The Mysterious Connection Between Physics and Mathematics

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Abstract

The relation between mathematics and physics is perplexing. In this article I have taken excerpts from history where both the subjects have played a major role and provided exceptional contribution to the growth of one another. I have provided elaborate examples pertaining to the various connections between mathematics and physics and how they helped evolve each other. Though seemingly unrelated areas of study, physics and mathematics share an incontroversial and an everlasting bond.

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I. INTRODUCTION

Physics and mathematics are elementary realms of "Natural Philosophy". We understand nature in realms of physics and formulate it in terms of mathematics. The physicist, in his study of natural phenomena, has two methods of making progress: 1) the method of experiment and observation and 2) the method of mathematical thinking[1]. One might describe the mathematical quality in Nature by saying the universe is so constituted that mathematics is a useful tool in its description. However, the connection goes far deeper than this and one can appreciate it only from a thorough examination of the various facts that make it up. One may describe the two subjects by saying that the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, the only difference being not all rules are known to the physicist. From the mathematicians point of view, physics is the physical visualization of the principles of mathematics.

Many discoveries in mathematics were made due to contributions of physics and vice versa. The earlier years of physics saw the inventions of powerful mathematical tools such as calculus, co-ordinate geometry to understand physical phenomena like motion and optics. The understanding of differential equations led to the study of vibrations, waves and heat. The algebraic solution of polynomials saw various applications in the fields of mechanics and quantum mechanics. The invention of the Dirac delta function, changed the concept of functions. The use of matrices as operators resulted in spectral theory and further development of quantum mechanics, matrices and geometry.

The real question that still eludes people is the mysterious connection between these branches of knowledge, to understand how they evolved to be inseparable and how they still play an important role in each others' development. In this article, I shall present various examples in both the fields of study which seem to have no relation with each other but have helped understand different areas and have improvised and advanced science to it's present state.

II. THEORY OF NUMBERS AND RANDOM MATRICES

In number theory, the zeta functon was discovered by Leonhard Euler in the eighteenth century who showed that it has a deep and a profound connection with the pattern of primes. In the mid-nineteenth century, Riemann explained that if one wants to understand a complex analytic function, one needs to understand the location of it's zeros. The problem that baffles every number theorist is the zeros of Riemann zeta function. Mathematicians have tried to prove the Riemann hypothesis, which states that the non-trivial zeros of the Riemann zeta function lie on the critical line with real part equal to 1/2. The statistical distribution of the zeros on the critical line of the Riemann zeta function has a property called "Montgomery's pair correlation conjecture", which explains that the zeros tend to repel between neighboring levels.

In 1960s Freeman Dyson had worked on random matrix theory, which was proposed by Eugene Wigner in 1951 to model the spectra of heavy atoms. Wigner asked, "If you take a random matrix, what do it's eigen values look like? Would they look different if you chose the numbers randomly?". Based on these questions he postulated that the spacings between the lines in the spectrum of a heavy atom should resemble the spacings between the eigenvalues of a random matrix, and should depend only on the symmetry class of the underlying evolution. Dyson realized that the statistical distribution found by Montgomery appeared to be the same as the pair correlation distribution for the eigenvalues of a random Hermitian matrix that he had discovered a decade later[2]. Dyson and Mehta identified three types of matrix ensembles with different correlations: Gaussian orthogonal ensemble (time reversal invariant and integer spin with weakest repulsion between neighboring levels), Gaussian unitary ensemble (no time reversal invariance with medium repulsion) and Gaussian symplectic ensemble (time reversal invariant with half integer spin and strongest repulsion), abbreviated as GOE, GUE and GSE respectively. Andrew Odlyzko confirmed the connection of GUE to the zeros of the Riemann zeta function. This tantalizing connection between prime numbers and mathematical physics still remains strange and mysterious.

