

Undecidability and indeterminism

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Abstract

The famous theorem of Bell (1964) left two loopholes for determinism underneath quantum mechanics, viz. *non-local* deterministic hidden variable theories (like Bohmian mechanics) or theories *denying free choice of experimental settings* (like 't Hooft's cellular automaton interpretation of quantum mechanics). However, a precise analysis of the role of randomness in quantum theory and especially its undecidability closes these loopholes, so that—accepting the statistical predictions of quantum mechanics—determinism is excluded full stop. The main point is that Bell's theorem does not exploit the full empirical content of quantum mechanics, which consists of long series of outcomes of repeated measurements (idealized as infinite binary sequences). It only extracts the long-run relative frequencies derived from such series, and hence merely asks hidden variable theories to reproduce certain single-case Born probabilities. For the full outcome sequences of a fair quantum coin flip, quantum mechanics predicts that these sequences (almost surely) have a typicality property called *1-randomness* in logic, which is definable via computational incompressibility à la Kolmogorov and is much stronger than e.g. uncomputability. Chaitin's remarkable version of Gödel's (first) incompleteness theorem implies that 1-randomness is unprovable (even in set theory). Combined with a change of emphasis from single-case Born probabilities to randomness properties of outcome sequences, this is the key to the above claim.

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1 Introduction: Gödel and Bell

While *prima facie* totally unrelated, Gödel’s theorem (1931) in mathematical logic and Bell’s theorem (1964) in physics share a number of fairly unusual features (for *theorems*):¹

- Despite their very considerable technical and conceptual difficulty, both results are extremely famous and have caught the popular imagination like few others in science.
- Though welcome in principle—in their teens, many people including the author were intrigued by books with titles like *Gödel, Escher Bach: An Eternal Golden Braid* and *The Dancing Wu-Li Masters: An Overview of the New Physics*, both of which appeared in 1979—this imagination has fostered wild claims to the effect that Gödel proved that the mind cannot be a computer or even that God exists, whilst Bell allegedly showed that reality does not exist. Both theorems (apparently through rather different means) supposedly also supported the validity of Zen Buddhism.²
- However, even among professional mathematicians (logicians excepted) few would be able to correctly state the content of Gödel’s theorem when asked on the spot, let alone provide a correct proof, and similarly for Bell’s theorem among physicists.
- Nonetheless, many professionals will be aware of the general feeling that Gödel in some sense shattered the great mathematician Hilbert’s dream of what the foundations of mathematics should look like, whilst there is similar consensus that Bell dealt a lethal blow to Einstein’s physical world view—though ironically, Gödel worked in the spirit and formalism of Hilbert’s Proof Theory, much as Bell largely *agreed* with Einstein’s views about quantum mechanics and about physics in general.
- Both experts and amateurs seem to agree that Gödel’s theorem and Bell’s theorem penetrate the very core of the respective disciplines of mathematics and physics.

In this light, anyone interested in both of these disciplines will want to know what these results have to do with each other, especially since mathematics underwrites physics (or at least is its language).³ At first sight this connection looks remote. Roughly speaking:⁴

1. Gödel proved that any consistent mathematical theory (formalized as an axiomatic-deductive system in which proofs could in principle be carried out mechanically by a computer) that contains enough arithmetic is incomplete (in that arithmetic sentences φ exist for which neither φ nor its negation can be proved).⁵
2. Bell showed that if a deterministic “hidden variable” theory underneath (and compatible with) quantum mechanics exists, then this theory cannot be local (in the sense that the hidden state, if known, could be used for superluminal signaling).

Both were triggered by a specific historical context. Gödel (1931) reflected on the recently developed formalizations of mathematics, of which he specifically mentions the *Principia Mathematica* of Russell and Whitehead and the axioms for set theory proposed earlier by Zermelo, Fraenkel, and von Neumann. Though relegated to a footnote, the shadow of *Hilbert’s Program*, aimed to prove the consistency of mathematics (ultimately based on Cantor’s set theory) using absolutely reliable, “finitist” means, clearly loomed large, too.

Bell, on the other hand, tried to understand if the *de Broglie–Bohm pilot wave theory*, which was meant to be a deterministic theory of particle motion reproducing all predictions of quantum mechanics, *necessarily* had to be non-local: Bell’s answer, then, was “yes.”⁶

In turn, the circumstances in which Gödel and Bell operated had a long pedigree in the quest for *certainty in mathematics* and for *determinism in physics*, respectively.⁷ The former had even been challenged at least three times:⁸ first, by the transition from Euclid’s mathematics to Newton’s; second, by the set-theoretic paradoxes discovered around 1900 by Russell and others (which ultimately resulted from attempts to make Newton’s calculus rigorous by grounding it in analysis, and in turn founding analysis in the real numbers and hence in set theory), and third, by Brouwer’s challenge to “classical” mathematics, which he tried to replace by “intuitionistic” mathematics (both Hilbert and Gödel were strongly influenced by Brouwer, though neither shared his overall philosophy of mathematics).

