How To Build Holographic Spacetime

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Is it possible to render holographic spacetime starting from simple discrete building blocks? I will demonstrate that not only is the answer *yes*, but also that to do so, surprisingly simple building blocks suffice. Holographic 2D toy models result that you can play with and investigate in terms of De-Bruijn-type sequences, dual descriptions and path sums.

Towards Countable Physics

The intuitive notion that ‘deep down’ a discrete and countable physical reality is hiding, is by no means new. Kronecker uttered his famous quote *"God made integers, all else is the work of man"* one and a half century ago. The statement was directed at mathematicians, but with math being our sole means to reach understanding of physical reality, the remark ricochets unabatedly into the realm of physics.

With the advent of quantum theory in the first half of the twentieth century, the idea of a countable physical reality raised its head more strongly. A mind-boggling world of discrete physical states and quantum jumps between these unfolded. Yet, the laws of quantum physics remain formulated against a background continuum formed by spacetime. So even if the wave functions describing the states accessible to a physical system form a countable set, each of these takes the shape of a continuous function.

Strong hints towards a countable spacetime emerged in the 1970's. Bekenstein's [1] and Hawking's [2] work on black holes thermodynamics made it clear that a consistent thermodynamics description of these highly curved, causally disconnected regions of spacetime requires a finite number of possible states to be associated with such regions. Surprisingly, the number of black hole states appeared to be linked not to a volumetric measure, but to the area of the causal horizon behind which the disconnected spacetime is hiding.

Years of confusion followed, as the thermodynamic picture that had emerged assigns a finite temperature to a black hole and therefore requires it to radiate. This thermal radiation is quantum mechanical in origin and takes place despite the lack of any causal paths from within the horizon to the outside world. As this radiation will ultimately cause the black hole to disappear, the question arises "what happens to the information that specifies the state of the black hole? Is this information carried away by the radiation, or does it simply disappear?"

Either way, the radiation process is at odds with the most fundamental principles of physics. If the information gets carried away, the laws of physics can no longer be local: the information that describes the physical state of a black hole must in some way reside inside as well as outside the black hole horizon. If on the other hand the information disappears, the laws of physics can not be information-preserving, and with the information that describes the past getting wiped out, dynamics can not be reversible and the very foundation of quantum physics (unitary evolution) becomes an illusion.

It was 't Hooft [3] who in the early 1990's took the bold step to formulate what nature seemed to be telling us: physical reality is holographic. The fundamental degrees of freedom that describe all that can happen in a given region of spacetime can be thought to reside as binary pixels on the boundary of the region. Susskind [4] further clarified this picture, elucidated its non-local aspects, and linked the holographic degrees of freedom to a string-theoretical description. The resulting view on physical reality salvages information conservation, but introduces non-local characteristics between degrees of freedom that are dramatically ‘thinned out’.

It From Bit

With a conceptual picture of strict information conservation and bits living on boundaries describing all of the physics that can happen in the space enclosed by the boundary, modern physics has dived deep into Wheeler's *'it from bit'* doctrine. [5] This approach to fundamental physics postulates all things physical to be information-theoretic in origin, and can be summarized by changing two words in Kronecker's quote: *"God made bits, all else is the perspective of man"*. Yet, despite all these slogans, the question remains, how can spacetime be holographic? What causes the ‘thinning out’ of degrees of freedom? How should we characterize the holographic degrees of freedom that survive this ‘thinning out’? Can we locate these?

In situations like this, it often helps to investigate an extreme case. The most extreme holographic thinning-out of degrees of freedom will take place in spacetimes with the fewest number of dimensions: 2D spacetime with one spatial and one temporal dimension. This can be understood as follows. Consider a spacetime with d-1 spatial dimensions and one temporal dimension. Let’s assume this spacetime is finite and consists of N elementary events. With deterministic evolution laws specified, the degrees of freedom are restricted to the starting events, and therefore the spacetime will have a number of degrees of freedom S that scales like $S \sim N^{(d-1)/d}$. Effectively, this means that the deterministic dynamics prevents the temporal dimension to contribute to the degrees of freedom. So, in the four-dimensional (d = 4) spacetime we live in that means that the number of degrees of freedom scales like $S \sim N^{3/4}$, i.e. like the total spatial volume. However, these considerations ignore any holographic ‘thinning out’ of the degrees of freedom. With ‘t Hooft’s and Susskind’s holographic principle incorporated, the degrees of freedom for d-dimensional spacetime would scale like $S \sim N^{(d-2)/d}$.

