TRICK OR TRUTH: THE MYSTERIOUS CONNECTION BETWEEN MATHEMATICS AND PHYSICS

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This essay takes the form of a dialogue between two participants whose names are "Lou" and "Rukhsan." Lou is a mathematician with an intense interest in physics, working on physical ideas and concepts from the side of mathematical thinking. Rukhsan is a condensed matter physicist with an intense interest in the nature of mathematics. This essay is inspired by a dream of John Wheeler [15] that the deepest reach of the physical world would be something like logic, like Boolean logic. He expressed this dream in the phrase *"It from Bit"*. Some have said that this should be modified to *"It from Qubit"*, and we shall get to that in the course of the essay. We beg the reader to be patient, as the dream is like the Cheshire cat and it is sometimes only possible to see its smile. Along with the references directly cited in the essay, the following references will be useful to a reader who wishes to follow up our themes and ideas [2, 6, 7, 8, 9, 10, 11].

Lou: Commonly, mathematics is regarded as a language that allows the analysis and classification of patterns and and the quantitative analysis of these patterns. We are all familiar with the remarkable matches between mathematical structures and physical law such as the fit with experiment of Newton's Law of Gravity which, expressed at the level of acceleration, is said with simple and elegant mathematics and encompasses a vast range of phenomena. The mysterious part of this correspondence of mathematics and Nature is that relatively simple mathematics, or simply stated mathematics, has been successful in describing models for complex aspects of the natural world.

We shall address the question "Does physics simply wear mathematics like a costume, or is mathematics a fundamental part of physical reality?".

In his book [1] , "Laws of Form", G. Spencer-Brown says:

"Now the physicist himself, who describes all this, is, in his own account, constructed of it. He is , in short, made of a conglomeration of the very particulars he describes, no more, no less, bound together by and obeying such general laws as he himself has managed to find and record. $\cdots$ Thus we cannot escape the fact that the world we know is constructed in order (and thus in such a way as to be able) to see itself. This is indeed amazing. Not so much in view of what it sees, although this may appear fantastic enough, but in respect of the fact that it *can* see *at all.* But in order to do so, evidently it must first cut itself up into at least one state which sees, and at least one other state which is seen. In this condition it will always partially elude itself."
([1], Notes to Chapter 12, p. 105)

Spencer-Brown reminds us that the Universe is constructed to be able to see itself, and we are constructed to be able to see the Universe. We begin, not with mathematics as a known formalism, or with physics as laws expressed in mathematical form, but with the condition of an observed world, a world in which it is possible to have a division of states into that which sees and that which is seen. One can begin with the idea of a distinction and naturally unfold the elements of mathematics. In set theory the simplest distinction is the empty set, denoted by

$$\{\ \}$$

The empty set is not nothing. It is the result of collecting nothing and so has nothing inside it. Sets are referential only to their contents (and not to their complements except indirectly). Two sets are equal exactly when they have the same members. For this reason there is only one empty set, and we can proceed to build more sets by the act of collection. Having created the empty set we can now create the set whose member is the empty set.

$$\{\{\ \}\}$$

Then we can create the set whose members are the empty set and the set consisting of the empty set.

$$\{\{\ \}, \{\{\ \}\}\}$$

These sets are representatives of the numbers $0, 1, 2$ since they have respectively zero, one and two members. We write:

$$0 = \{\ \}$$
$$1 = \{\{\ \}\} = \{0\}$$
$$2 = \{\{\ \}, \{\{\ \}\}\} = \{0, 1\}$$
$$3 = \{\{\ \}, \{\{\ \}\}, \{\{\ \}, \{\{\ \}\}\}\} = \{0, 1, 2\}$$

The rest of the natural numbers are created by the inductive formula

$$n + 1 = \{0, 1, 2, \cdots, n - 1, n\}.$$

That is, to create $n + 1$ make a collection of the previously created sets up to n. The *number concepts* $1, 2, 3$ are not identical with sets that have that many members. In fact, to take a case in point, 2 refers to any double. Thus I can say that I have 2 arms. If we use the substitution of $\{\{\ \}, \{\{\ \}\}\}$ for 2, then I would say

"I have $\{\{\ \}, \{\{\ \}\}\}$ arms!"

This is comprehensible. It means the same as if I had said

"I have [holds up two fingers] arms."