In physics (and more generally), what Hacking (1990, Chapter 2) calls the *doctrine of necessity*, which thus far—barring a few exceptions—had pervaded European thought, began to erode in the 19th century, culminating in the invention of quantum mechanics between 1900–1930 and notably in its probability interpretation as expressed by Born (1926):

Thus Schrödinger’s quantum mechanics gives a very definite answer to the question of the outcome of a collision; however, this does not involve any causal relationship. One obtains *no* answer to the question “what is the state after the collision,” but only to the question “how probable is a specific outcome of the collision”. (...) This raises the entire problem of determinism. From the standpoint of our quantum mechanics, there is no quantity that could causally establish the outcome of a collision in each individual case; however, so far we are not aware of any experimental clue to the effect that there are internal properties of atoms that enforce some particular outcome. Should we hope to discover such properties that determine individual outcomes later (perhaps phases of the internal atomic motions)? (...) I myself tend to relinquish determinism in the atomic world. (Born, 1926, p. 866, translation by the present author)

In a letter to Born dated December 4, 1926, Einstein’s famously replied that ‘God does not play dice’ (‘Jedenfalls bin ich überzeugt, daß *der* nicht würfelt’). Within a few years he saw a link with (non) locality, and Bell’s (1964) and later papers followed up on this.

This precise history has a major impact on my argument,⁹ since it shows that right from the beginning the kind of randomness that Born (probably preceded by Pauli and followed by Bohr, Heisenberg, Jordan, Dirac, von Neumann, and most of the other pioneers of quantum mechanics except Einstein, de Broglie, and Schrödinger) argued for as being produced by quantum mechanics, was antipodal to *determinism*.¹⁰ In other words, as the above quotation and the subsequent history including Bell’s (1964) theorem show, randomness in quantum mechanics was identified with *indeterminism*, and hence attempts (like the de Broglie–Bohm pilot wave theory) to undermine the “Copenhagen” claim of randomness looked for *deterministic* theories underneath quantum mechanics. I will come to the precise notion of determinism that is used in such theories, as well as the way they are supposed to be compatible with quantum mechanics, concluding that this is impossible.

Although “undecidability” may sound a bit like “indeterminism”, the analogy between the quests for certainty in mathematics and for determinism in physics (and their alleged undermining by Gödel’s and Bell’s theorems, respectively) may sound rather superficial. To find common ground more effort is needed to bringing these theorems together.

First, some of its “romantic” aspects have to be removed from Gödel’s theorem, notably its reliance on self-reference, although admittedly this *was* the key to both Gödel’s original example of an undecidable sentence φ (which in a cryptic way expresses its own unprovability) and his proof, in which an axiomatic theory that includes arithmetic is arithmetized through a numerical encoding scheme so as to be able to “talk about itself”.

2 Randomness and its unprovability

Though later proofs of Gödel’s theorem also use numerical encodings of mathematical expressions (such as symbols, sentences, proofs, and computer programs), this is done in order to make recursion theory (initially a theory of numerical functions $f : \mathbb{N} \rightarrow \mathbb{N}$) available to a much wider context, rather than to exploit self-reference. Each computably enumerable but undecidable subset $E \subset \mathbb{N}$ leads to undecidable statements (sometimes even in mainstream mathematics),¹¹ namely those for which the sentence $n \notin E$ is true but unprovable. *Chaitin’s incompleteness theorem* (paraphrased as Theorem 2.1 below) is an example of this. To understand this theorem, we first need some notation and concepts.

A *string* is a *finite* succession of bits (i.e. zeros and ones). The length of a string σ is denoted by $|\sigma|$. The set of all strings of length N is denoted by 2^N , where $2 = \{0, 1\}$, and $2^* = \bigcup_{N \in \mathbb{N}} 2^N$ denotes the set of all strings. For a fair coin flip the probability of each string σ within $2^{|\sigma|}$ is $2^{-|\sigma|}$, so that an apparently “random” string like $\sigma = 0011010101110100$ is as probable as a “deterministic” string like $\sigma = 11111111111111$. Therefore, whatever its definition, the “randomness” of a string cannot be defined via its probability.¹² Even the much more refined statistical test of requiring the relative frequency of each group $x_1 \cdots x_n$ of digits in σ to be 2^{-n} for all $n \leq N$ (which for $N \rightarrow \infty$, i.e. for sequences, is called *Borel normality* in base 2) cannot define randomness.¹³

To remedy this, in the 1960s Solomonoff, Kolmogorov, and Chaitin defined randomness through what is now called *Kolmogorov complexity* or *algorithmic information*; we refer to Li & Vitányi (2008), Calude (2010), and Gács (2013), as well as to Svozil (1993)—for physicists—and Downey & Hirschfeldt (2010)—for mathematicians—for more detail about what follows. *Kolmogorov randomness* (or simply *randomness*) is defined as *incompressibility*, in that the shortest *computational* description of σ cannot improve over just listing σ itself.¹⁴ Simple counting arguments show that as $|\sigma| = N$ gets large, the overwhelming majority of strings in 2^N (and hence in 2^*) is random.¹⁵ The following theorem therefore shows that although most of the world (of strings) is random, this randomness is elusive.