III. RIEMANNIAN GEOMETRY AND THEORY OF RELATIVITY

A second example of this famous connection lies again in the past two centuries. Riemannian geometry was first put forward in generality by Bernhard Riemann in the nineteenth century. It deals with a broad range of geometries whose metric properties vary from point to point, including the standard types of Non-Euclidean geometry. Any smooth manifold admits a Riemannian metric, which often helps to solve problems of differential topology. It also serves as an entry level for the more complicated structure of pseudo-Riemannian manifolds. When this theory was proposed, it was considered to be "a beautiful theory without any practical applications".

By the beginning of the 20th century, Newton's law of universal gravitation, which describes the mutual attraction experienced by bodies due to their mass, had been accepted for more than two hundred years as a valid description of the gravitational force between masses. In Newton's model, gravity is the result of an attractive force between massive objects. Although, even Newton was troubled by the unknown nature of that force, the basic framework was extremely successful at describing motion. Even after being successful, the theory was unable to reconcile with the theory of electrodynamics (the interaction between objects with electric charge) proposed by James Maxwell.

In September 1905, Albert Einstein published his theory of special relativity, which reconciles Newton's laws of motion with electrodynamics. Special relativity introduced a new framework for all of physics by proposing new concepts of space and time. Several physicists, including Einstein, searched for a theory that would reconcile Newton's law of gravity and special relativity. Only Einstein's theory proved to be consistent with experiments and observations. His simple thought experiment involving an observer in free fall led to his fully geometric theory of gravity.

In exploring the equivalence of gravity and acceleration as well as the role of tidal forces, Einstein discovered several analogies with the geometry of surfaces. An example is the transition from an inertial reference frame (in which free particles coast along straight paths at constant speeds) to a rotating reference frame (in which extra terms corresponding to fictitious forces have to be introduced in order to explain particle motion): this is analogous to the transition from a Cartesian coordinate system (in which the coordinate lines are straight lines) to a curved coordinate system (where coordinate lines need not be straight).

A deeper analogy relates tidal forces with a property of surfaces called curvature. For gravitational fields, the absence or presence of tidal forces determines whether or not the influence of gravity can be eliminated by choosing a freely falling reference frame. Similarly, the absence or presence of curvature determines whether or not a surface is equivalent to a plane. In the summer of 1912, inspired by these analogies, Einstein searched for a geometric formulation of gravity[3].

The elementary objects of geometry points, lines, triangles are traditionally defined in three-dimensional space or on two-dimensional surfaces. In 1907, Hermann Minkowski, Einstein's former mathematics professor at the Swiss Federal Polytechnic, introduced a geometric formulation of Einstein's special theory of relativity where the geometry included not only space but also time. The basic entity of this new geometry is four-dimensional spacetime. The orbits of moving bodies are curves in spacetime; the orbits of bodies moving at constant speed without changing direction correspond to straight lines.

For surfaces, the generalization from the geometry of a plane a flat surface to that of a general curved surface had been described in the early 19th century by Carl Friedrich Gauss. This description had in turn been generalized to higher-dimensional spaces in a mathematical formalism introduced by Bernhard Riemann in the 1850s. With the help of Riemannian geometry, Einstein formulated a geometric description of gravity in which Minkowski's spacetime is replaced by distorted, curved spacetime, just as curved surfaces are a generalization of ordinary plane surfaces.

After he had realized the validity of this geometric analogy, it took Einstein a further three years to find the missing cornerstone of his theory: the equations describing how matter influences spacetime's curvature. Having formulated what are now known as Einstein's equations (or, more precisely, his field equations of gravity), he presented his new theory of gravity at several sessions of the Prussian Academy of Sciences in late 1915, culminating in his final presentation on November 25, 1915.

Experiments and observations show that Einstein's description of gravitation accounts for several effects that are unexplained by Newton's law, such as minute anomalies in the orbits of Mercury and other planets. General relativity also predicts novel effects of gravity, such as gravitational waves, gravitational lensing and an effect of gravity on time known as gravitational time dilation. Many of these predictions have been confirmed by experiment, while others are the subject of ongoing research. This is an example of the second type of development of physics theory, which emerged purely out of imagination and mathematical reasoning and verily tested by experimenters time and again.