For a four-dimensional holographic spacetime, it follows that the degrees of freedom S scale with the square root of the number of events: $S \sim N^{1/2}$. Hence, the number of degrees of freedom represents a tiny fraction of the total number of events. For lower-dimensional spacetimes, however, for the same total number of events N, the number of degrees of freedom forms an even smaller portion of N. The limit $d \to 2$ is an interesting one. Based on the above scaling arguments, one would expect the number of degrees of freedom S for a two-dimensional holographic spacetime to be constant and independent of size N. However, this ignores any higher order (non-leading) terms in S(N) for given dimensionality d. Including such terms, one arrives at the conclusion that the number of degrees of freedom for d=2 can be expected to scale as S ~ log(N). When holographic, 2D spacetime is so to the extreme, with a number of degrees of freedom that grows with the number of events no faster than logarithmically.

Can we build such ‘extremely holographic’ two-dimensional spacetimes? The answer is *yes*. Moreover, very little is needed in terms of definition at a micro level to achieve this. Models for holographic 2D spacetimes can emerge from causal structures containing nothing more than simple logical gates. Such 2D spacetimes can be realized in the form of simple tiling puzzles.

The approach presented in the following is similar in philosophy to what drove Conway to construct his 'Game of Life'. Conway’s toy model does not give us a realistic description of complex systems and certainly not of any life forms. It was never intended to do so. It does, however, demonstrate what little structure is needed at small scales to allow complex behavior including self-replication [6] to emerge at large scales. Similarly, the 'xor universes' that I will describe here give us the insight that austerely simple cause-and-effect structures suffice for the emergence of spacetime structures with holographic degrees of freedom.

From Tiling Puzzles To Holographic Automata

Consider the tessellation puzzle depicted in figure 1. Can you tile a 2D plane having only these four types of pieces available? There is a trivial solution that I will refer to as the 'vacuum solution': tiling the plane using solely yellow rectangles. Can you create a tiling that contains at least one tile with a blue disk segment?

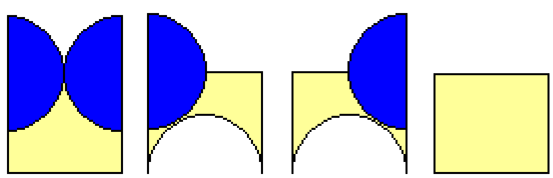

Figure 1. *How to tile an infinite plane with these four pieces? Can you find solutions other than the trivial ‘all yellow’ tiling?*

The simplest non-vacuum solution is shown on the left-hand side of figure 2. More complex solution patterns are shown in the middle and to the right-hand side. In each of these cases the solution that emerges takes the shape of a pattern of hexagonal cells with each of these cells being empty, or occupied by a blue circular disk. The pattern formed by the occupied and empty hexagonal cells is repeating. The solutions shown in figure 2 consist of repeating patterns of three, seven and twenty-one hexagonal cells, respectively.

What does this tiling puzzle have to do with holographic spacetimes?

A lot. Let me explain.

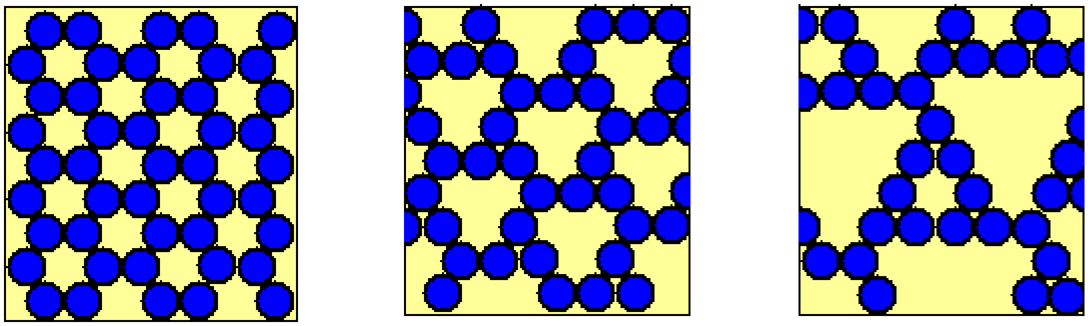

Figure 2. *Progressively more complex solutions of the tiling puzzle depicted in figure 1.*