Theorem 2.1 *Although countably many strings are in fact random, any sound axiomatic theory T containing arithmetic (however powerful, such as ZFC, i.e. Zermelo–Fraenkel Set Theory with the Axiom of Choice), can only prove randomness of finitely many strings.*¹⁶

As an idealization of a long string, a *sequence* $x = x_1x_2\cdots$ is an infinite succession of bits, with finite truncations $x_{|N} = x_1 \cdots x_N \in 2^N$ for each $N \in \mathbb{N}$. We then say that x is *1-random* or *typical* if asymptotically (as $N \rightarrow \infty$) each truncation $x_{|N}$ is (Kolmogorov) random.¹⁷ This definition is intuitively appealing enough, and it is further strengthened by the fact that it turns out to be equivalent to two other intuitive notions of randomness, namely *patternlessness* and *unpredictability*, both also defined computationally.¹⁸ A typical sequence is Borel normal and hence contains any finite string infinitely often.¹⁹

Corollary 2.2 *Not a single 1-random sequence can be proved to be 1-random (in ZFC).*

The thrust of the corollary comes from Theorem 3.2 below, according to which there are in fact uncountably many 1-random sequences; heuristically, since the number of computer programs (which are necessary to compress the truncations of an atypical sequence) is countable, the typical sequences form an uncountable subset of the set of all sequences.

To get a rough idea of the proof of Theorem 2.1, let me just say that for any sufficiently long string σ (depending on the given axiomatic theory T) a proof within T that σ is random would necessarily identify it and as such would give a shorter description of σ than its incompressibility allows.²⁰ Corollary 2.2 easily follows by *reductio ad absurdum*.²¹

3 Rethinking Bell's theorem

Also in Bell's theorem (and perhaps in quantum theory as a whole) the emphasis should shift, namely from the single-case *theoretical* Born probabilities the underlying deterministic theory must reproduce, to the *empirical* randomness properties of entire outcome sequences (i.e. of long—ideally infinite—runs of repeated experiments). As a start, we have:

Theorem 3.1 *In the usual approach based on the Born rule and the combination of independent systems by taking tensor products of the underlying Hilbert spaces or operator algebras, the following procedures for repeated identical measurements are equivalent (in giving exactly the same outcome sequences with the same probabilities):*

1. *Quantum mechanics is applied to the whole run (with classically recorded outcomes).*
2. *Quantum mechanics is just applied to single experiments (with classically recorded outcomes), upon which classical probability theory takes over to combine these.²²*

Hence *purely probabilistically* a fair quantum coin is indistinguishable from a fair classical coin (note that in my view the latter cannot physically exist, as I will explain in §5).²³ The idealization of infinitely many repetitions is described by the set $2^{\mathbb{N}}$ of binary sequences, carrying the unique probability measure P^{∞} induced by the single-case 50-50 probabilities on the outcome space $2 = \{0, 1\}$. A fundamental result (due to Martin-Löf) is then:²⁴

Theorem 3.2 *With respect to P^{∞} almost every outcome sequence $x \in 2^{\mathbb{N}}$ is 1-random.*

Corollary 2.2 and Theorems 3.1 and 3.2 imply the following about quantum mechanics:

Corollary 3.3 *With respect to P^{∞} almost every outcome sequence x of an infinitely repeated fair quantum coin flip is typical (though this cannot be proved of any given x).*

Before moving on, as an aside we note that this corollary makes the life of commercial providers of Quantum Random Number Generators (such as IdQuantique in Geneva) quite difficult: they cannot certify that all sequences their devices provide are typical (i.e. random in arguably the most desirable way), firstly because this is not even true, and secondly in the cases where it is true it cannot be proved in principle. What should be possible is to prove weaker randomness properties (such as uncomputability) for *all* sequences, but even this has so far been accomplished only under strong additional assumptions.²⁵

On the up side, the single-case Born probabilities can be recovered almost surely from any outcome sequence via the law of large numbers, i.e. for almost every sequence $x = x_1x_2\cdots$ (with respect to P^{∞}) one has $\lim_{N \rightarrow \infty} (x_1 + \cdots + x_N)/N = \frac{1}{2}$. This holds for each typical sequence, but many other sequences have this property, too (think of 101010 \cdots or 00110011 \cdots), so it is a rather weak randomness property. Thus my argument will be based on the idea that *in order to be genuinely compatible with quantum mechanics, deterministic hidden variable theories should not merely reproduce the Born probabilities, but should also produce typical outcome sequences, including their unprovability as such.*

The point is that Bell's (1964) theorem constrains such theories in a rather different way:

- The theory in question is only required to reproduce single-case (Born) probabilities;
- This requirement is imposed for joint Born probabilities defined by entangled states.

