are $\mathbf{R}$-related to any set containing elements of Y. Let us now consider the set $P(T_Y)$, the power set of T_Y. The elements of $P(T_Y)$ are all of them subsets of T_Y and therefore sets of sets that do not contain elements of Y nor $\mathbf{R}$-related to sets containing elements of Y. That is to say:

$$\forall D \in P(T_Y):\ \ D \cap Y = \emptyset \wedge \neg\exists V(V \cap Y \neq \emptyset \wedge D\,\mathbf{R}\,V) \tag{13}$$

Consequently, it holds:

$$\forall D \in P(T_Y):\ \ D \in T_Y \tag{14}$$

And then:

$$P(T_Y) \subset T_Y \tag{15}$$

Accordingly, we can write:

$$|P(T_Y)| \leq |T_Y| \tag{16}$$

which contradicts Cantor's theorem on the power set:

$$|P(T_Y)| > |T_Y| \tag{17}$$

But X is any set of cardinal greater than 1 and Y any one of its nonempty subsets. We can therefore conclude that every set of cardinal C gives rise to at least 2^C inconsistent totalities. ←

[19]This argument is a variant of Cantor's first proof of the uncountable nature of the real numbers that was published in a short paper in which he also proved the countable nature of the algebraic numbers, and then of the rational numbers [7]. ←

[20]An ω-ordered sequence is one in which there is a first element and each element has an immediate successor and an immediate predecessor, except the first one that only have an immediate successor. ←

[21]If it were impossible to perform all possible x-definitions we would be in the face of the elementary contradiction of an impossible possibility. ←

[22]Evidently, contradictory results do not invalidate one another, they simply prove the existence of contradictions. This obviousness is often ignored in the discussions on the actual infinity. An argument cannot be refuted by other argument. An argument can only be refuted by indicating where and why *that* argument fails. ←

[23]A Jordan curve is a simple closed curve that is topologically equivalent to the unit circle, i.e. a curve that does not intersect itself. ←

[24]In fact, all we have to do is to pick a different rational number within each (r_i, r_{i+1}), being $r_1 = x_1$ and $r_{i+1} = r_i + L(x_i, x_{i+1})$, to get an impossible uncountable sequence of rationals [8]. ←

[25]The ordinal of the first class are finite while those of the second class are infinite [9]. ←

[26]Ordinals of the second class first kind have immediate predecessor and immediate successor, while those of the second kind have immediate successor but not immediate predecessor. ←

[27]α is the limit of a sequence of first kind ordinals, and being the limit of a sequence, it is not a term of the sequence. It cannot have, therefore, an immediate predecessor. ←

[28]From this conclusion, it could easily be proved that any open line of infinite length is also inconsistent. ←

[29][7]. ←

[30][35]. ←

[31]See [27], [29] for a general background. ←

[32]The length of a sit, and the duration of a tit, are needed in order to transform between digital and analog expressions. ←

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