What if Natural Numbers are *not* constant?

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Abstract

Mathematics, via model theory, gives us the possibility that natural numbers could be understood as varying objects. We analyze this from the point of view of physics were standard models of natural and real numbers are not always absolute or fixed. The extended equivalence principle appears covering the changes of the numbers. As the consequence strange exotic geometry emerges with which a kind of gravity is assigned. Taking such perspective, from the foundations of mathematics, sheds completely new light on the nature and construction of a theory of quantum gravity.

1 Introduction

Modelling and explaining the world is one of the tasks of everyday practice of theoretical physicists. Mathematics is a tool box one can always reach to, for picking up a suitable tool. The tools are always unchanged and in the same place in the box and we know well which should be used in what situation. Language in which mathematical theories are formulated is irrelevant, but necessary, supply. It is transparent from the point of view of practicing physicist. For example, natural numbers are just mathematical 'bricks' or 'concrete blocks' which serve as absolute measure for any construction. Real numbers are similarly fixed. Every input of a physical experiment is represented by such understood numbers in this or another way.

The above statements, even they are oversimplified, seem to be accurate generally. Even though there exist some mathematical subtleties regarding the numbers and the language, these subtleties are finally irrelevant to physics. Physicists are looking for a simple and universal layer of the reality. But is such simplification always a reasonable rule? In fact, I show that owing many difficulties arising in mathematics and physics in particular dimension 4 the statements we started with might be completely wrong (maybe, with the exception of the first one). History of mathematics and physics teaches us constantly that even most solid fundamentals can become one day just illusion or approximation at best. Let me mention only relativization of absolute space and time in one unified and dynamical space-time entity, non-absoluteness of coincident events, quantum phenomena etc., etc..

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My story begins with space-time. Thinking about models for space-time, or our Universe, heavily relies on the concept of mathematical spaces - differentiable manifolds. Differential manifold in physical dimension 4 is the space which locally looks like the simplest 4-d piece, i.e. $\mathbb{R}^4$. This last, however, apparently is built of real numbers belonging to the real line $\mathbb{R}$. We are so accustomed to using the reals in various physical theories modelling reality, that almost nobody is cast to doubt in their uniqueness or the right to use them. In fact, we do. It does not mean I am so brave to do so, it simply means that such possibility is well founded in mathematics and has profound consequences for physics. Yes, indeed, the approach is not popular. It is mainly because of the very fundamental and constant meaning assigned to natural and real numbers especially in physical theories. We discussed that point above and will come back again to it later. The numbers are involved in the constructions as absolute entities with well understood and constant meaning assigned. Similarly as Newton's space was serving as the unchanged hence, absolute, box for apparently external events and in external time. Perhaps now the time has come to consider numbers in physics in a relativistic way.

This possibility would not be a big surprise, if one realizes that mathematics involved in both, quantum gravity and classical physics, should be applicable in all the scales of energy, from big bang, black holes to the everyday's scales. The mathematical formalism seems to be extremely invariant fixing the absolute point of view of the 'observer' with her/his absolute mathematical tools, like classical logic and real or natural numbers, that have to maintain relevance in all those extremely different physical regimes. It would be quite a surprise if there were no need, whatever, to modify those basic tools, and the absolute observer point of view, when trying to describe such extremely different limits properly. And it would be surprising if mathematics itself had not supplied suitable tools.

However, many questions arise. What is the proper stage for observing changes in natural or real numbers? What is the meaning of such changes? One probably needs natural and real numbers again to grasp these changes. Or, maybe, physical theories are built, such that, the effects due to the varying numbers are not physically observable, or even valid, in any sense. However, owing the variations of the numbers in mathematics, they potantially could affect the formalism of theories of physics. To make them irrelevant we need some action. This would be similar to the effects of a free falling lift seen (locally) from the point of view of observer in this lift. The observer can not decide which is true: gravity is switched off or the lift falling. However, locality is crucial here. From the global point of view one can not switched off gravity completely in entire space-time by the choice of any suitable reference frame. More technically, we say that general covariance is spoiled by the non-tensorial nature of Christoffel symbols and gravity effects enter the stage. One can rephrase this by saying that general covariance of some expressions in physical theories is broken and thus, the effects of gravity can be included. If all the expressions (including Christoffel symbols) in a theory were independent of the choice of reference frame, gravity effects would not be seen in such a theory.

