Mathematical investigation reveals a new understanding of quantum theory

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Some mathematical theories in physics have claimed explanatory superiority even in the absence of new predictions. This was justified by the clarity of their postulates. In particular, mathematical derivations drive home the importance of the composition rule and the continuity assumption as two pillars of quantum theory. Our approach sits on these pillars but it combines new mathematics with a testable prediction. If the observer is defined by a limit on string complexity, information dynamics leads to an emergent continuous model in the critical regime. Restricting it to a family of binary codes describing 'bipartite systems,' we find strong evidence of an upper bound on bipartite correlations equal to 2.82537. This is measurably different from the Tsirelson bound. The Hilbert space formalism emerges from this mathematical investigation as but an effective description of a fundamental discrete theory in the critical regime.

I. MATHEMATICS GUIDES UNDERSTANDING

Max Born happened to know matrix multiplication. He told Werner Heisenberg about a mathematical theory that contained precisely Heisenberg's own rules for calculating atomic spectra [1]. Matrix mechanics was born.

A little later John von Neumann helped to replace matrix and wave mechanics by a theory of operators on the Hilbert space [2]. Operator theory underwrote both von Neumann's and Dirac's canonical books [3, 4]. Its success had two reasons: Heisenberg's empirically motivated rules fit the Hilbert space formalism very well, and the new mathematical framework produced novel experimental predictions. With these predictions confirmed, the position of the Hilbert space at the foundation of quantum theory became unshakeable.

The pragmatism and simplicity of Heisenberg's rules were not sufficient to install them as the ultimate ingredient of quantum theory. But was the Hilbert space fundamental? Von Neumann did not think so. In 1935, he wrote to Garrett Birkhoff: "I would like to make a confession which may seem immoral. I do not believe absolutely in Hilbert space any more" [5, p. 59]. Obviously von Neumann did not mean that the Hilbert space formalism of quantum theory was wrong; he merely believed that vector spaces were not essential. Together with Birkhoff, he initiated a program of quantum logic, which sought to replace the Hilbert space axioms by (what was later called) an orthomodular lattice.

Von Neumann's work was mathematical. Many physicists at the time thought that it did not improve physical theory. This applied, too, to quantum logic: lattices are more difficult to manipulate than operators. Yet von Neumann's program did make a lasting contribution to physics. His and other mathematical reconstructions each provided a set of fundamental axioms. Although none made new predictions, the hope was that a convincing axiomatic approach would lead to a better understanding of the existing theory. When this happened many years later, the reconstruction program could vindicate a vision of what was truly important in quantum theory [6].

A similar case appears in the history of relativity. Einstein's general theory of relativity is a textbook example of a successful application of new mathematics to physics. The Riemannian geometry, to use Einstein's own famous phrase, "told us what we can observe," and these predictions were experimentally confirmed by Eddington. His theory was based on the metric tensor. In opposition to Einstein, Hermann Weyl pursued an approach based on what we now call the Weyl connection [7]. Weyl's was still a version of Riemannian geometry but it had a different pivot concept. This approach yielded no new physical predictions but Weyl kept arguing for its explanatory superiority. While he has gradually surrendered all claims in favor of his theory as a practical replacement of Einstein's, Weyl still "marshaled a number of aesthetic and 'philosophical' arguments that he believed recommended his theory over that of Einstein" [8, p. 160].

One can draw two lessons from this historic episode. First, mathematical improvements do not always supply a more practical method of doing calculations. Their chief goal is to achieve explanatory superiority. If Weyl had to surrender all claims to the pragmatic advantage of his approach over Einstein's, perhaps the very expectation was vain. Both von Neumann's quantum logic and Weyl's theory lost to earlier formalisms on the pragmatic side, while still claiming superiority in explaining physics.

Second, one should ask why, despite Weyl's perseverance, his approach never really took off. Its aesthetic merits and conceptual integrity are beyond doubt. Was it Einstein's reputation that secured his theory's victory over Weyl's? Maybe, but Weyl's formalism was also too close to Einstein's. It used the same fundamental mathematics of Riemannian geometry, even if the metric tensor was replaced with a connection. Most physicists saw in Weyl's idea only a minor improvement. It was not impressive enough. Whenever mathematics purports a new explanation of physical theory, it better use a framework profoundly different from all previous ones. This new mathematics must create a real surprise among the physi-

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cists.