The second point is the real thrust of Bell’s reasoning, and it is powerful enough to lead to serious constraints on deterministic hidden variable theories. The hidden variable theory is not asked to produce specific outcome sequences, but instead the quantum-mechanical probabilities are obtained by formally averaging over the set of hidden variables, i.e.,

$$P_\psi(F = x, G = y \mid A = a, B = b) = \int_\Lambda d\mu_\psi(\lambda) P_\lambda(F = x, G = y \mid A = a, B = b). \quad (3.1)$$

Here, in the usual bipartite (Alice & Bob) setting of the EPR–Bohm experiment: F is an observable measured by Alice defined by her choice of setting a , likewise G for Bob defined by his setting b , with possible outcomes $x \in \underline{2}$, likewise $y \in \underline{2}$ for Bob; the left-hand side is the Born probability for the outcome (x, y) if the correlated system has been prepared in a known quantum state ψ ; the expression $P_\lambda(\cdots)$ on the right-hand side is the probability of the outcome (x, y) if the unknown hidden variable or state equals λ (in a deterministic hidden variable theory $P_\lambda(\cdots)$ equals 0 or 1); and finally, μ_ψ is some probability measure on the space Λ of hidden states supposedly provided by the theory for each ψ .

The initial claim of Bell was that in a deterministic theory the compatibility condition (3.1) cannot be satisfied for suitable F , G , and ψ if $P_\lambda(\cdots)$ is *local*. Defining the expressions

$$P_\lambda(F = x \mid A = a, B = b) = \sum_{y=0,1} P_\lambda(F = x, G = y \mid A = a, B = b); \quad (3.2)$$

$$P_\lambda(G = y \mid A = a, B = b) = \sum_{x=0,1} P_\lambda(F = x, G = y \mid A = a, B = b), \quad (3.3)$$

for *deterministic theories* this means that (3.2) is independent of b whilst (3.3) is independent of a ; that is, the probability of Alice’s outcomes are independent of Bob’s settings, and *vice versa*. This follows from special relativity, for if Bob chooses his settings just before his measurement, there is a frame of reference in which Alice measures before Bob has chosen his settings, and *vice versa*. In turn, this is equivalent to the property that even if she knew the value of λ , Alice could not signal to Bob, and *vice versa*.²⁶ Making Bell’s *tacit* assumption that experimental settings can be “freely” chosen *explicit*, we obtain:²⁷

Theorem 3.4 *The conjunction of the following properties is inconsistent:*

1. Determinism (*in the sense of the existence of a deterministic hidden variable theory*)
2. The Born rule (*as either theoretically given or experimentally confirmed*)
3. Locality (*e.g. as a ban on superluminal communication using the hidden variable*)
4. Free choice, i.e. independence of the choice of measurement settings from the state (*either hidden or manifest*) of the system one measures using these settings.

Deterministic hidden variable theories compatible with the Born rule therefore have to choose between giving up either *Locality* or *Free Choice*. Two representative examples are *Bohmian mechanics*,²⁸ which gives up *Locality*, and the *cellular automata interpretation* of quantum mechanics developed by ’t Hooft (2016), which gives up *Free Choice*. In both cases the Born rule of quantum mechanics is indeed recovered by averaging the hidden variable or state with respect to a probability measure μ_ψ on the space of hidden variables, given some (pure) quantum state ψ . The difference is that in Bohmian mechanics the total state (which consists of the hidden configuration plus the “pilot wave” ψ) determines the measurement outcomes *given the settings*, whereas in ’t Hooft’s theory the hidden state (see below) all by itself determines the outcomes as well as the settings. In more detail:

- In Bohmian mechanics the hidden variable is position q , and $d\mu_\psi = |\psi(q)|^2 dx$ is the Born probability for outcome q with respect to the expansion $|\psi\rangle = \int dq \psi(q)|q\rangle$.
- In 't Hooft's theory the hidden state is identified with a basis vector $|m\rangle$ in some Hilbert space H ($m \in \mathbb{N}$), and once again the measure $\mu_\psi(m) = |c_m|^2$ is given by the Born probability for outcome m with respect to the expansion $|\psi\rangle = \sum_m c_m |m\rangle$.

Such theories only manage to pass the sieve of Bell's Theorem 3.4 because of the specific way the compatibility condition with quantum mechanics is stated. Instead, I submit:

Theorem 3.5 *The conjunction of the following properties is inconsistent:*

1. Determinism (*in the sense of the existence of a deterministic hidden variable theory*)
2. Typicality of almost every outcome sequence of an infinitely repeated fair quantum coin flip, including the unprovability of this property in ZFC (cf. Corollary 3.3).

For simplicity, I will argue this just for Bohmian mechanics and for 't Hooft's theory (as prototypes of hidden variable theories that give up locality and free choice, respectively): the argument is easier for these theories, since as just shown, their hidden variables (i.e. $q \in Q$ and $n \in \mathbb{N}$, respectively) have familiar quantum-mechanical interpretations and also their compatibility measures are precisely the Born measures for the quantum state ψ . Since the argument does not rely on entanglement and hence on a bipartite experiment, we may as well work with a quantum coin flip. The setting is then simply fixed,²⁹ so if we interpret the assumption of determinism (for fixed settings) as the existence of functions

$$B : Q \rightarrow \underline{2}; \quad B(q) = 0, 1 \quad (\text{Bohm}) \quad (3.4)$$