By the analogy, we will argue in this essay that the incomplete invariance of a theory with respect to the different representations (models) of the systems of numbers, is the mechanism for the inclusion of important mathematical effects. As we will see they are highly relevant to physics and have especially dramatic consequences in dimension 4.

2 Model theory and varying numbers

First, let us approach the changes of numbers from suitable mathematical perspective. To grasp what the changes are we need to have clear meaning what the numbers are. To this end one formulates the theory of natural and real numbers. To formulate a theory we need a formal language with its alphabet of symbols. Without diving into too much details let us assume that a theory of natural numbers is formalized as the Peano arithmetic (PA) with its, say, 10 axioms expressing the fundamental properties of natural numbers. Some properties of natural numbers require reference to *sets*. That is why, again following the historical development of the subject, let us assume that a theory of all sets is formalized as the Zermelo-Frankel set theory (ZF). It is axiomatic formal system. However, the crucial property of the both axiomatizations, PA and ZF, is the language. The language in which both theories are axiomatized is the *first-order language*, which roughly means that one does not refer to the subsets of the fixed 'universe of discourse', or does not quantify over predicates. Only quantification over individual points of that universe are allowed. This has, however, profound consequences. Suppose that a theory T (a set of sentences in the first-order language $\mathcal{L}$) is not contradictory (is consistent). Then, as was shown by Gödel, T has a *model* M. By model we mean a relational system, or the carefully constructed universe for the theory T, where every sentence from T is valid. Let $|\mathcal{L}|$ be the cardinality (the number of sentences and formulas, usually infinite) of the language $\mathcal{L}$.

Theorem 1 (Löwenheim-Skolem, 1920)**.** *If a set T of sentences in the first-order language $\mathcal{L}$ has an infinite model, then it has a model of arbitrary cardinality $\geq |\mathcal{L}|$.*

Both theories, PA and ZF, are formulated in the countable language, which means they have countable models. For PA it is not very peculiar since we know that standard natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$ constitute the countable set which is a model for PA (with the suitable relations between the numbers). But Th. 1 states that there are also universa of 'natural numbers' which are not countable. In fact there is plenty of such universa and together with countable non-isomorphic models, we call them non-standard models of PA. The case of ZF is even more surprising: all properties which one can prove in ZF about arbitrarily high cardinality sets are fulfilled in a countable model. This is more-or-less what is called Skolem paradox. Regarding real numbers their full topological structure is not expressible in the first order language, though all their 1-st order properties are also expressible in some non-standard models. The non-standard models have to exist and can not be just removed from 1-st order theories. So, even the theory of natural numbers, PA, in mathematics is ambiguous in the sense that it can not refer uniquely to the standard model. For the nonstandard models of natural and real numbers we use $^\star\mathbb{N}$ and $^\star\mathbb{R}$ symbols, respectively.

Even though there are non-standard models of PA, one can show that all are conservative extensions of the standard numbers. This means that finite initial segments of $\mathbb{N}$ are preserved in every $^\star\mathbb{N}$. However, taking into account also *informal* ingredients ever-present in every formal construction, it was proposed that the following hypothesis (main hypo) should be considered as a low level assumption [2]:

For any formal theory there exists a pair of non-isomorphic models $(\mathbb{N}, {}^\star\mathbb{N})$ *of PA which are indistinguishable, meaning in any formal extension of the theory of any order there does not exist any formula in this extension which would express any difference or non-isomorphism of the models.*

We use $\mathbb{N} \simeq {}^\star\mathbb{N}$ to denote this relation. Now, starting from such background assumption one could try to give a formal meaning to it. One consequence is that the initial segments of naturals should be somehow modified... On such modified natural numbers there are spanned corresponding reals. The point is that there should exist a 'limit' where the above indistinguishability takes place and one where it does not. Some mathematical objects require both limits to be properly described.