This was what von Neumann intended when he rejected the Hilbert space in favor of a discrete structure. The mathematical formalism of orthomodular lattices was an innovation fuelling a hope for new insights. The insights have indeed arrived but not where von Neumann expected them. It took about half a century to realize that the main finding of quantum logic did not belong to the routine business of deriving the structure of the Hilbert space from an orthomodular lattice with additional assumptions. In this sense the quantum logical programs of Mackey [9] or Piron [10] pursued a vain goal. One had to wait till the 1990s to fully appreciate that the composition of systems is crucial in quantum theory.

In quantum logic, a straightforward product of two orthomodular lattices is not orthomodular. Various fixes are needed; had Weyl lived to study quantum logic, he would be appalled at their inelegance. Because these tricks look so artificial, they seem to point at something deep: even if quantum logic can be used to reconstruct the Hilbert space of an individual system, it fails to do so in a natural way for a pair of subsystems. Hence the importance of composition. Quantum theory emerged from quantum logic, not only as a description of individual systems, but also with a clear need of an axiom about how such systems compose.

The first fundamental insight from the mathematical reconstructions of quantum theory bears on the role of composition. It must be complemented with a quantitative bound on the amount of correlations, given by the Bell inequalities [11] and explored in postquantum models [12–15]. This is a fundamental fact about nature: the amount of correlations between distant subsystems is limited by a non-classical bound, e.g., the Tsirelson bound in quantum theory [16]. It is as important as the values of fundamental constants or the mass of the Higgs boson. All mathematical alternatives to the Hilbert space must fit the experimental value of this bound.

The mathematical reconstructions provide a second insight relative to individual systems. If one looks at the many sets of postulates devised as alternatives to the Hilbert space axioms, all of them agree on one point: a continuity assumption is mandatory [17–20]. It may come in different disguises, but it is present in every model. A prominent representative is the existence of a continuous reversible transformation between any two pure states of the system [21, 22].

Thus the mathematical investigation initiated by von Neumann yields two major findings about quantum theory: the role of continuity and the amount of correlations.

II. NIELS BOHR'S VIEW

Another avenue leading to the question about the mathematics of quantum theory begins with the quantum observer. Quantum theory says nothing about its physical composition. It only describes the observer's information, which must be somehow registered. Hugh Everett argued that observers are characterized by their memory, i.e., "parts... whose states are in correspondence with past experience" [23]. It seems reasonable to assume that differently constituted observers with the same memory size will have equal capacity to register measurement results. This is because quantum mechanics uses abstract mathematics: it deals with *observers* possessing *information* about *systems*, not with storage in the imperfect human brain or computer tape. Can one use such abstract mathematics to further describe observers?

A somewhat puzzling answer was attempted by Niels Bohr. Over time Bohr "became more and more convinced of the need of a symbolization if one wants to express the latest results of physics" [24]. He had already written extensively on the problem of objective description, but he only connected it in 1958 with the choice of mathematical formalism for quantum theory: "The use of mathematical symbols secures the unambiguity of definition required for objective description" [25]. What exactly Bohr meant remains unclear. It is unlikely that his point was that quantum theory must rely on the Hilbert space, by then a standard tool. Had it been so, Bohr could have named the Hilbert space explicitly, yet he only vaguely referred to "mathematical symbols". In the same text he rejected Schrödinger's wave function as a candidate mathematical tool. It is conceivable that Bohr's view was that such mathematical symbols remained to be found. If so, their discovery would purportedly guarantee the unambiguity of communication and secure the objectivity of description. It would then account to a "common-language" basis of physical theory, in line with Bohr's well-known insistence on the role of classical concepts [26].

Any attempt to clarify the meaning of Bohr's statement begs two questions. First, is there a mathematical framework that includes both ambiguous and unambiguous descriptions? In Section III we introduce such a framework based on the algebraic coding theory. This theory provides a general model of communication and deals mathematically with errors or ambiguity. Second, how is quantum theory different from all other unambiguous descriptions? As emphasized in Section I, it must obey a condition of continuity and a bound on bipartite correlations.

The use of coding theory is enabled by the definition of observer in information-theoretic terms. Introduced in Section IV, it involves a limit on the complexity of strings, which (to use a common-language expression) the observer can 'store and handle'. Strings contain the descriptions of states allowed by quantum theory but also much more: they may not refer to systems or bear any semantics in terms of preparations or measurements. Using the work of Manin and Marcolli, we show that symbolic dynamics on such strings leads to an emergent continuous model in the critical regime (Section V). Restricting this model to a subfamily of 'quantum' binary codes describing 'bipartite systems' (Section VI), we find strong evidence of an upper bound on bipartite correlations equal to 2.82537. The difference between this number and the Tsirelson bound $2\sqrt{2}$ can be tested. The Hilbert space formalism emerges from this mathematical approach as an effective description of a fundamental discrete theory of quantum languages in the critical regime, somewhat similarly to the description of phase transitions by the Landau theory.