$$H : \mathbb{N} \rightarrow \underline{2}; \quad H(m) = 0, 1 \quad (\text{'t Hooft}), \quad (3.5)$$

which state the outcome given the value of the hidden variable, then the outcome of each run $n \in \mathbb{N}$ of the quantum coin flip is determined if in addition we provide functions

$$\mathbf{q} : \mathbb{N} \rightarrow Q; \quad \mathbf{q}(n) = q_n \quad (\text{Bohm}) \quad (3.6)$$

$$\mathbf{m} : \mathbb{N} \rightarrow \mathbb{N}; \quad \mathbf{m}(n) = m_n \quad (\text{'t Hooft}), \quad (3.7)$$

that give the value of the hidden variable for each run. Indeed, we define the functions

$$B : \mathbb{N} \rightarrow \underline{2}; \quad B = B \circ \mathbf{q} \quad (\text{Bohm}) \quad (3.8)$$

$$H : \mathbb{N} \rightarrow \underline{2}; \quad H = H \circ \mathbf{m} \quad (\text{'t Hooft}), \quad (3.9)$$

which mathematically are simply binary sequences, then the outcome of coin flip no. n is equal to $B(n)$ for Bohm and to $H(n)$ for 't Hooft. The key point is now that in order to recover the predictions of quantum mechanics as meant in Theorem 3.5, i.e. the typicality of generic outcome sequences of quantum mechanics, the functions \mathbf{q} and \mathbf{m} must sample the Born measure (in its guise of the compatibility measure μ_ψ) on Q and \mathbb{N} , respectively, in the sense of “randomly” picking elements from Q or \mathbb{N} according to the probability measure on the infinite product $Q^\mathbb{N}$ or $\mathbb{N}^\mathbb{N}$. This, in turn, should guarantee that the sequences defined by B or H themselves mimic fair coin flips. Since the “outcome” functions B and H are supposed to be given by the deterministic hidden variable theories in question, this implies that the (outcome) randomness properties of quantum mechanics that are to be reproduced must entirely originate in the sampling functions \mathbf{q} and \mathbf{m} .

4 Proof of Theorem 3.5

This origin introduces an element of randomness into the hidden variable theories in question, which contradicts their deterministic character. For there are just two possibilities:

1. *The sampling is provided by the hidden variable theory.* In that case, the sampling functions by nature must also be deterministic, which leads to a contradiction. If “deterministic” means computable,³⁰ then the contradiction arises straight away, since in that case all outcome sequences are computable, whereas Corollary 3.3 implies that *almost every* outcome sequence should be typical, and one takes the contrapositive of the implication *typical* \Rightarrow *uncomputable*, i.e. *computable* \Rightarrow *atypical*.

The general case rests on the unprovability of typicality. We assume that the sampling functions q and m are explicitly described in the theory T used in Theorem 2.1 (which is hardly an assumption since T may be taken to be ZFC set theory, which formalizes all mathematics used in physics). Then *any* outcome sequence is also explicitly described by T via (3.8) - (3.9), and by Corollary 3.3 *almost all* outcome sequences thus described are typical. If some such sequence is indeed typical, this can be proved in T from its description.³¹ But this would contradict Corollary 2.2.

2. *The sampling is not provided by the theory.* In that case, the theory fails to determine the outcome of any specific experiment and just provides averages of outcomes. Since the source of the sampling cannot lie in quantum mechanics either (which does not know anything about hidden variables), to obtain it in a deterministic world picture one needs to invoke a second deterministic theory and ends up in the previous case.

Bohmians as well as ’t Hooft go for the second option and blame the randomness in question on the initial conditions of the experiment (which in my notation would be the choice of the sampling functions q and m), whose specification is seen as lying outside the scope of a deterministic theory (which only feels responsible for providing the functions B and H above). As argued by both parties (Dürr, Goldstein, & Zanghi, 1992; ’t Hooft, 2016), the randomness in the outcomes of measurement on quantum system, including the Born rule, is then a consequence of the above randomness in initial conditions.³² I find this extremely unsatisfactory, not just because Bohmians and ’t Hooft cannot complete their deterministic world picture in view of the above dilemma, but also because in a (Laplacian) deterministic theory prediction and retrodiction should be equivalent. Now such a theory attributes the origin of randomness to the *initial conditions* for measurement, whereas standard (Copenhagen) quantum mechanics attributes it to the *outcomes* of measurement. But *for deterministic theories* this cannot make any difference. The difference between “standard” and “hidden variable” quantum mechanics fades even more if we realize that:

- In standard quantum mechanics the Born measure (on the single-case outcome space) *defined by the quantum state* is sampled in arriving at an outcome sequence;
- In “deterministic” hidden variable theories the compatibility measure on the space of hidden variables *defined by the quantum state* is sampled for the same purpose.