When attempting to tile the plane with the puzzle pieces, you will soon find yourself creating rows of pieces that on one side have convex edges and on the other side concave edges. Given such a row, there is no choice in how to extend the pattern along the piecewise concave side. With a row defined, the pattern evolves in a deterministic way. The patterns in figure 2 are built from such rows with the piecewise concave edges pointing downward. The deterministic evolution becomes clear when inspecting in figure 2 rows of disks and empty spaces. You will notice that each disk has immediately above it one disk and one empty slot. Any empty slot has either two empty slots or two disks above it. There are no exceptions. The state of each row of cells is fully determined by the state of the row above it, with the occurrence of a disk in a certain cell being determined by the exclusive-or (xor) of the two neighboring cells in the row above it. So when looking at figure 2 we are looking at binary cellular automata with time running from top to bottom, and the single spatial dimension stretching horizontally.

Ok, solutions to the tiling puzzle contain an evolution element and can be thought to represent a binary spacetime. But where is the holography in this model?

Holography is all about degrees of freedom. So we need to ask the question “given the xor evolution rule, how many degrees of freedom can we assign to the patterns in figure 2?” Considering that a single row initiates an evolution that assigns occupancies to all cells in each of the rows below it, one might be tempted to answer “the number of degrees of freedom equals the number of cells in the starting row”. That answer obviously needs some modification, as the patterns depicted in figure 2 are repeating horizontally. Let’s take the middle picture as an example. The pattern depicted consists of a 7-cell pattern that repeats. In figure 3 red lines demarcate such 7-cell repeating units. Indicated on the left-hand side is the situation when 3 cells in the top of each repeating unit have known content (filled with disks). The other cells are marked gray with a question mark, denoting “content unknown”. Applying the xor rule, we can immediately eliminate two question marks in each 7-cell unit as their value is determined by the xor of cells with known content (middle picture). Now we can repeat once more this process, as the remaining gray cells are fully determined by the xor of cells with known content. We now have achieved a situation in which all cells have known content (rightmost situation in figure 3). Clearly, assigning content to three cells suffices to get an evolution that determines the state of each cell. So we arrive at the conclusion that the 7-cell pattern has three degrees of freedom.

This conclusion is wrong. It is true that defining just three cells leads to a complete pattern, but

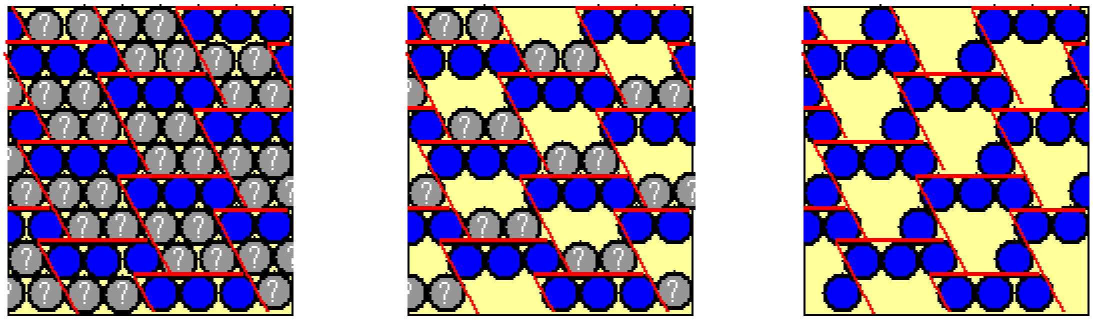

Figure 3. *Evolution of the 7-cell pattern starting from three known cells in each repeat unit.*

what is not taken into account is that these three cells can have any initial state. You might give that a try. Change the rows of three blue cells on the left-hand side of figure 3 into any other combination containing at least one blue cell, and you will again recover the full solution pattern. Each time you change the initial three-cell pattern, the same final pattern results, but each time shifted in a different way.