3 Consequences: geometry, gravity and quantum physics

Now, we can come back to the discussion of invariance of mathematics with respect to the choice of (non-standard) model of arithmetics. That would mean, in particular, that given a local coordinate atlas $\{\mathbb{R}^n_\alpha\}_{\alpha\in I}$ for a manifold M^n with the transition functions $\phi_{\alpha\beta} : \mathbb{R}^n_\alpha \to \mathbb{R}^n_\beta$, allows now for the non-standard extension: $\phi^\star_{\alpha\beta} : \mathbb{R}^n_\alpha \to {}^\star\mathbb{R}^n_\beta$, and the resulting structure of M^n is still the standard smoothness structure. The description of M^n by such non-standard patches might be seen as redundant or even irrelevant extension of the standard smooth M^n. However, let us allow for the possibility that the formalism used to describe smooth manifolds is not completely invariant with respect to the choice of varying numbers. Extra effects, comparing with the standard case, can appear and the resulting structure becomes different, in some non-redundant sense. First, it can happen that the resulting structure is still smooth. In fact we are assuming this. Next, the structure appears as 'classical' in a sense that it is not just an artifact or remnant of the non-standard constructions involved, but rather the structure can be understood purely in standard terms and it exists as classical object, though different from the initial one. The resulting smooth structure is equivalent to a smooth structure described without any reference to non-standard $^\star\mathbb{N}$ or $^\star\mathbb{R}$. However, what does different smooth structure, which would be non-equivalent to the initial one, mean in this context, and is it *real* at all? Let us be more systematic. Based on the indistinguishability rule from the previous section we can define:

i. We say that some mathematical object (construction) survives in the limit where the indistinguishability $\mathbb{R} \simeq {}^\star\mathbb{R}$ holds, provided the object has descriptions both in the theory where $\mathbb{R} \simeq {}^\star\mathbb{R}$ and in the limit where $\mathbb{R} \neq {}^\star\mathbb{R}$. The object surviving the limit, can be considered in both situations.

ii. We say that an object survives in the above limits as the same object if it survives the limit, and if it is the same object, up to a natural equivalence of the structures, before and after taking the limit. In the case of smooth structures the natural equivalence is given by diffeomorphisms.

iii. The object surviving the duality as the same object will be called model-theoretically self-dual one (MTSD).

The following, quite obvious, theorem explains the role of the model-theoretic duality

Theorem 2 (Th. 4, [2]). *Let us suppose that a smooth differential structure on $\mathbb{R}^4$ is the object surviving in the $\mathbb{R} \simeq {}^\star\mathbb{R}$ limit as the smooth differential structure on $\mathbb{R}^4$ (it is model-theoretically self-dual). Then this structure cannot be standard one.*

Now, we conjecture that MTSD smooth $\mathbb{R}^4$ does not refer to any non-standard constructions and it is different 'real' smooth $\mathbb{R}^4$. Here mathematics, or Nature, comes with help: mathematicians were able to prove in 1980's, that indeed there exists *exotic* $\mathbb{R}^4$ which is non-diffeomorphic to the standard smooth $\mathbb{R}^4$, though, still it is the same topological $\mathbb{R}^4$. The history of the proof of its existence is a fascinating story in itself and finally it appeared that there are *infinite continuum many* (more than just infinite countably many) different smooth structures on $\mathbb{R}^4$. And only for $\mathbb{R}^4$ this is true - any other $\mathbb{R}^n, n \neq 4$ is smooth in exactly unique way. Let us note the important thing – all exotic $\mathbb{R}^4$s are smooth Riemannian 4-manifolds, hence plenty of smooth metrics should exist on each of them. However, none of these metrics is known explicitly today. Mathematics suitable for this purpose is partly unknown.

In our model-theoretic approach, we arrive at the point where the extension (*EP) of the relativistic equivalence principle (EP) emerges such that the choice of non-standard coordinate frames is now allowed. One mathematical consequence of such *EP is the possible inclusion of the effects due to exotic smooth geometries on $\mathbb{R}^4$ as the result of the non-complete invariance of a formalism of a theory with respect to the choice of different model of natural numbers. From the point of view of physics we would be including gravitation to a theory due to the fact that exotic $\mathbb{R}^4$s are non-flat 4-dimensional smooth manifolds.