Therefore, “deterministic” hidden variable theories fit the same bill as quantum mechanics: though deterministic at first sight,³³ in their accounting for measurement outcomes they cannot be, at least if compatibility with the Born rule is required. I conclude that deterministic hidden variable theories compatible with quantum mechanics cannot exist; Bell’s theorem leaves two loopholes for determinism (i.e. nonlocality or no choice) because its compatibility condition with quantum mechanics is not stated strongly enough.³⁴

5 Some musings on (in)determinism

In order to understand Theorem 3.5 and its proof it may be helpful to note that in classical coin tossing the role of the hidden state is also played by the initial conditions (cf. Diaconis & Skyrms, 2018 Chapter 1, Appendix 2). The 50-50 chances (allegedly) making the coin fair are obtained by averaging over the initial conditions, i.e., by sampling. By my arguments, this sampling cannot be deterministic, for otherwise the outcome sequences appropriate to a fair coin would not obtain: it must be done in a genuinely random way. This is impossible classically, so that—unless they have a quantum-mechanical seed—*fair classical coins do not exist*, as confirmed by Diaconis & Skyrms (2018, Chapter 1).

Let me also note that the fundamental incompleteness result in Corollary 2.2 (i.e. the unprovability of typicality or 1-randomness) has not only been used in the proof of Theorem 3.5 (namely in dealing with the case where determinism may not be conflated with computability); it has an interesting additional consequence for this discussion. Namely, although it is *true* that deterministic hidden variable theories necessarily fail to recover the randomness properties of quantum mechanics as detailed in Corollary 3.3, this cannot be *proved* in any formal mathematical theory to which Theorem 2.1 applies!

Indeed, let us look at the proof of Bell’s theorem for inspiration as to what such a proof should look like. As we all know, in the context of the EPR–Bohm experiment local deterministic hidden variable theories predict correlations that *satisfy* the Bell inequalities,³⁵ whereas on suitable settings quantum mechanics predicts (and experiment shows) that (through their associated single-case Born probabilities) typical outcome sequences *violate* these inequalities. Now a proof of the above true statement about deterministic hidden variable theories should perhaps not be expected to show that *all* typical quantum-mechanical outcome sequences violate the predictions of the hidden variable theory, but it should identify at least an uncountable number of such typical sequences—for finding a countable number, let alone a mere finite number, would make the contradiction with quantum mechanics happen with probability zero. However, already the identification of a single typical sequences is impossible by Corollary 2.2. Thus the unprovability of their falsehood condemns deterministic hidden variable theories, and perhaps even determinism as a whole, to a zombie-like existence in a twilight zone comparable with the Dutch situation around selling soft drugs: although this is forbidden by law, it is not prosecuted.

The situation would change drastically if deterministic hidden variable theories gave up their compatibility with the Born rule (on which my entire reasoning is based), as for example Valentini (2019) has argued in case of the de Broglie–Bohm pilot wave theory. For it is this compatibility requirement that kills such theories, which could leave zombie-dom if only they were brave enough to challenge the Born rule. This might open the door to superluminal signaling and worse, but on the other hand the possibility of violating the Born rule would also provide a new context for deriving it, e.g. as a dynamical equilibrium condition (as may be the case for the Broglie–Bohm theory, if Valentini is right).

I would personally expect that a valid theory of the Planck scale (including quantum gravity or string theory, though these words are misleading here), far from assuming the Born rule and the rest of quantum mechanics (as these theories normally do), would derive quantum mechanics as an emergent theory (instead, the opposite seems to be the majority goal, i.e. deriving gravity as an emergent phenomenon from quantum theory). Thus quantum mechanics would typically be a limiting case of something else, which would, then, also render the Born rule valid in some limit only, rather than absolutely.

Notes

¹It might be more appropriate to speak of Gödel’s theorems and Bell’s theorems, since there are two incompleteness theorems in logic due to Gödel and two non-locality theorems in physics due to Bell, but in this essay I am mainly interested in the first ones, of both authors, except for a few side remarks.

²See Franzén (2005) for an excellent (first) introduction to Gödel’s theorems, combined with a fair and detailed critique of its abuses, including serious overstatements by highly influential scientist like Chaitin and Penrose (a similar guide to the use and abuse of Bell’s theorems remains to be written).

³Yanofsky (2013) nicely discusses both theorems in the context of the limits of science and reason.

⁴Both reformulations are a bit anachronistic and purpose-made. See Gödel (1931) and Bell (1964)!

⁵Gödel’s *second* incompleteness theorem shows that one example of φ is the (coded) statement that the consistency of the theory can be proved within the theory. This is often taken to refute Hilbert’s Program, but even among experts it seems controversial if it really does so. For Hilbert’s Program and its role in Gödel’s theorems see e.g. Sieg (2013) as well as numerous works by W. Tait, C. Tapp, R. Zach, and others.

⁶For a recent (popular) book on the history and interpretation of Bell’s work see e.g. Greenstein (2019).

⁷Some vocal researchers in the history and philosophy of physics insist that Bell and Einstein were primarily interested in locality and realism, determinism being a secondary issue, and that Bell’s theorem(s) have nothing to do with hidden variables but show that Nature itself is non-local. The historical record is actually mixed and may be interpreted either way—cf. Landsman (2020) or even Bell (1964)—more generally, over 10,000 papers about Bell’s theorems show that Bell can be interpreted in almost equally many ways. But this controversy is a moot point: whatever his own (or Einstein’s) intentions, Bell’s (1964) theorem puts constraints on possible deterministic underpinnings of quantum mechanics, and that is how I take it.

