Math Matters

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Abstract: Gödel taught us that mathematics is incomplete. Turing taught us some problems are undecidable. Lorenz taught us that, try as we might, some things will remain unpredictable. Are such theorems relevant for the real world or are they merely academic curiosities? In this essay, I first explain why one can rightfully be skeptical of the scientific relevance of mathematically proved impossibilities, but that, upon closer inspection, they are both interesting and important.

1 Physics Isn't Math

Mathematics has worked incredibly well for physicists; the proof is in front of your eyes. Whether you read this essay on a screen or laser-printed on paper, it was brought to you by physicists who dug deep into the math of quantum mechanics on which modern technology relies. You may not know the math, you may not understand it, or you may not like it, but there's no doubt that it works.

And yet, physics isn't math. Physics is science and as such has the purpose of describing observations of natural phenomena. Yes, we use mathematics in physics, and plenty of that, as I'm sure you have noticed. But we do this not because we know the world is truly mathematics. It may be mathematics, but Platonism is a philosophical position, not a scientific one. You are free to believe that reality is math, but since this belief is scientifically indefensible I will not defend it. We also don't need it: Physicists use mathematics simply because it is useful. Reality may not *be* math, but it surely can be well described by math.

So, math works, alright. But its use in science has limits. In physics, theories are sets of axioms together with instructions for how to map mathematical structures to observables. We can experimentally test whether a theory correctly describes observations, but no matter how many measurements we make, no matter how precisely we measure, and no matter how smart we are, we will never find out whether the mathematical assumptions themselves are correct. A theory may have given us the most extraordinarily accurate predictions until today, and still tomorrow we could discover that it doesn't explain the next measurement.

This is why, in physics, any mathematical theorem must be taken with a grain of salt. Any proof is only as good as its assumptions, and since the assumptions of physical theories themselves are unprovable, any proof of impossibility is only as good as our faith in the assumptions. Science shouldn't rest on faith. For this reason, the topic of this essay contest – undecidability, uncomputability, and unpredictability – sounds very academic indeed. Who cares whether a big-brained scientist proved that a certain mathematical problem is unsolvable if we can never know whether this math is fundamentally the correct description of nature?

Moreover, the three un's that encapsulate this contest's theme seem utterly divorced from practical consideration. If something doesn't compute in our lives, the reason is usually not that the problem is uncomputable; the reason is that we don't know how to compute it. The same goes for unpredictability: The major difficulty we face in making predictions is that we either don't have sufficient data or don't have the math for handling the data, not that there's a mathematical theorem preventing us from making predictions. And for what undecidability is concerned, well, in real life, not making a decision is also a decision. If it's not a deadline that sets an end to your hesitation, then the heat death of the universe certainly will.

2 Living In A Material World

The scientific relevance of impossibility-theorems becomes even more questionable if we look at the underlying mathematics. Let us start with Gödel's famous theorem [4].

Gödel's theorem says that any sufficiently complex set of mathematical axioms allows formulating statements whose truth-value cannot be decided within the set of axioms. However, it is always possible to extend the original set of axioms with an additional axiom that simply states whether the previously undecidable statement is true or false. For physical theories this means that if we had an undecidable statement, we could simply experimentally determine whether it is true and false, and then add a corresponding axiom to the theory. And if the undecidable statement has no experimentally testable consequences, we may as well ignore it. Physics isn't math, and Gödel's theorem is irrelevant for scientific practice.

This conclusion carries over to theorems based on or related to Gödel's, like Chaitin's incompleteness theorem [2]. This theorem says, in a nutshell, that within a given set of axioms you can't decide whether a string of numbers can be expressed in any simpler way, for example by writing down an algorithm that produces the string. But so what? Looking for patterns that allow us to express data in simpler ways is pretty much all that scientists have ever done. They have never been deterred by not knowing whether what they aspire to is even possible, and hopefully they never will.

Let us then move on to Turing and various related but slightly different notions of uncomputability. Turing's best-known elaboration on mathematical impossibility is the Halting Problem [3]. An algorithm will either finish running at finite time or continue calculating forever. The problem is, Turing showed, that there is no meta-algorithm that can decide for any given algorithm whether the algorithm will or won't halt.