IV. THE MATHEMATICS OF GROUPS

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right.

Group theory has three main historical sources: number theory, the theory of algebraic equations and geometry. The number-theoretic strand was begun by Leonhard Euler, and developed by Gauss's work on modular arithmetic and additive and multiplicative groups related to quadratic fields. Early results about permutation groups were obtained by Lagrange, Ruffini, and Abel in their quest for general solutions of polynomial equations of high degree. Évariste Galois coined the term "group" and established a connection, now known as Galois theory, between the nascent theory of groups and field theory. In geometry, groups first became important in projective geometry and later, non-Euclidean geometry. Felix Klein's Erlangen program proclaimed group theory to be the organizing principle of geometry.

Galois, in the 1830s, was the first to employ groups to determine the solvability of polynomial equations. Arthur Cayley and Augustin Louis Cauchy pushed these investigations further by creating the theory of permutation groups. The second historical source for groups stems from geometrical situations. In an attempt to come to grips with possible geometries (such as euclidean, hyperbolic or projective geometry) using group theory, Felix Klein initiated the Erlangen programme. Sophus Lie, in 1884, started using groups (now called Lie groups) attached to analytic problems. Thirdly, groups were, at first implicitly and later explicitly, used in algebraic number theory.

In physics, groups are important because they describe the symmetries which the laws of physics seem to obey. According to Noether's theorem, every continuous symmetry of a physical system corresponds to a conservation law of the system. Physicists are very interested in group representations, especially of Lie groups, since these representations often point the way to the "possible" physical theories. Examples of the use of groups in physics include:

- **The Standard Model:** The Standard Model of particle physics is a theory concerning the electromagnetic, weak, and strong nuclear interactions, as well as classifying all the subatomic particles known. The construction of the Standard Model proceeds following the modern method of constructing most field theories: by first postulating a set of symmetries of the system, and then by writing down the most general renormalizable Lagrangian from its particle (field) content that observes these symmetries. The global Poincaré symmetry is postulated for all relativistic quantum field theories. It consists of the familiar translational symmetry, rotational symmetry and the inertial reference frame invariance central to the theory of special relativity. The local $SU(3) \times SU(2) \times U(1)$ gauge symmetry is an internal symmetry that essentially defines the Standard Model. Roughly, the three factors of the gauge symmetry give rise to the three fundamental interactions. The fields fall into different representations of the various symmetry groups of the Standard Model.

- **Gauge theory:** In physics, a gauge theory is a type of field theory in which the Lagrangian is invariant under a continuous group of local transformations. The term gauge ("scale ") refers to redundant degrees of freedom in the Lagrangian. The transformations between possible gauges, called gauge transformations, form a Lie group, referred to as the symmetry group or the gauge group. If the symmetry group is non-commutative, the gauge theory is referred to as non-abelian, the usual example being the Yang-Mills theory.

  Gauge theories are important as the successful field theories explaining the dynamics of elementary particles. Quantum electrodynamics is an abelian gauge theory with the symmetry group $U(1)$ and has one gauge field, the electromagnetic four-potential, with the photon being the gauge boson. The Standard Model, as seen above, is a non-abelian gauge theory with the symmetry group $SU(3) \times SU(2) \times U(1)$ and has a total of twelve gauge bosons: the photon, three weak bosons and eight gluons.

  Gauge theories are also important in explaining gravitation in the theory of general relativity. Its case is somewhat unique in that the gauge field is a tensor called the Lanczos tensor. Theories of quantum gravity, beginning with gauge gravitation theory, also postulate the existence of a gauge boson known as the graviton. Gauge symmetries can be viewed as analogues of the principle of general covariance of general relativity in which the coordinate system can be chosen freely under arbitrary diffeomorphisms of spacetime. Both gauge invariance and diffeomorphism invariance reflect a redundancy in the description of the system. An alternative theory of gravitation, gauge theory gravity, replaces the principle of general covariance with a true gauge principle with new gauge fields.