⁸For an overall survey of this theme see Kline (1980).

⁹This phase in the history of quantum mechanics is described by Mehra & H. Reichenberg (2000).

¹⁰See Landsman (2020) for a general analysis of randomness as an antipodally defined family resemblance.

¹¹A subset $E \subset \mathbb{N}$ is *computably enumerable* (c.e.) if it is the image of a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$, and *decidable* if both E and its complement are c.e., i.e. iff its characteristic function 1_E is computable.

¹²The idea of defining randomness prior to probability goes back to the flawed attempts of von Mises in 1919 to define random sequences through unpredictability, correctly formalized by Schnorr only in 1971 as the impossibility of establishing a successful computational betting strategy on the successive digits of x . Von Mises was a strict frequentist for whom probability was a derived concept, predicated on first having a good notion of a random sequence from which relative frequencies *defining* probability could be extracted. Kolmogorov, on the other hand, started from an axiomatic *a priori* notion of probability from which a suitable mathematical concept of randomness was subsequently to be extracted. His initial failure to achieve this follow-up led to his later notion of algorithmic randomness, which was to become one of the three equivalent definitions of 1-randomness. See e.g. Diaconis & Skyrms (2018) for this history.

¹³A sequence x is *Borel normal* in base k if each string σ has frequency $k^{-|\sigma|}$ in x . Any hope of defining randomness as Borel normality in base 10 is blocked by *Champernowne’s number* 012345678910111... , which is Borel normal but is clearly not random in any reasonable sense (this is also true in base 2, i.e. for 01101110010111000100110101011...). The decimal expansion of π is also conjectured to be Borel normal on base 10 (with huge numerical support), although π clearly is not random either.

¹⁴To make this precise, fix some universal (prefix-free) Turing machine U , and define the *Kolmogorov complexity* $K(\sigma)$ of σ as the length $|P|$ (in bits) of the shortest computer program P run on U (including its input) that computes σ (a Turing machine T is *prefix-free* if its domain $D(T)$ consists of a prefix-free subset of \mathbb{N}^* , i.e., if $\sigma \in D(T)$ then $\sigma\tau \notin D(T)$ for any $\sigma, \tau \in \mathbb{N}^*$, where $\sigma\tau$ is the concatenation of σ and τ ; the prefix-free version is only needed for the limit $N \rightarrow \infty$ and may be avoided for finite strings). The choice of U affects $K(\sigma)$ only up to a σ -independent constant, and to take this dependency into account we say that σ is *c-Kolmogorov random*, for some σ -independent constant $c \in \mathbb{N}$, if $K(\sigma) \geq |\sigma| - c$, and *Kolmogorov random* if it is *c-Kolmogorov random* for all $c \in \mathbb{N}$, that is, if $K(\sigma) \geq |\sigma|$. Since $K(\sigma) \leq |\sigma| + |P|$ by definition of $K(\cdot)$, this suggests that for very long strings σ is Kolmogorov random if $K(\sigma) \approx |\sigma|$. Foreshadowing Chaitin’s incompleteness theorem, the function $K : \mathbb{N}^* \rightarrow \mathbb{N}$ is uncomputable.

¹⁵It is easy to show that least $2^N - 2^{N-c+1} + 1$ strings σ of length $|\sigma| = N$ are *c-Kolmogorov random*.

¹⁶More technically: *For any sound mathematical theory T (formalized, as in Gödel’s theorem, as an axiomatic-deductive system in which proofs could in principle be carried out mechanically by a computer) that contains enough arithmetic, there is a constant $C \in \mathbb{N}$ such that T cannot prove any sentence of the form $K(\sigma) > C$ (although infinitely many such sentences are true).* Here “sound” means that all theorems proved by T are true; this is a stronger assumption than consistency (in fact only the arithmetic fragment of T needs to be sound). See Chaitin (1987) for his own presentation and analysis of his incompleteness

theorem. Raatikainen (1998) also gives a detailed presentation of the theorem, including a critique of the far-reaching conceptual conclusions Chaitin has drawn from it. See also Franzén (2005) and Gács (1989).

¹⁷ In more detail, a sequence x is 1-random if there exists $c \in \mathbb{N}$ such that each truncation $x_{|N}$ satisfies $K(x_{|N}) \geq N - c$. As shown in Calude (2010), Theorem 6.38, this is equivalent to $\lim_{N \rightarrow \infty} K(x_{|N})/N = 1$.

¹⁸ Any pattern in a sequence x would make it compressible, but one has to define the notion of a pattern very carefully to make this work in the same computational setting where compressibility is defined. This was accomplished by Martin-Löf in 1966, who defined a pattern as a specific kind of probability-zero subset T of $2^{\mathbb{N}}$ (called a “test”) that can be computably approximated by subsets $T_n \subset 2^{\mathbb{N}}$ of increasingly small probability 2^{-n} ; if $x \in T$, then x displays some pattern and it is patternless iff $x \notin T$ for all such tests.