However, the major difficulty in solving the Halting Problem is that the meta-algorithm must work for all possible input, which is an infinitely large class of algorithms. Nothing real is infinite [4], therefore the whole formulation of the problem is scientifically meaningless. In practice, we never need an algorithm that can correctly answer infinitely many questions.

This lack of scientific relevance also affects attempts to map the Halting Problem to physically existing systems, for example in the proposals put forward in [5] and [6]. In both cases the identification between math and reality works only if the physical system is infinite, or has an infinite number of (addressable) degrees of freedom. This is just never the case in reality.

Again, we conclude that impossibility-theorems are mathematical curiosities without scientific relevance.

Similar considerations apply to statements about uncomputable numbers or uncomputable properties of mathematical structures. Let us take for example Wang's Domino Problem [7]. The Domino Problem, in a nutshell, is the question whether a given set of tiles will cover an infinite plane without gaps. The answer, it turns out, is undecidable (given certain assumptions). Again, however, what causes the problem is an infinity – here, the infinite size of the plane – and again this is not a situation we would ever encounter in reality.

We further known that almost all real numbers are uncomputable, in the sense that there is no algorithm that will approximate the number to arbitrary but finite precision in finite time. But in science we don't deal with real numbers, ever. We deal with rational numbers that have quantifiable error-estimates [8].

So much about undecidability and uncomputability. What about unpredictability? Measurement outcomes in quantum mechanics are unpredictable, but rather uninterestingly so, since quantum mechanics is unpredictable by assumption, not by theorem. The best example for surprisingly unpredictable systems comes from chaos theory, epitomized by Lorenz's "butterfly effect" [9].

Let me be clear that I am here referring to what was dubbed the "real butterfly effect" by Palmer *et al* [10]. The common butterfly effect has it that the time-evolution of a chaotic system is exquisitely sensitive to the initial conditions; smallest errors (a butterfly flap in China) can make a large difference later (a tornado in Texas). The real butterfly effect, in contrast, means that even arbitrarily precise initial data only allow predictions for a finite amount of time.

The example that Lorenz gave in [9] is that improving the precision of initial data by a factor 2 will only increase the original time-window for accurate predictions, T, by adding $1/2\, T$. If we sum up the prediction-times all the way to infinitely accurate initial data we get $(1+1/2+1/4+1/8+\ldots)\, T$, which converges to a finite value, $2T$. The behavior of a non-linear system, therefore, may be deterministic and yet be unpredictable beyond a limited time. For illustration, see Figure 1.

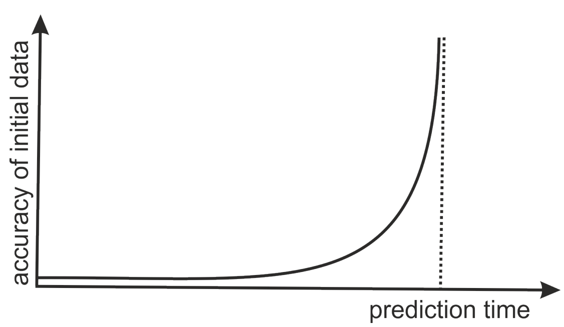

Figure 1: Sketch of Lorenz's real butterfly effect [9]. Even with arbitrarily well-known initial data, predictions are possible only for a limited amount of time.

However, while we know some differential equations that have such behavior [11] it is presently unclear whether these apply to any physically real system. The collapse of a black hole, described by Einstein's Field Equations, gives rise to a geodesically incomplete space-time (see Figure 2). In this case, trajectories of infalling particles terminate at the singularity in finite proper time. However, Einstein's Field Equations are widely believed to break down near the singularity because quantum gravitational effects should become important. And since the singularity is hidden behind the event horizon, we can't just go and measure what is happening. This means that ultimately we don't know whether or not black hole collapse (and other singular solutions of Einstein's Field Equations) cease to become predictable at finite time. And, looking at the literature on black hole collapse, I fear we may not answer this question in finite time either.