III. CODES

Communication is based on encoding messages that are transmitted in suitable codes using an alphabet shared between communicating parties. An alphabet is a finite set A of cardinality $q \geq 2$. A code is a subset $C \subset A^n$ consisting of some of the words of length $n \geq 1$. A language is an ensemble of codes of different lengths using the same alphabet. As an example, take the alphabet $A = F_q$, the finite field of q elements. Linear subspaces of F_q^n gives rise to codes called linear. Linearity provides such codes with extra structure. Another example is given by binary codes of length n based on a two-letter alphabet, say, $\{0, 1\}$. Strings of zeros and ones of arbitrary length belong to a language formed by binary codes with different values of n.

In full generality, nothing can be stipulated about the semantics of the message, the material support of the encoding and decoding operations, or their practical efficiency. One can observe, however, that decoding a message is less prone to error if the number of words in the code is small. On the other hand, reducing the number of code words requires the words to be longer. The number

$$R = \frac{\log_q \#C}{n} \tag{1}$$

is called the (transmission) rate of code C.

One can associate a fractal to any code in the following way [27, 28]. Define a rarified interval $(0,1)_q = [0,1] \setminus \{m/q^n | m, n \in \mathbb{Z}\}$. Points $x = (x_1, \ldots, x_n) \in (0,1)_q^n$ can be identified with $(\infty \times n)$ matrices whose k-th column is the q-ary decomposition of x_k. The Sierpinski fractal set $S_C \subset (0,1)_q^n$ of Hausdorff dimension R has all rows of these matrices in C. The closure of S_C inside the cube $[0,1]^n$ includes the rational points with q-ary digits. This new fractal $\hat{S}_C$ is a metric space in the induced topology from $[0,1]^n$. Now consider a family of codes C_r of $k_r = \log_q \#C_r$ words of length n_r, with rate R:

$$\frac{k_r}{n_r} \nearrow R. \tag{2}$$

They define a fractal $S_R = \bigcup_r S_{C_r}$ of Hausdorff dimension $\dim_H(S_R) = R$.

IV. BOUNDED COMPLEXITY

Any observer's memory is limited in size. While their material constitution may be radically different, different observers with the same memory size should demonstrate similar performance in handling information. This intuition serves as a motivation for the following information-theoretic definition of observer.

Definition IV.1. An observer is a subset of strings of bounded complexity, i.e., strings compressible below a certain threshold.

This limit can be viewed as the length of observer's memory. If a string has high complexity, it cannot describe an observer with a memory smaller that the minimal length required to store it; but it remains admissible for an observer with a larger memory.

Definition IV.1 requires a notion of string complexity independent of the observer's material organization. Kolmogorov complexity is a suitable candidate. It has already been used in fundamental physics, e.g. by Zurek who argued that physical entropy should be defined as a sum of Shannon entropy H and algorithmic randomness of available information [29–31]. The latter was to be understood as information contained in a 'binary image' of the state of the system, defined as Kolmogorov complexity K of the shortest program able to generate it. When the state of the system is known sufficiently well, K supplies the main contribution to entropy. In particular, it is hard to extract work from a state with large K, which stands as a motivation for counting K in physical entropy. Zurek argued that this algorithmic component of physical entropy can be made observer-independent by discretizing the system on a family of grids that are concisely describable by a universal computer. We reinterpret his 'binary image' as a string of symbols in a given alphabet used by an observer who 'reads' it as information about the state of the system. Such strings are information-theoretic primitives; taken together, they define the observer. Unlike in Zurek's work, the strings do not necessarily describe states: they may not 'refer' to any system. They may not have any semantics at all. For a set of strings that are code words of code C with rate R, the lower Kolmogorov complexity satisfies [28]:

$$\sup_{x \in \hat{S}_C} \kappa(x) = R. \tag{3}$$

For all words $x \in S_R$ in a language formed by codes C_r, the lower Kolmogorov complexity is bounded by $\kappa(x) \leq R$. Hence the closure $\hat{S}_R$ of the fractal S_R is a metric space that describes the handling of words of bounded Kolmogorov complexity. It is a 'minimal' geometric structure corresponding, in Zurek's terms, to the space of 'states' assigned by the observer to a 'system'.