One would wonder why we are using yet another relativistic principle to generate Einstein gravity which is already successfully generated and described by the usual EP of general relativity (GR). The answer seems to be surprisingly essential: Einstein GR is inherently classical theory. *The extension of the EP* *such that it generates exotic smoothness on $\mathbb{R}^4$, allows for crossing the border line between classical and quantum gravity in dimension 4.* This is due to the remarkable connection of exotic 4-geometries on $\mathbb{R}^4$ with quantum theories. Let me mention here just some results without presenting details.

a. The quantization of electric charge can be explained by allowing for existence a region $\mathbb{R}^4$ in 4-space-time which would be exotic smooth rather than standard smooth. The effect is the same as if there exists a magnetic monopole [7, 10].

b. The gravitational corrections to (quantum) Seiberg-Witten theory (SW) on flat $\mathbb{R}^4$ are generated by formulation of SW on exotic $\mathbb{R}^4$ [15].

c. Superstring theory (ST) is a theory of quantum gravity (QG) in 10 dimensions. Some configurations of D-branes and NS-branes correspond to exotic $\mathbb{R}^4$ appearing in well-defined regions of the string background [12]. Also flux on S^3-part of some string backgrounds can be generated by exotic $\mathbb{R}^4$ where this S^3 is suitably embedded.

d. Also in ST, some quantum branes correspond to exotic $\mathbb{R}^4$ [14]. Moreover, a new class of quantum topological branes emerges and is connected with the so called wild embeddings of spaces in topology [11].

e. Some states of effective quantum matter, as Kondo state, correspond to exotic 4-geometry appearing in 4-space-time [13].

Now, let us turn to the EP of GR which roughly says:

The observer in 4-d space-time can locally cancel gravity effects by the choice of special coordinate frame. In manifold's space-time language this means the choice of the local patch in the atlas, i.e. the standard flat $\mathbb{R}^4$.

Owing the above connections with quantum gravity and quantum field theory the extended version of EP, *EP, could read:

In 4-d space-time a (mathematical) observer, facing some quantum or QG effects, can locally replace some of these effects by the choice of non-flat exotic 4-geometry on $\mathbb{R}^4$.

Whether this mathematical observer can become physical one, or whether the exotic 4-geometry patch becomes valid physical coordinate frame, we leave as open possibility here. In principle, however, it is possible. Anyway, dual descriptions in terms of exotic geometry or quantum (gravity) effects can be always applied leading to new insights.

Such extended *EP formulation works in analogy with ordinary EP in GR where non-tensorial nature of Christoffel symbols includes/excludes gravity effects into a theory. Here, we have rather non-invariance of mathematical structures of a theory with respect to the limits where some models of Peano arithmetics are different or are indistinguishable. This means that we extend the allowed local patches over non-standard $^\star\mathbb{R}^4$. The mathematical consequence is the appearance of MTSD smoothness on $\mathbb{R}^4$ which is conjectured to be exotic smooth $\mathbb{R}^4$. Physically, it can generates/cancells some quantum, also QG, effects in a theory.

Let us try to grasp more clearly what this all means for our current understanding of the subjects involved. First, we do not have yet satisfactory and effective constructions of exotic smooth structures on open 4-manifolds like $\mathbb{R}^4$, nor we know the mathematics suitable for. The model-theory gives some impact.

Second, exotic $\mathbb{R}^4$s exist only in dimension 4, i.e. none $\mathbb{R}^n$ for $n \neq 4$ is exotic. 4 is the physical dimension where in particular standard model of particles and GR - a classical theory of gravity, are formulated. So, maybe partially, the problems with exotic $\mathbb{R}^4$ are physically valid. The extended EP gives a method, inspired by GR and model theory, to include effects of exotic $\mathbb{R}^4$s into a theory.