So, there are no degrees of freedom associated with the blue-yellow 7-cell pattern depicted in figure 2. Specifying that we are dealing with a repeating 7-cell pattern, is specifying the whole pattern. If this would hold without any change for all other patterns for this puzzle, the conclusion would be that all these patterns have vanishing degrees of freedom regardless of size. In reality larger patterns typically show multiple realizations, with the number of patterns growing modestly (no faster than algebraically) with size. This results in a logarithmic growth of degrees of freedom with pattern size, the hallmark of holographic behavior in 2D.

This can be made more precise as follows. Look again at figure 2, and carefully compare the left (3-cell) and center (7-cell) pattern with the rightmost (21-cell) pattern. Do you spot a correlation? Well, there is a very strong correlation: the 21-cell pattern is a superposition of the two other patterns. This superposition is determined by xor addition: if corresponding cells in the 3-cell and 7-cell pattern have the same occupancy (both empty or both filled with a blue disk), then the same cell in the rightmost pattern is empty. If, however, corresponding cells in the 3-cell and 7-cell pattern have different occupancies, then the cell in the 21-pattern is occupied.

When we superimpose patterns of different sizes (different number of cells per repeat unit), we get back a pattern of yet another size, typically equal to the product of the two original sizes. This scaling behavior under superposition also carries a holographic signature. If patterns of N cells are characterized by S(N) degrees of freedom, then if the superposition of two patterns of sizes N_1 and N_2 results in a new pattern of size N_1 times N_2, the number of independent degrees of freedom of this sum pattern can not exceed the sum of the number of degrees of freedom of the original two patterns. So:

$$S(N_1 \cdot N_2) \leq S(N_1) + S(N_2)$$

It follows that the number of degrees of freedom S scales with pattern size N at most as $S(N) \sim \log(N)$. As already indicated, this is the holographic scaling behavior expected for 2D spacetimes.

Sub-DeBruijn Sequences And Emergent Spacetime

At the very heart of the holographic behavior of the tiling puzzle is the occurrence of what I shall refer to as 'sub-DeBruijn sequences' [7]. I define sub-DeBruijn sequences of order n as two or more cyclic binary sequences that add up to a total length of 2^n and in which every sequence of n consecutive bits appear. For instance, allowing the single-bit sequence (0) as well as the three-bit sequence (1,0,1) to repeat, you can get all two bit sequences: 00 from repeating (0), and 01, 10, and 11 from the cycle (1,0,1). This can be summarized by stating that ((0), (1,0,1)) forms a sub-DeBruijn pair of order 2. Similarly, ((0), (0,1,1,1,0,1,0)) forms a sub-DeBruijn pair of order 3.

Now look at figure 2. The order two sub-DeBruijn sequence (1,0,1) summarizes the pattern on the left-hand side: the presence (1), absence (0), and presence (1) of a blue disk is repeated along the rows. The order 3 sub-DeBruijn sequence (0,1,1,1,0,1,0) you see repeated along the rows in the middle of figure 2. Reversing this observation, we can start from higher order sub-DeBruijn sequences such as the order four quadruplet ((0), (1,0,0,01), (0,1,0,1,0), (1,1,0,1,1)) and check whether these leads to solutions of the xor tiling puzzle. This is indeed the case (see figure 4).