I make reference to an instance of 2 in the course of asserting that my arms are two. In mathematics we often take the elements of this collection hierarchy as the standard instances of the numbers and give to these sets the names of the numbers.

It is common to learn a counting system such as the arithmetic we all learned in school and to regard the formalities of this system as the "mathematics." We work with this and it is part of our understanding. Thus I can find 17^2 by mental arithmetic by thinking 17 is 10 plus 7. And 10 squared is 100, 7 squared is 49, twice 7 times 10 is 140. So 17 squared is equal to 100 plus 49 plus 140 which is 289. So 17 squared is 289. Mathematical forms can be manipulated like objects. Mathematics is the art of doing this sort of work. But that is not all. After all, we are now confident that a square array of dots 17×17 will have 289 marks in the square array. We are sure that if we *count* the marks in the array *in some order* then we will get 289. We are confident that the meaning of the numbers includes the regularity of the correspondence with counting and matching that is their meaning. The numbers are not just counters in a game that we play using the rules of arithmetic. Each number has a conceptual meaning in relation to all possible multiplicities of that order (to which it can be mapped by a one-to-one correspondence), and it is this conceptual meaning that allows us the use arithmetic in the world. In fact, without the formalities and the concepts of arithmetic we would not be able to work with multiplicities in the world. One could say that without the language of arithmetic and the understanding of its meanings, the world (perceived by us) would not have multiplicity.

A world without multiplicity would have quite a different physics than the one we know. Now you can begin to see how physics and mathematics are related. As we grow mathematics, we expand the possibilities for physics. As we make new observations, we grow new possibilities for mathematics. Counting came from physical necessity. Persons needed to count their sheep and not just to go to sleep!

But I must talk about non-numerical worlds. These days everyone is familiar with the formalities of Boolean algebra. It is the mathematics of computers, of patterns of logic, patterns of sets, the properties of switching circuits and transistors. It underlies all of our practical mathematics. In the Boolean arithmetic that everyone knows there are two values 0 and 1 and an operator that switches them: $\bar{0} = 1$ and $\bar{1} = 0$. There are two ways that the elemental opposites 0 and 1 interact, called *plus* $(+)$ and *times* $(\times)$ and we have that 0 times anything is 0 while 1 times anything is *that thing*. So multiplication behaves just like the old numerical values of zero and one. Addition is almost the same. 0 plus anything is that thing. But *one plus one is one!*

$$1 + 1 = 1$$

Now you see how the world of Boolean arithmetic has no multiplicity beyond one. It is the world of one only. Try to count beyond one, and you are returned to 1, returned to unity! Of course we understand this from the examples. Two closed switches in parallel are no different that one closed switch. Either closed switch will carry the current. Two calls of your name are the same as one call of your name. The other is redundant. But go further! Live in the Boolean world! You have logic, reason, either, or, and neither. But you have no time, and you have no ability to count beyond 1. Lewis Carroll in his classics "Alice in Wonderland" and "Through the Looking Glass" played with his heroine Alice, projecting her into just such worlds. Worlds where you have logic, you have the Boolean, but you do not have the customary multiplicities and regularities. Mathematicians are people who live half their lives in the eternal timeless Boolean world. For them it is the Reality, the holocosm from which emerges (just as we have see above with the multiplicity of numbers coming from the distinctions of set creation) at once and inevitably worlds of unimaginable multiplicity. Because mathematicians are exploring the holocosm beyond mulitiplicity, when they do create multiplicity that multiplicity is not constrained by any prejudice from the physical world. Thus you hear mathematicians working with higher orders of infinity, and even hoping to approach the meaning of absolute infinity. All this is possible by working from the holocosm where there are no numbers, no multiplicities, no infinities and no finity other than zero and one, nothing and something. This is the freedom of mathematical thought and to the mathematician, the question of the possible relation of mathematics and "physics" is utterly inevitable. By leaving the holocosm for worlds of possible multiplicity, the mathematician understands that he may meet examples of these phenomena. If they can be imagined they may be encountered, and if so we will point to an amazing correspondence between mathematics and physics.

Rukhsan: You are trying to say too much! Let me slow you down and ask some questions. You say the mathematician has no use for time, but what about recursions and calculations. They all take a lot of time and seem to need multiplicity.