V. CRITICAL LANGUAGE DYNAMICS

A change in observer's information can be represented via dynamical evolution on the fractal set S_R. As proposed in [28], this involves a change in the 'occupation numbers' λ_a of words $a_i \in \bigcup_r C_r$ under Hamiltonian dynamics in the Fock space:

$$H\epsilon_{a_1\ldots a_m} = (\lambda_{a_1} + \ldots + \lambda_{a_m})\epsilon_{a_1\ldots a_m}, \qquad (4)$$

with the Keane 'ergodicity' condition:

$$\sum_{a\in\cup_r C_r} e^{-R\lambda_a} = 1. \qquad (5)$$

The Keane condition gives a meaning to the weights λ_a as normalized logarithms of inverse probabilities that a is stored in the observer's memory. This evolution has a partition function:

$$Z(\beta) = \frac{1}{1 - \sum\limits_{a\in\cup_r C_r} e^{-\beta\lambda_a}}. \qquad (6)$$

Manin and Marcolli have shown that at the critical temperature (equivalently, string complexity) $\beta = R$, the behaviour of this system is given by a KMS state on an algebra respecting unitarity [28]. Consider characteristic functions $\chi_{\hat{S}_C(w)}$, where $w = w_1 \ldots w_m$ runs over finite words composed of $w_i \in C$ and $\hat{S}_C(w)$ denotes the subset of infinite words $x \in \hat{S}_C$ that begin with w. These functions can be identified with the range projections

$$P_w = T_w T_w^* = T_{w_1}\ldots T_{w_m} T_{w_m}^* \ldots T_{w_1}^*. \qquad (7)$$

At the low temperature $\beta > R$ there exists a unique type I_∞ KMS-state ϕ_R on the statistical system of codes, which is a Toeplitz-Cuntz algebra with time evolution:

$$\sigma_t(T_w) = q^{itn} T_w. \qquad (8)$$

The partition function is:

$$Z_C(\beta) = (1 - q^{(R-\beta)n})^{-1}. \qquad (9)$$

However the isometries in the algebra do not add up to unity. Only at the critical temperature $\beta = R$, where a phase transition occurs for all codes C_r, is there a unique KMS state on the Cuntz algebra, i.e., an algebra such that isometries add up to unity: $\sum_a T_a T_a^* = 1$.

Critical behavior of the original discrete linguistic model is described at $\beta = R$ by a field theory on the metric space $\hat{S}_R$, which obeys unitarity. Recall that, by construction, this fractal also has scaling symmetry. Consequently, we have a field theory respecting scaling and unitarity. While there has been some discussion of theories that are scale invariant but not conformal, we assume that, in agreement with Polyakov's general conjecture [32], this field theory is conformal. Due to its property of continuity and to the geometric character of its state space, it is a tentative candidate for the reconstruction of quantum theory.

VI. AMOUNT OF CORRELATIONS

The conformal model likely contains more than a description of 'quantum' languages. The C^*-algebraic reconstructions of quantum theory demonstrate that an extra condition is required to select quantum theory from other continuous models [33]. In this section we pick out a class of models corresponding to the critical regime of binary codes describing measurements on bipartite quantum systems in the usual 3-dimensional Euclidean space. First we define 'bipartite' informationally. In quantum theory, subsystems that are entangled can be materially different, but they are described by the same number of entangled degrees of freedom. Their informational content is represented by strings of identical complexity. For example, measurements in a CHSH-type experiment produce binary strings of results for a choice of $\sigma_x, \sigma_y, \sigma_z$ measurements. The no-signalling condition implies that the probability of 0 on Alice's side is independent of the Bob's settings, and vice versa. Hence the strings resemble Bernoulli distributions with a Kolmogorov complexity equal to the binary entropy of the probability of 0, plus a correction due to the existence of non-zero mutual information between Alice's and Bob's outputs. Since both sides enter symmetrically in the CHSH inequality, this correction to Kolmogorov complexity is *a priori* the same on Alice's and Bob's side. We use this argument to replace Eq. (4) with a class of Hamiltonians that describe a 'bipartite system' in the framework of codes.