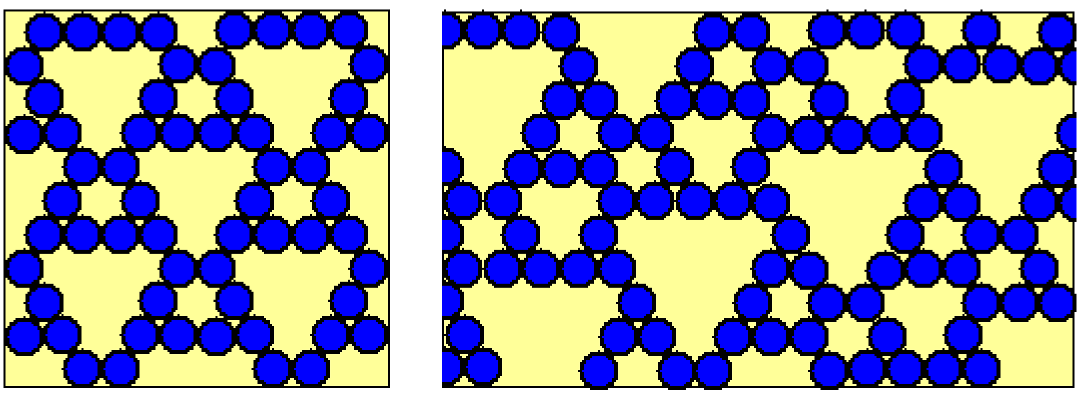

Figure 4. *The 15-cell tiling solution (left) composed of the order 4 sub-DeBruijn sequences (1,0,0,0,1), (0,1,0,1,0), and (1,1,0,1,1), and a 31-cell tiling based on an order 5 sequence.*

Like DeBruijn sequences, also sub-DeBruijn sequences have associated graph representations. These sub-DeBruijn graphs are shown in figure 5 for the order-2 and –3 sequences. Mathematician would refer to these graphs as strongly regular, balanced, bi-colored digraphs of degree two. Mathspeak for abstract networks of nodes of two colors joined by directed links.

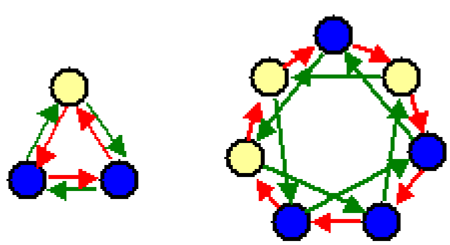

Figure 5. *Sub-DeBruijn graphs for sub-DeBruijn sequences of order 2 (left) and 3 (right). These graphs correspond to the 3-cell and 7-cell tiling solutions depicted in figure 2.*

In these graphs, node colors are selected according to the xor description, a node color is blue ('filled') if the two nodes it is connected to via incoming links (the 'parent nodes') have different colors. A node color is yellow ('empty') if the parent nodes have the same color. The links in figure 5 are colored such that each node is connected to its parents via an incoming red and green link. Furthermore, the link coloring is such that at least one color forms a cycle that visits all nodes (a so-called Hamiltonian cycle). Such coloring appears always to be possible. Given these coloring characteristics, the red and green links in the digraph representation correspond to left and right causations in the corresponding cellular automaton representation. By comparing digraphs with corresponding cellular automata, it can readily be seen that the red and green links in the digraph correspond to opposing 'light rays' in the cellular automaton (see figure 6).

We have landed on a dual description of the puzzle solutions. On the one hand we can represent these as cellular automata defined on a 2D spacetime. This is a representation that renders the cause-effect relation strictly local. On the other hand, we can also describe these as graphs that represent the 1D 'causal strings' that are present in the cellular automata. This abstract network description in itself contains no notions related to space or time. Concepts of distance other than node adjacencies defined by the link connectivity are absent from the graph representation. Yet, both representations describe one and the same system.

Now the time has come to reverse the whole train of thought. Rather than describing the puzzle solutions in cellular automata terms, and subsequently in more abstract digraph terms, we take as starting point the abstract digraph description. The digraph description defines a closed causal structure consisting of nothing more than a finite number of binary events (blue and yellow nodes) on which a xor cause-effect relationship (pairs of red and green links) operates. If we link this abstract digraph representation to its dual cellular-automaton representation, a notion of spacetime emerges. As discussed above, this spacetime has holographic characteristics. Note that space nor time is explicitely defined. Holographic spacetime emerges from the closed causal structure contained within the graph. The causal structure also introduces to this spacetime light cones that form the boundaries between timelike and spacelike regions. It is even possible to define on the graph a metric, a notion of distance between the nodes, that renders the cellular automaton spacetime Minkowskian. This is achieved by a metric that link the square of the distance between two cellular automata events to the size of the causal area (the product of the number of red and green links) between these events.