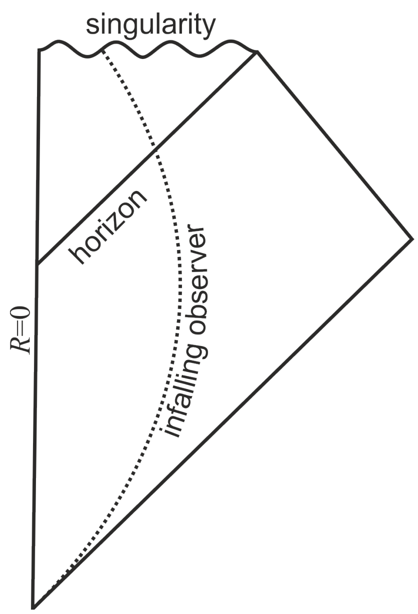

Figure 2: Penrose-Carter diagram of a black-hole collapse. An infalling observer hits the singularity at finite proper-time.

Edward Lorenz, needless to say, was not concerned with black holes, but with the Navier-Stokes equation that determines the weather. Whether the Navier-Stokes equation always permits predictable solutions or whether it suffers from the real butterfly effect is currently an open question. Indeed, it is number four on the list of the Clay Institute's Millenum Problems [12].

But wait, I can hear the particle physicists among you object, that's rubbish. We *do* know that the weather is predictable, at least in principle. Indeed, if only we could measure initial conditions precisely enough, the weather would be predictable for arbitrarily long times ahead. That's because the Navier-Stokes equation is not fundamental; it is an approximation for the flow of gases and fluids. Strictly speaking, gases and fluids should be described by the quantum field theories which underpin the Standard Model of particle physics. And these theories are linear. They don't suffer from butterfly effects, common or otherwise[1].

True, but then, maybe quantum field theory is itself not fundamental. So who really knows.

What have we learned from all this talk about the three un's? Not a terrible lot. As anticipated, mathematical theorems are not of much use for the messy science of real-world systems.

3 Want What You Do

Now that I have made a case against the scientific relevance of undecidability, uncomputability, and unpredictability, I will do my best to convince you that they are relevant after all. While what I previously said is of course correct – we will never know whether the three un's apply to nature on the most fundamental level – impossibility theorems have uses nevertheless. That's because they (a) tell us what we can do with a given theoretical framework and (b) allow us stay clear of the three un's, or to deliberately institute them, should we desire to. And this has practical consequences.

[1]Note that the phenomenon of "quantum chaos" rests on a different definition of chaos that does not require non-linearity and is therefore not in conflict with predictability.

To see this, let us return to Lorenz's problem of weather forecast. We are not going to solve the fourth Millenium Problem here, so, for the sake of the argument, let us just assume that he was right and the Navier-Stokes equation gives rise to a real butterfly effect. As I explained above, there may be an underlying theory in which predictability is restored, so we will never know whether the weather is truly unpredictable. But understanding the limits of the presently available theory allows us to draw conclusions about what we can reasonably do with it.

If Lorenz was right, we would for example conclude that it doesn't make sense to invest huge amounts of money into additional measurement stations because we could cover every inch of the planet with apparatuses and never get a weather forecast that's good for more than two weeks ahead. Whether the Navier-Stokes equation is fundamentally the right equation doesn't make this investment-advice any less sound; it only matters that it's the equation we use in practice.

Let us pursue this thought a little further. Suppose we got really good at doing the two-week weather forecast, so good that we could figure out exactly when the Navier-Stokes equation is about to run into an unpredictable situation. This could then allow us to find out which small interventions in the weather system have the effect of changing the weather to our liking[2]. By understanding unpredictability, thus, we could learn how to avoid it.

This idea is hopelessly futuristic when it comes to the weather – I dare you to track all the butterflies in China – but it is not quite as hopeless for other (partially) chaotic systems. Think for example of plasma-control in a nuclear fusion plant. The hot plasma sometimes develops instabilities that can greatly damage the containment vessel. If an instability is coming on, therefore, the fusion process must be rapidly interrupted. This is one of the main reasons why it is so difficult to run a fusion reactor energy-efficiently.