Lou: Yes. I said that the mathematician spends half his life in the timeless holocosm of the non-numerical Boolean world. The other half is spent "out here" in the world where we can build computers and calculate and remember and take our time to tease out the consequences. But you will notice how it is done. Each bit of mathematics that seems to have something to do with time is expressed as structure and can be comprehended in a definition that does not *require* any definite amount of time. For example. *I say that the Mandelbrot set M is the set of complex numbers z such that the iteration of the function $f(w) = w^2 + z$ starting at $w = 0$ goes to infinity.* This is a statement that is understandable in the timeless eternity of the holocosm. No calculations are necessary to grasp the definition, once it is grasped. Oho! You almost caught me there. I heard you about to object!! Yes it takes time to get straight what the definition says. You have to read and reread it. But it is all a matter of logic, and once you have it you have it in eternity! That is why mathematicians are skittish about computers. They want to take it with them back into the holocosm, and if a result depends on temporal computation and cannot be eventually instantly understood, then it is suspect. But oh my, there is a wealth of phenomena in that Mandelbrot set that we need the computer to see. It is all implicitly there, but we are as curious as the next person and so we are pulled out of the realm of pure logic and find ourselves glued to computer screens.

Rukhsan: So would you say that when you are not in the eternity of the holocosm, then you are in the physical world?

Lou: When you leave the holocosm, you enter a realm where multiplicities can be instantiated. It is not obvious that such a world would exist, but it does and we experience that world. We understand from a number of points of view in the holocosm how such worlds may be structured. For example, we have the forms of recursion in the holocosm. In the holocosm, we can write an equation that refers to itself, an equation of higher degree. For example we can write $G = F(G)$. And in the holocosm, this is just an equation (actually to write it at all, we are peeking through into a world of memory but let that be and understand the amphibian nature of the mathematician!). There is no need to resort to recursion here. But in the outer world we are can recurse and then

$$G = F(G) = F(F(G)) = F(F(F(G))) = F(F(F(F(G)))) = F(F(F(F(F(G))))) = \cdots .$$

At once the exit from the holocosm releases not only some multiplicity, but a potential infinity that cannot be fulfilled in the present known world. From the mathematician's point of view there is no reason why the phenomenal world might not allow an "actual" passage to infinity. Some cosmologists imagine that the physical world does indeed contain passages to infinity or infinitely many parallel worlds! There is no evidence for such things, but this does not bother the mathematician. From his point of view, just leaving the holocosm, anything is possible.

Rukhsan: You did not answer my question. What do you think is the composition of the physical world? Is it made out of mathematics?

Lou: I was not talking about the composition of the physical world, except indirectly. The question of the composition of the world requires us to back up and ask what every word in that question means. Look carefully at what you know and at your actions. When we say that this wall is composed of bricks, we know it because we know how to build the wall by taking entities called bricks and putting them together in a certain way. But if I say that numbers are made by forming certain sets, this is obviously just a way to get to the concept of a number. And the concept of a number is made from thought. Thought is not composed of pieces like bricks. Mathematics is composed of concepts, particular languages that embody those concepts and many examples, some concpetual and some physical. The physical examples are still conceptual, but they involve actions in the world that we regard as physical. Archimedes weighed an irregular object by noting how much water it would displace. Here mathematics and actions in the world combine to give us new informatiion. But this does not indicate that the world is composed of mathematics! In fact after all this saying just now, I have to admit that I do not know what it could possibly mean to say that a 'physical object' is 'composed of' mathematics!

Rukhsan: I still feel that you are avoiding the question. As a physicist, I work with the notion that substances are composed of atoms, atoms are composed of electrons and protons, protons are composed of quarks and so on. Furthermore, as we delve more and more deeply into elementary particles we find that they seem to be characterized entirely by their mathematical properties. It would seem that in the end, all that is left of matter is mathematics!