The Kolmogorov order is an arrangement of words $a_i \in \bigcup_r C_r$ in the increasing order of complexity [34]. It is not computable and it differs radically from any numbering of a_i based on the Hamming distance in the codes C_r. Words that are adjacent in the Kolmogorov order have the same complexity. This motivates the choice of an Ising-type Hamiltonian

$$H = -\sum_{ij} a_i \times a_j, \qquad (10)$$

as a dynamical model on the language that describes bipartite quantum systems. The sum is taken over N neighbors in the Kolmogorov order, i.e. all strings of identical complexity. The result of multiplication on binary words is a new word with letters isomorphic to multiplication results in a two-element group $\{\pm 1\}$. Hence, for a two-letter alphabet $\{a, b\}$,

$$a \times a = b \times b = b, \quad a \times b = b \times a = a. \qquad (11)$$

A binary language with $N = 6$ using the Hamiltonian (10) gives rises to information dynamics equivalent to the dynamics of a 3-dim Ising model. The class of Hamiltonians that lead to such dynamics includes all weight distributions on the strings that describe bipartite systems with $N = 6$, but its precise characteristics are uncomputable due to the properties of Kolmogorov complexity. Plainly, we do not know which binary codes give rise to the $N = 6$ situation. However, as it is usually the case in statistical mechanics, their critical regime can be studied without knowing the details. The equivalence with a 3-dim Ising model entails that all these Hamiltonian models exhibit critical behaviour described by a conformal field theory. Bipartite correlations in this regime are described by the lowest-dimension even primary scalar $\epsilon = \sigma \times \sigma$ in the CFT. This field is symmetric under the switch between Alice and Bob. It provides a conformal description of 'bipartiteness', which is encoded in the Ising interaction. The operator dimension of ϵ is

$$\Delta_\epsilon = 3 - \frac{1}{\nu}, \tag{12}$$

where ν is a well-known critical exponent describing the correlation length.

The 3-dim spatial analogy has its limitations since the true metric space of code evolution is the fractal $\hat{S}_C$. Still it provides significant evidence that, if the Hamiltonian (4) is to be replaced by (10), there emerges a critical regime, whereby the symmetric bipartite correlations are described in the CFT by an analogue of the 3-dim Ising primary even scalar ϵ. Further, the exponential character of the mapping that links the fractal with a Euclidean model hints at the existence of a connection between their critical behaviours. The correlation length in the fractal representation of a language describes a logarithmic distance from which words are brought in clusters of equal complexity by the Kolmogorov reordering. It diverges in the critical regime at string complexity $\beta = R$. However, in the Ising analogy the amount of correlations on the words of equal complexity remains limited by the scaling of the correlator of the lowest primary even field:

$$\langle \epsilon(a)\epsilon(0) \rangle \sim a^{-2\Delta_\epsilon}. \tag{13}$$

We conjecture that the corresponding correlations in the fractal are limited by the logarithm of the RHS of (13). Their maximum strength $2\Delta_\epsilon$ can be computed based on the value $\nu = 0.62999(5)$ in [35]:

$$2\Delta_\epsilon = 2.82537(2). \tag{14}$$

VII. CONCLUSION

Mathematical investigation of quantum theory drives home the importance of both the composition rule and the continuity assumption. These are two pillars of quantum theory. Freely interpreting Bohr's dictum that the unambiguous communication of measurement results requires a mathematical formulation, we proposed a mathematical framework that sits on these pillars and the idea that the observer is defined by the limited string complexity. The result is a conjecture on the amount of bipartite correlations slightly different from the Tsirelson bound, but consistent with available experimental results $S = \Delta_\epsilon + 2 \simeq 3.41267 \leq 3.426 \pm 0.016$ [36] and $2\Delta_\epsilon \simeq 2.82534 \leq 2.827 \pm 0.017$ [37].

The crucial analogy with the Ising model relies on the assumption that the number of strings possessing the same complexity after uncomputable Kolmogorov reordering is $N = 6$. This does not need to be so for all codes. Codes with $N = 4$ correspond to 2-dim Ising interaction ($\nu = 1$) and give rise to the classical bound on bipartite correlations $2\Delta_\epsilon = 2 \cdot 1 = 2$. Codes with $N = 8$ correspond to 4-dim Ising interaction ($\nu = 1/2$) and give rise to the Popescu-Rorlich maximum correlation $2\Delta_\epsilon = 2\cdot(4-\frac{1}{1/2}) = 4$. It is not clear whether binary codes endowed with critical dynamics exist for other values of N and, if they do, what meaning they may have.

Although our model is highly speculative, we believe that it demonstrates the interest to explore quantum theory mathematically. Von Neumann's quantum logic and Weyl's relativity theory did not yield new testable predictions. Our model does. As Wittgenstein said, "A particular method of symbolizing may be unimportant, but it is always important that this is a possible method of symbolizing. [This] possibility... reveals something about the nature of the world" [38]. If something is indeed revealed, it is thanks to the application of new mathematics to physical theory.

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