¹⁹ For details and proofs see Calude (2010), Corollary 6.32 in §6.3 and almost all of §6.4.

²⁰ The proof of the theorem as stated in endnote 16 is based on the existence of a c.e. list $T = (\tau_1, \tau_2, \dots)$ of the theorems of T , and on the fact (which is well known from Gödel’s theorem) that after Gödelian encoding by numbers, theorems of any given grammatical form can be computably searched for in this list and will eventually be found. In particular, there exists a program P (running on the universal prefix-free Turing machine U of endnote 14) such that $P(n)$ halts iff there exists a string σ for which $K(\sigma) > n$ is a theorem of T . If there is such a theorem the output is $P(n) = \sigma$, where σ appears in the first such theorem of the kind (according to the list T). By definition of the function $K(\cdot)$, this means that $K(\sigma) \leq |P| + |n|$. Now suppose that no C as in the above statement of the theorem exists. Then there is $n \in \mathbb{N}$ large enough that $n > |P| + |n|$ and there is a string $\sigma \in 2^*$ such that T proves $K(\sigma) > n$. Since T is sound by assumption this is actually true, which gives a contradiction between $K(\sigma) > n > |P| + |n|$ and $K(\sigma) \leq |P| + |n|$ (this contradiction can of course be made more dramatic by taking n such that $n \gg \gg |P| + |n|$). It is very important that this argument shows that a *proof* in T of $K(\sigma) > n$ (if true) would also *identify* σ .

²¹ Since T is sound, Corollary 2.2 states that the property of 1-randomness cannot be proved for any sequence x . Indeed, if for some $c \in \mathbb{N}$ we could prove (in ZFC) for all truncations $x_{|N}$ that $K(x_{|N}) \geq N - c$, then we would have proved (Kolmogorov) randomness for an infinite number of strings $\sigma_N = x_{|N}$, contradicting Theorem 2.1. In other words, of the countably many clauses $K(x_{|N}) \geq N - c$ that are needed to prove that a sequence x is 1-random (if it is), at most a finite number could be proved.

²² For a proof see Landsman (2017), §8.4, summarized in Landsman (2020), Appendix A.

²³ Mathematically, an example of a fair quantum coin flip would be measuring the third Pauli matrix $\sigma_z = \text{diag}(1, -1)$ in a state like $\psi = (1, 1)/\sqrt{2}$. Physically, one may think of a spin measurement on an electron along some given direction, or of measuring the (yes/no) passage of light through a polaroid filter.

²⁴ For a proof see e.g. Calude (2010), Corollary 6.32.

²⁵ See for example Abbott, Calude, & Conder (2012), whose proof (by contradiction) relies on the debatable assumption that computable sequences originate in some non-contextual hidden-variable theory.

²⁶ In quantum mechanics the left-hand side of (3.1) satisfies this locality condition.

²⁷ See Landsman (2017), §6.5 for details, or Landsman (2020, Appendix C) for a summary.

²⁸ There is a subtle difference between Bohmian mechanics as reviewed by e.g. Goldstein (2017), and de Broglie’s original pilot wave theory (Valentini, 2019). This difference is immaterial for my discussion.

²⁹ In Bohmian mechanics, the hidden state $q \in Q$ just pertains to the particles undergoing measurement, whilst the settings a are supposed to be “freely chosen” for each measurement (and in particular are independent of q). The outcome is then fixed by a and q . In ’t Hooft’s theory, the hidden state $x \in X$ of “the world” determines the settings as well as the outcomes. Beyond the issue raised in the main text, Bohmians (but not ’t Hooft!) therefore have an additional problem, namely the origin of the settings (which are simply left out of the theory). This weakens their case for determinism even further.

³⁰ This identification is highly unlikely unless “the universe is a computation”, as some people claim.

³¹ See endnote 20. Any specific description of a string σ in T (for example by a formula) identifies the digits of σ and hence makes it atypical by providing a Martin-Löf test for randomness that σ fails.

³² The Bohmians are divided on the origin of their compatibility measure, referred to in this context as the *quantum equilibrium distribution*, cf. Dürr, Goldstein, & Zanghi (1992) against Valentini (2019). The origin of μ_ψ is not my concern, which is the need to randomly sample it and the justification for doing so.

³³ The Schrödinger equation is as deterministic as you can have it: by Stone’s theorem its solutions are even guaranteed to exist for all times. In classical physics this completeness is rather the exception!

³⁴ By completely different arguments, Colbeck & Renner (2012) argue that any hidden variable theory that satisfies Locality and Free choice and is compatible with quantum mechanics in the same way as in Bell’s theorem, adds nothing to quantum mechanics, concluding that quantum mechanics “cannot be extended”, or “is complete”. For an analysis see Landsman (2017), §6.6 (and similar work by G. Leegwater); as far as I can see, their claims can only be proved on far stronger assumptions than those they make.

³⁵ For Bell’s proof it is irrelevant whether or not some hidden variable is able to sample the compatibility measure, since the Bell inequalities follow from pointwise bounds, cf. Landsman (2017), eq. (6.119).

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