Lou: One could look for pure form in a physical object. I imagine Michelangelo felt that his David embodied the pure form of the rock of which it was composed and was not just an imposition of the David form on this particular block of stone. Physicists hypothesize that all electrons are the same and that the mathematical form of the electron *is* the electron. But by the time you get to speaking of electrons you are actually talking about certain modes of observation that sometimes exhibit particulate properties (e.g. dots on a phosphor screen) and sometimes other less local properties. A host of ideas and mathematical ways of geometrizing are combined to make the concept of the electron useful and matching with the actions and observations of experimentalists. Simple localized objects have disappeared. It is as if Michelangelo had carved the David so well out of rock that all the rock vanished and there was still something left to appreciate. I can give you an example that is closer to home. Consider a closed loop of rope that is knotted. The knot is not the rope but the rope forms a knot in space. The very same knot can be woven in many different types of rope. A mathematician studying the topology of knots (as opposed to friction and other matters) does not care about the substrate of the knot. He cares only about the form of the knot. And the form can be represented by any or all ropes that are sufficiently long and sufficiently flexible. But you have never seen a knot that was not tied in something, even if that something was an imaginary curve of the sort that mathematicians like sketch on a blackboard. But for the mathematician that knot exists in the eternal holocosm of non-numerical forms. There is a desire to make this holocosm the basis of the physical world. I cannot assent to that unless we explore how ideal entities like numbers and knots are related to our experiences.

Rukhsan: So you would identify the physical world with our experiences. Is not mathematics and concept the basis of our experience?

Lou: We have experiences and we have thoughts and concepts. They happen together, but thought is singular in that thought can examine itself. It is this that lets the mathematicians proceed from the holocosm, but he still needs experiences. Each experience is accompanied by some concept. I look at the light in my room and 'see' a lamp. The concept lamp fits that light and fits a web of explanation about it going through all I know about electricity and too much to say. We meet experiences that do not meet their best concepts until later. It took a long time to understand the laws for falling bodies or the unity of electricity and magnetism. The concepts that unify bring us closer to the core of our experience. I take experience to be a mystery that we explore and it is through concept and particularly mathematical concepts that we can come to terms with that experience. In this sense, mathematical concepts are the basis of our experience.

Rukhsan: Certainly there are experiences that are beyond mathematics. Shakespeare says "Juliet is the Sun.". Surely we understand this metaphor, but we do not pretend that it is mathematics. Oh. Well. We also do not pretend that it is physics. Maybe physics is exactly that part of our experience that can be understood with mathematics. If that is so, then our discussion would seem to be over.

Lou: Even if physics were that part of our experience that can be understood with mathematics, we shall have to look at that word "understood." Some who like the calculational side of mathematics best, have said that a few pages of code should suffice to simulate the universe. Suppose we had this simulation in hand. Would we be satisfied? No! To use only the simulation would require 'running the Universe' and seeing how 'it' behaved. We would have lost the unification that conceptual work gives. We would not have understood. The understanding that we seek is in the interface of the conceptual, non-numerical holocosm and the realms of experience, calculation and simulation. This is a continuing mystery.

Rukhsan: You started by characterizing the holocosm as the place best described by logic alone, or thought alone or perhaps the Boolean arithmetic. I recall that long ago John Wheeler suggested that the universe could emerge from just that. He said “It from bit.” Since then we have seen much work on informational bases for quantum theory and it may be that the Wheeler motto has evolved “It from Qubit.”. Comment?

Lou: Look at the Boolean arithmetic. It has two values, zero and one. And it has an operator that switches these values. In logic the operator is negation, $\sim$, and the values are true (T) and false (F). Ludwig Wittgenstein [16] had this to say about negation:

“*How can the all-embracing logic which mirrors the world use such special catches and manipulations? Only because all these are connected into an infinitely fine network, to the great mirror.*” ([16], Tractatus 5.511) ‘‘ *... the sign $\sim$ corresponds to nothing in reality.*”. ([16], Tractatus 4.0621).

I prefer that the mirror be seen as the amphibian boundary between the logical timeless world and the world of experience and action.

Rukhsan: How will you modify the already very simple Boolean arithmetic. It seems you want to remove the separation between the operator of negation and the two states of 0 and 1.

Lou: Yes. I want to do that. And I have to give up something, but it is something that I really do not mind giving up! There are two symbols for the two Boolean states. Lets give up one of them! We can call that one the *unmarked state*. If we want to name it, we will use the letter U, but remember that state is actually not indicated by anything. The other state will be called the marked state and denoted by M. These states can be 0 and 1 or F and T, or whatever distinct states you have in mind.

Rukhsan: And this will make a difference in the boundary structure? You will include the great mirror in the world itself and also in the holocosm?