Third, exotic smooth $\mathbb{R}^4$s are *all* topologically the simplest Euclidean 4-space, $\mathbb{R}^4$, and *all* have to be non-flat smooth Riemannian 4-manifold, hence a kind of gravity could be, and, in fact, is present on such spaces.

Fourth, we do not have any satisfactory theory of QG in dimension 4. Even worse, we know that something fundamental is missing in our present understanding of the subject. But QG in 4-d should also deal with exotic $\mathbb{R}^4$s and their effects have been mostly ignored by existing theories of QG. The extended EP is a way, following GR, for inclusion certain quantum effects to a theory.

Fifth, the connection of exotic 4-geometries with quantum physics and string theory serves as a new aspect of the quantum theories applicable exclusively in dimension 4.

We see that very profound problems of physics and also purely mathematical are gathered in dimension 4. Possibly, building the successful theory of QG in dimension 4 requires a ground level rethinking of the nature of natural numbers. Here we have a direct indication and explanation of how it could be. The presented approach is not very technical at every part, but rather conceptual in many aspects. However, based on deep and rigorous mathematics, it questions assumed basic a'priori absoluteness of numbers in physical theories. Moreover, it is rather necessary step to be considered on the way to a final QG theory in 4-d. A fruitful and intriguing perspective for physics, thus, emerges.

We are just at the beginning of the programm of uncovering various relations of exotic $\mathbb{R}^4$s with quantum phenomena and foundations of mathematics. Following this programm can shed light also on the foundations of mathematics. This is not a big surprise at all that physics and mathematics could be connected (again) also at this very basic level. The exceptional historical coincidences in mathematics and physics, and the inherent nature of physical world itself, focus the attention and effort of many researchers at difficulties emerging particularly in dimension 4. The understanding of the interplay between physics and mathematics in this specific domain, serves as a way to uncover certain mysteries of QG in 4-d.

4 Remarks

Let me close this short presentation with some remarks which are relevant here but were not included in the main part of the essay.

The presented approach is just scratching of the surface in some aspects. Namely the model-theoretic arguments yield their different mathematical perspective when turning to category theory, especially topos theory, see e.g. [5, 4]. However, I refrained using the categorical methods here, since the approach is not reducible completely and the universality of category theory would hide some ideas. On the other hand, I did not make any use of many other, attractive from the point of view of model theory and exotic $\mathbb{R}^4$, constructions. Let me mention just Isham and Doering or Landsman, and their co-workers, approaches on the application of topos theory to theories of quantum physics, or the Takeuti's Boolean-valued analysis, where real numbers are realized as self-adjoint operators on a Hilbert space in a Boolean topos, or the Moerdijk-Reyes smooth toposes. Moreover, there exist categorical versions of relativity principle which are based rather on the invariance of mathematics with respect to the choice of a topos with natural numbers object (tovariance principle by Landsman, Heunen and Spitters [6], or Bell invariance [1]). Though, they should be again somehow broken to include special geometry effects into a theory.

There is the substantial recent activity on relating string theory with exotic 4-geometries. The reason is the following difficulty. Superstring theory is the best, by now, candidate for the theory of QG. However, ST is formulated in 10 or 11 dimensions and the intensive attempts to bridge it with 4-dimensional world of our physics seems to be problematic. In fact, the ambiguity with obtaining the 4-d worlds is estimated, by some researchers, to be of the order of 10^{500}. For the fundamental theory of our physical world it is too many. Even 2 different possibilities would be a problem. This is the place where new 4-d connection can help. Superstring theory serves as very rich mathematics by itself, so, maybe it can describe also exotic $\mathbb{R}^4$s. Moreover, ST is a theory of QG and exotic $\mathbb{R}^4$s are non-flat, gravitational, 4-spaces. Thus, expecting some relation of exotic $\mathbb{R}^4$s with QG (see, e.g. [9]) it is natural to look for this from ST where exotic $\mathbb{R}^4$ could be a part of string backgrounds. Some interesting results were thus obtained. Conversely, exotic $\mathbb{R}^4$s can presumably say something new and interesting about ST itself. Such programm has indeed been proposed [8] and then developed [12, 10, 14]. In the essay I related these attempts with model-theoretic origins of 4-exoticness.