However, plasma instability is in principle avoidable if we can predict when an unpredictable situation is about to come up, and if we can control the plasma so that the situation is averted. In other words, if we understand when a solution to the equations becomes unpredictable, we can use that knowledge to change the boundary conditions and prevent it from happening in the first place.

This is not just the fantasy of a theorist; a recent study looked into exactly this. The authors of [13] trained an artificially intelligent system to recognize data-patterns that signal an impending plasma instability. They were able to do this with good success, using only data that are in the public record. A second ahead, they correctly identified an imminent instability in somewhat more than 80% of cases; 30 ms ahead, they saw allmost all instabilities coming. Granted, theirs was a hindsight analysis, with no option of active control. However, should we become good enough with such predictions, active control might be possible in the future. We here see that far from being an academic dilemma, unpredictability can be a real-world problem. Who wouldn't want to have an energy-efficient fusion plant?

A similar consideration applies for a superficially entirely different system that, however, has many parallels to plasma blow-ups and weather forcast: The stock market. Today, a whole

[2]Hopefully for more peaceful purposes than those envisioned by the US Airforce, according to which weather modification "offers the war fighter a wide range of possible options to defeat or coerce an adversary" [18].

army of financial analysts makes money by trying to predict selling and buying of stocks and financial instruments, a task that now includes predicting their competitors' predictions. But every once in a while, even they get caught by surprise. A stock market crashes, vendors panic, everyone blames everyone else and the world slumps into a phase of recession.

But imagine we could tell in advance when trouble is on the front door; we might be able to close the door. Again, note that the question here is not whether the stock market is fundamentally predictable or not. We may simply want to avoid situations in which it becomes unpredictable for us.

This conclusion – that unpredictability is something we might want to recognize to avoid it – also holds for uncomputability. Take the economic system. It is a self-organized, adaptive system with the task of optimizing the distribution of resources. But some economists have argued [14] that this optimization is partly uncomputable. That's clearly not good for it means that the economic system cannot do its job. Or rather, we as agents in the economic system cannot do our job, because trading does not have the desired result. Creating an economic system that can actually do the desired optimization in finite time has motivated the research line of "computable economics." And as with unpredictability, what makes impossibility-theorems relevant for computable economics is not proving that a problem (here: how to best distribute resources) is fundamentally uncomputable – it may or may not be – but merely that it is uncomputable with the means that we currently have.

In other situations, however, unpredictability is something that we may want to trigger rather than avoid, for the same reason that randomness can sometimes be beneficial. Machine learning algorithms that search for optimal solutions can get stuck in local minima and fail to find the globally best solution. Adding stochastic noise can prevent this from happening because the algorithm then has a chance to coincidentally discover a better solution. Counterintuitively, therefore, an element of randomness can improve the performance of mathematical code.

In a machine learning algorithm, randomness can be implemented by a (pseudo) random number generator without drawing on complicated mathematical theorems. But unpredictability might well be useful for optimization in other circumstances. Eg, small doses could aid the efficiency of the economic system. Even more interestingly, unpredictability might be an essential element of creativity (see [15] and references therein), and thus something that artificial intelligence could draw on in the future.

In summary, the scientific relevance of undecidability, uncomputability, and unpredictability does not stem from its fundamental role in natural law, but from their presence in the theories that we currently have. Recognizing when they appear allows us to control them in practice. Or, as my mother likes to put it: "You may not always be able to do what you want, but you should always want what you do."

4 A Cure For Physics Envy

Let me finally come to an aspect of the three un's that is close to my own research interest, namely the role of unpredictability for reductionism. Countless experiments have shown that large things are made of smaller things and the natural laws for the large things derive from the laws for the small things. In most cases, we cannot presently execute this calculation, but the formalism – effective field theory [16] – is well understood, and there is nothing in principle preventing such a derivation. For illustration, see Figure 3.