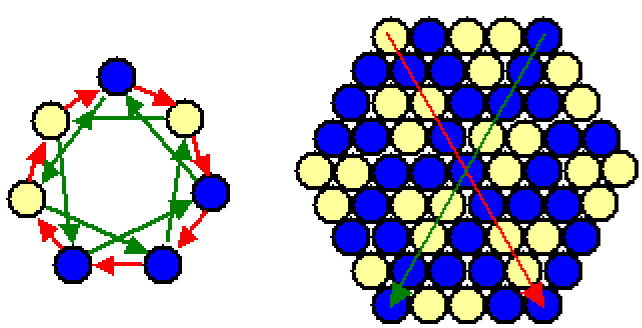

Figure 6. *Dual representations of the 7-node xor universe. Left: Hamiltonian digraph (non-local 1D) representation. Right: spacetime (local 2D) representation, with two light rays corresponding to the red and green Hamiltonian cycles in the digraph.*

Causal Loops And Path Sums

We have encountered 3-, 7-, 15-, 21- and 31-cell solutions for the xor puzzle. Each of these represent a unique (modulo translations and reflections) solution to the tiling puzzle. No other n-cell solutions exist for 1 < n < 32. The sparseness of solutions is obviously related to the fact that the only solutions that form are those composed of sub-DeBruijn sequences. And it is this very sparseness that is required to render the tiling holographic.

What is the ‘mechanism’ behind this sparseness?

It appears that xor puzzles can be described by path sums. When represented in cellular automaton representation, these xor path sums are akin to the path sums in the Feynman checkerboard model that describes a spin ½ particle in 2D spacetime. [8] In the abstract digraph representation, the sum runs over all directed paths that do not visit a node more than once. Two nodes are causally connected (in mathematical physics jargon: are connected via a nonzero propagator) when there exist an odd number of paths between these two points. One type of paths are most relevant: closed loops. For a digraph to represent a xor puzzle, it should allow a xor coloring different from the 'all yellow' vacuum coloring. For this to be possible, the propagator from each node to itself needs to be nonzero. This means that the total number of closed loops going through any given node needs to be even.[1]

It is easy to check in figure 5 that the simplest (3- and 7-node) xor digraph realizations indeed satisfy this 'loop rule'.[2] Experimenting with 4-, 5- and 6-node digraphs (or graphs with any other number of nodes that does not allow a xor node-coloring) should make it clear that this loop rule is indeed responsible for thinning out the xor puzzle solutions, and effectively causes the associated digraphs to come only in specific sizes.

Outlook

The traditional approach towards building models of reality takes the shape of a locally reversible (locally information preserving) dynamics. However, theoretical considerations reveal that such models can not provide a consistent description of black hole thermodynamics. The toy model presented here points to a potential solution: models in which the local dynamics is irreversible, yet in which the global dynamics recovers information preservation. This can be realized by rendering the dynamics to close upon itself. The closure condition not only yields information preservation, but also introduces non-local characteristics and holographic scaling into the global behavior of the model.

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[1] This might seem to be in contradiction with the requirement for the path sum between two distinct points to be odd. The parity difference is related to the exclusion of the trivial zero step path from the loop sum. When included in the path sum, this single additional 'path' would lead to a loop path sum parity requirement in agreement with that for distinct points.

[2] For example: each node in the seven-node digraph is part of 3 period-three loops, 5 period-five loops, and 2 period-seven loops, for a total of 10 loops.

It is utterly straightforward to define closed causal structures based on irreversible cause-effect relationships. The simple xor digraphs presented here provide a clear example. These appear to behave holographically with logarithmically few degrees of freedom. When mapped out such that the cause-effect structures become explicitly local, 2D spacetime structures with local dynamics emerges. The binary spacetime patterns that result, no matter how complex, have low-entropy. Independent of the size of the repeating unit, these patterns can be described in few bits.

These observations suggest that for spacetime to be emergent, a computational structure is required. [9] This result is in line with the Machian notion that spacetime void of any content is a meaningless concept. A minimalistic content to render spacetime existent, is provided by binary fields on which a causal structure operates. Closure of this causal structure yields information preservation with holography as ‘side effect’.

It is tempting to view such closed causal structures as discrete toy models for black holes as well as for the universe as a whole. However, we should keep in mind that the models described here contain only one spatial dimension. To generate closed causal structures resulting in higher-dimensional holographic models, the current digraph model needs to be generalized beyond strongly regular graphs. The realization of such models could be a decisive step towards revealing a countable physical reality.

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