Lou: Yes. Here I will explain the discovery of the Calculus of Indications of G. Spencer-Brown [1]. I will take the following notation for negation.

$$\sim a = \overline{\;a\;}\!\rceil .$$

Then we have

$$\overline{\;U\;}\!\rceil = M$$

and

$$\overline{\;M\;}\!\rceil = U.$$

But now you have to take me seriously and allow the unmarked state to be unmarked! Then we have

$$\overline{}\big| = M$$

and

$$\overline{M}\big| = \ .$$

Something incredible and wonderful has happened here! We see that the marked state can be identified with the operator symbol itself! And so we can further write

$$\overline{\overline{}\big|}\Big| = \ .$$

What is on the right is unmarked and we have a fundamental equation for a deeper version of Boolean arithmetic where the operator symbol is seen as a distinction itself, as the marked value in the arithmetic *and* as the operator that switches between the arithmetic values. There is one more fundamental equation. It is

$$\overline{}\big|\,\overline{}\big| = \overline{}\big| \, .$$

This is just the statement that "Marked or Marked is Marked," or "A rose is a rose is a rose." It is most economical to also give up the sign for "or" and use "AB" for "A or B."

Rukhsan: Now the negation operator is also the marked value in the arithmetic. The two ways of seeing this are right there when you examine the symbol.

$$\overline{}\big|$$

This is a mark itself making a distinction in the plane. This is an operator, operating on the unmarked state to produce the marked state. This is both of those notions at once. The negation sign is not separate from the world. The Great Mirror is more like a Great Permeable Membrane or cell boundary, with logic,mathematics and the timeless holocosm on one side and the world of calculation and experience on the other. From here we can begin to explore anew how physics and mathematics are entwined.

Lou: Yes. And I will give you a hint about how we can proceed. The mark $\overline{}\big|$ can be seen as a "logical particle" whose counterpart in the mathematical physical world is a Majorana Particle [3, 13, 14]. A Majorana particle is the simplest of possible particles. It is its own anti-particle and it interacts with itself either by annihilating itself as in

$$\overline{\overline{}\big|}\Big| \longrightarrow$$

or interacting with itself to produce itself as in

$$\overline{}\big|\,\overline{}\big| \longrightarrow \overline{}\big| \, .$$

Rukhsan: I see that you have used a process arrow in this description of the interactive properties of our logical particle. The arrow has shifted us from the holocosm of timeless form to the world of process. But are there any real Majorana Fermions? It seems that this may be so. The mathematics of electrons suggests that each electron is a linked pair of Majorana particles, and recent experiments [5] appear to confirm this indirectly. Also, vortices, collectivities of electrons confined to flat surfaces as in the quantum Hall effect [4, 12] can be understood by viewing the collectivities as quasi-particles that are

Majorana particles. These quasi-particles are just like our logical particle *and* they have non-trivial phase relationships in the plane when then move around one another. This means that there is significant braiding of the world lines of these quasi-particles spinning around one another, and it is hoped that these systems will produce the first topological quantum computers. Up to density all finite dimensional unitary transformations can be produced from the phase relationships of these vortices. Topological phases of matter are described by mathematics that is just this side of the holocosm!

Lou: It is very simple. The mark $\overline{\quad}|$ is a logical Majorana particle. It is the *"bit"* from which the pre-geometry of topological matter will emerge. But in the emergence that *"bit"* becomes a *"qubit"*, a quantum Majorana particle that can be in superposition with the unmarked state and undergo quantum interaction. Phases and braiding fill in the picture and start to give us all the background for quantum physics and topological quantum computing. Perhaps we are witnessing the very thing we have been discussing - new physics and new mathematics arising through the permeable boundary between the conceptual worlds and the worlds of experience. Wittgenstein's Great Mirror has been replaced by that permeable boundary and we are at the beginnnings of a great adventure.

Rukhsan: As we move away from the purity of the marked and unmarked states, complexities occur. The braiding of Majorana particles is a first step into pre-geometry (topology). We want our models to match the observed complexities of condensed matter physics. We hope to be able to use this knowledge to extend calculation into the quantum domain to create topological quantum computation. Neither all the mathematics, nor all the concepts have yet come forth. It is work in progress that will change our understanding of the world, our understanding of mathematics and our understanding of ourselves.

Lou: This is an adventure at the very foundations of physics, mathematics and the roots of thought. The physicist is inseparable from the Universe herself. It is the Universe that studies herself through the articulations of mathematics and the observation of experience. The key lives in that moving permeable boundary of the holocosm, allowing the world to come to awareness of itself.

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