Historically, these ideas were first stated in the context of classical electromagnetism and later in general relativity. However, the modern importance of gauge symmetries appeared first in the relativistic quantum mechanics of electrons quantum electrodynamics. Today, gauge theories are useful in condensed matter, nuclear and high energy physics among other subfields.

- **The Lorentz group and the Poincaré group:** In physics and mathematics, the Lorentz group is the group of all Lorentz transformations of Minkowski spacetime, the classical setting for all (nongravitational) physical phenomena. The Lorentz group is named for the Dutch physicist Hendrik Lorentz.

  The mathematical form of the kinematical laws of special relativity, Maxwell's field equations in the theory of electromagnetism and the Dirac equation in the theory of the electron are each invariant under the Lorentz transformations. Therefore, the Lorentz group is said to express the fundamental symmetry of many of the known fundamental laws of nature.

  The restricted Lorentz group $SO^+(1,3)$ is isomorphic to the projective special linear group $PSL(2,\mathbb{C})$, which is in turn isomorphic to the Möbius group, the symmetry group of conformal geometry on the Riemann sphere. (This observation was utilized by Roger Penrose as the starting point of twistor theory.)

  The Poincaré group, named after Henri Poincaré, is the group of Minkowski spacetime isometries. It is a ten-generator non-abelian Lie group of fundamental importance in physics. The Poincaré symmetry is the full symmetry of special relativity. It includes, translations (displacements) in time and space ($\mathbf{P}$), forming the abelian Lie group of translations on space-time; rotations in space, forming the non-Abelian Lie group of three-dimensional rotations ($\mathbf{J}$); and boosts, transformations connecting two uniformly moving bodies ($\mathbf{K}$).

- **Material science:** In material science, groups are used to classify crystal structures, regular polyhedra and the symmetries of molecules. The assigned point groups can then be used to determine physical properties (such as polarity and chirality), spectroscopic properties (particularly useful for Raman spectroscopy and infrared spectroscopy) and to construct molecular orbitals. Molecular symmetry is responsible for many physical and spectroscopic properties of compounds and provides relevant information about how chemical reactions occur.

Although the theory of groups started with a different purpose, it has found many applications in physics and has become a necessity to understanding nature and it's ways.

V. KNOT THEORY AND NUCLEAR PHYSICS

In topology, knot theory is the study of mathematical knots. While inspired by knots which appear in daily life in shoelaces and rope, a mathematician's knot differs in that the ends are joined together so that it cannot be undone. In the 1860s, Lord Kelvin's theory that atoms were knots in the aether led to Peter Guthrie Tait's creation of the first knot tables for complete classification. Tait, in 1885, published a table of knots with up to ten crossings, and what came to be known as the Tait conjectures. This record motivated the early knot theorists, but knot theory eventually became part of the emerging subject of topology.

In the late 20th century, mathematicians and physicists discovered that a major tool in knot theory, applied as well to elementary particle physics. The connection was unexpected because particle physics seemed far removed from knot theory, a branch of topology. They asserted that the mathematics of the Jones polynomial, a polynomial describing knots, was exactly the mathematics of elementary particle physics.

In 2013, L. Kauffman [4] showed how knots were related not just to braiding and quantum operators, but to quantum set theoretical foundations and algebras of fermions. He showed how the operation of negation in logic, seen as both a value and an operator, could generate the fusion algebra for a Majorana fermion, a particle that is its own anti-particle and interacts with itself either to annihilate itself or to produce itself.

VI. CONCLUSION

With these and many other examples, it is clear for sure that the relation between mathematics and physics is an everlasting one. It is remarkable to see how the elegant features of mathematics have helped understand physical phenomena. We find that big domains of pure mathematics have been brought in to deal with the advances in fundamental physics. It is possible in the future that the two subjects may unify and provide new methods of research into the foundations of the subjects. Eventually, we all know that nature is beautiful and that our target is to understand this beauty, mainly in the form of mathematical beauty.

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