One could wonder whether there are inherent reasons for model-theoretic constructions in particular dimension 4 generating exoticness. Partial answer is given by the classic, set-theoretic, forcing which was used originally by Paul Cohen in 1963 to prove independence of the Axiom of Choice and the Continuum Hypothesis from other ZF axioms of set theory. In our case we can develop *forcing adding Casson handless* which is non-trivial in dim 4 due to the infinite layer structure of topological Casson handless.

There is also an interesting possibility to look at string theory at the set-theoretic level directly, however respecting 1-st order model-theory properties. Let us call it *set string theory.* This is yet another level of string theory which complements ordinary, geometric or differential strings and their topological twist, i.e. topological string theory. In set-theoretic strings each string corresponds to real numbers in some countable transitive model of ZFC. Then, a natural relation between strings at this level is, again, the forcing relation, adding some reals to the models. One can show that such structure interprets quantum mechanics. So, starting from classical strings in 10-d space-time, we arrive at gravitons in the spectrum, but also, at the set-theory level of the string theory, QM is being already present.

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References

[1] J. L. Bell, From absolute to local mathematics, *Synthese* **69**, 409-426, 1986.

[2] J. Król, Background Independence in quantum gravity and forcing constructions, *Found. Phys.* **34**, 361, 2004.

[3] J. Król, Exotic smoothness and non-commutative spaces. The model-theoretic approach, *Found. Phys.* **34**, 843, 2004.

[4] J. Król, Model theory and the AdS/CFT correspondence, presented at the IPM String School and Workshop, Queshm Island, Iran, 05-14. 01. 2005, arXiv:hep-th/0506003.

[5] J. Król, A Model for Spacetime: The role of interpretation in some Grothendieck topoi, *Found. Phys.* **36**, 1070, 2006.

[6] N. P. Landsman, C. Heunen and B. Spitters, The principle of general tovariance, in Geometry and Physics, Proc. XVI Int. Fall Workshop, pp. 93-102, Lisbon 2007.

[7] T. Asselmeyer-Maluga and J. Król, Abelian Gerbes, generalized geometries and Exotic R^4, arXiv: 0904.1276, 2009.

[8] T. Asselmeyer-Maluga and J. Król, Small exotic smooth R^4 and string theory, in *International Congress of Mathematicians ICM, Hyderabad, India 2010, Short Communications Abstracts Book*, R. Bathia (Ed.), Hindustan Book Agency, p. 400 (2010).

[9] J. Król, (Quantum) gravity effects via exotic $\mathbb{R}^4$, *Ann. Phys. (Berlin)*, **19**, 3-5, 355-358, 2010.

[10] T. Asselmeyer-Maluga, P. Gusin and J. Król, The modification of the energy spectrum of charged particles by exotic open 4-smoothness via superstring theory, will appear in *Int. J. Geom. Meth. Mod. Phys.* **10** No 1, 2013, arXiv: 1109.1973.

[11] T. Asselmeyer-Maluga and J. Król, Topological quantum D-branes and wild embeddings from exotic smooth R^4, *Int. J. Mod. Phys. A* **26**, 3421-3437, 2011, arXiv:1105.1557.

[12] T. Asselmeyer-Maluga and J. Król, Exotic Smooth R^4 and certain configurations of NS and D branes in string theory, *Int. J. Mod. Phys. A* **26**, 1375-1388, arXiv:1101.3169, 2011

[13] T. Asselmeyer-Maluga and J. Król, Pure 4-geometry of quantum interacting spin matter from Kondo effect, arXiv:1107.0650, 2011.

[14] T. Asselmeyer-Maluga and J. Król, Quantum D-branes and exotic smooth R^4, *Int. J. Geom. Meth. Mod. Phys.*, **9**, 2012, arXiv:1102.3274.

[15] T. Asselmeyer-Maluga and J. Król, Towards superconformal and quasi-modular representation of exotic smooth $\mathbb{R}^4$ from superstring theory II, arXiv:1207.4603.