This means that large objects – basket balls, cucumbers, you – behave according to the same laws as elementary particles. Even though no one in their right mind would use the Standard Model of particle physics to predict the growth of a cucumber, there is nothing, in principle, standing on the way of doing so. Given a sufficiently large and powerful computer, even human decisions could be calculated in finite time[3]. The consequence is that, according to our best current knowledge, the future is already determined, up to the occasional random interference from quantum events[4]. And really all of science is just a sloppy version of physics.

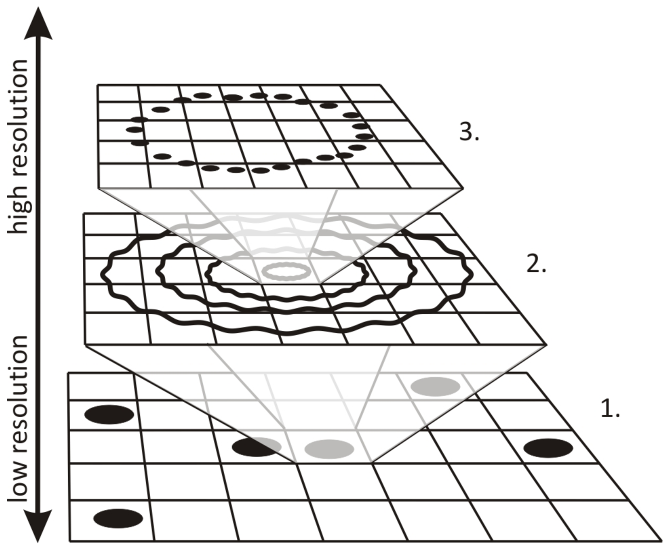

Figure 3: Sketch of reductionism. The theory at low resolution/large scales follows from the theory at high resolution/short scales.

This conclusion, however, entirely changes if the formalism of effective field theory can break down for some reason. I argued in an earlier essay [17] that this might well be the case if the equations which relate the theory on small scales with the theory on long scales run into a singular point. A singular point, I must emphasize, does not necessarily mean that something observable becomes infinitely large, which would be unphysical (as, eg, a Landau pole is). It merely means that the integration of the equation cannot be continued beyond a finite interval.

Indeed, I understood belatedly, the situation I was thinking of in my earlier essay is analogous to the real butterfly effect that Lorenz wrote about in 1969. The difference is that I, the particle physicist, was thinking of equations that are integrated over energy in theory space, whereas Lorenz, the meteorologist, was thinking of equations that are integrated over time in phase-space.

Like for the Navier-Stokes equations, for effective field theory too it is presently unclear whether such a breakdown actually occurs for any real-world system. But if it did occur, it

[3] Whether this calculation could be completed before the human actually makes the decision is unclear, so it may not be a prediction in the literal sense.

[4] What you think this implies for the existence of free will depends on your definition of free will. Personally, I would argue this means free will does not exist in any sensible definition of the term, though, of course, no one is forced to use a sensible definition. But regardless of what you think free will means, the statement that the future is determined up to quantum fluctuations remains correct.

would mean that natural laws for large objects cannot be derived from the laws on small scales. Reductionism would be demonstrably false, and other scientific disciplines would count as equally fundamental as physics.

As I explained above, if this turned out to be the case, it would be a statement only about the theories that we currently have. We still wouldn't know whether one day we'd find a theory more fundamental than the Standard Model, in which case the pendulum could swing back from unpredictable to predictable. This limitation, however, does not make it any less interesting to know whether, according to the scientific state-of-the-art, reductionism is or isn't correct. It might not be of practical use to know whether the future is determined, but then practical use is not the only thing we care about.

5 Physics Isn't Math, But

Physics is not math. But since the theories we use in physics (and other disciplines like economics) are mathematical, the limits posed by impossibility-theorems are of practical relevance. Understanding when undecidable, uncomputable, and unpredictable situations occur is key to avoiding or harnessing them. Equations aren't everything, but as long as we rely on them to understand nature, math matters.

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