A HISTORICAL APPROACH TO RESEARCH IN FUNDAMENTAL PHYSICS

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ABSTRACT

Research that aims at identifying new fundamental ideas in physics can greatly profit from a historical approach. The present essay develops this idea by conceptually analyzing the major physical theories created since antiquity and by distilling from them the research trends that have been unmistakably successful. The author's approach to research is based on extrapolating these trends into the future. It is a method that led to a unification of quantum mechanics and relativity based on a new number system structurally located between the complex numbers and the quaternions. Following a brief description of the concrete results obtained so far, the question of what's ultimately possible in physics is addressed by speculatively generalizing the results in question.

The discussions that follow are based on Kuhn's analysis of the history of science [1]. According to Kuhn, science (physics in the sequel) evolves in an alternating sequence of bursts of fundamental research that culminate in "paradigm shifts", and periods of accretion and consolidation of knowledge, referred to as "normal science".

Since doing research in fundamental physics is equivalent to trying to predict the next paradigm shift, a historical approach suggests itself. It consists in analyzing the sequence of past paradigm shifts with the objective of discovering trends. Extrapolation of these trends beyond current physics helps us identify the ideas most likely to succeed.

The next six sections discuss the major paradigm shifts of the past from a conceptual point of view that brings out the trends in question.

The Ptolemaic model (2nd century)

The precursor of all physical theories is the "Ptolemaic model", whose success in predicting the positions of planets and the timing of eclipses remained unsurpassed until the 17th century. This paradigm marks the transition from a mythological to a rational conception of natural phenomena.

The idea was to condense all relevant astronomical observations into a kinematical "model" that could be run into the future. It was a model built on a single object: the circle. The conic sections (ellipses) were not considered, even though they had been known for 500 years and were later identified by Kepler as the orbits of celestial bodies. This may have been for lack of mathematical means to assign a kinematic role to these curves. The circle, however, easily plays such a role because a single number specifies a constant speed of rotation, something easily visualized as a frictionless flywheel.

A philosophically perfect solution for planetary orbits was thus available in the circle, but it did not fit observations. The Ptolemaic model reconciled philosophy with observations by affixing the axis of one flywheel to the rim of another one. Being recursive, this approach could model the wildest motions to any desired accuracy.

Ptolemy's solution amounts to expanding the apparent motion of a planet into a general trigonometric sum. It is an "engineered solution" that consists of a large number of simple elements organized into a complex structure. Agreement with observations is obtained by individually adjusting the numerical values of a large number of radii and speeds. Such an empirical approach is being used to this day in all sciences until a more elegant model can be found — meaning a model in which very few parameters have to be determined experimentally. After a long hiatus in science, an elegant model of planetary motions was discovered by Kepler.

Kepler's contributions (early 17th century)

In the second half of the 16th century, Tycho Brahe made extensive astronomical measurements later analyzed by Kepler in search of patterns. This analysis resulted in three laws of planetary motions. The first identified ellipses as planetary orbits. The second generalized the flywheel idea: It is not the angular speed of the sun-to-planet vector that is constant, but the speed at which this vector sweeps the ellipse's area. These two laws supplanted the complicated Ptolemaic model by a simple one in which one ellipse per planet does the work of dozens of flywheels mounted on flywheels.

Kepler also noticed a third regularity: If Y denotes the duration of a planet's year and A the length of the ellipse's long axis, the ratio A^3/Y^2 is the same for all planets. This is a qualitatively different insight: It tells us that *something interrelates the planets.*

These beautiful results were nevertheless puzzling: By what mechanism could a planet 'know', at millions of leagues from the sun, how to trace a perfect ellipse while (1) keeping the sun in a focus, (2) the areal velocity of the radius vector constant in time, and (3) the ratio A^3/Y^2 constant across the solar system? Let's refer to these questions as "the puzzle of global laws".

Kepler also conjectured why there were five planets — a number believed to be exact. The planets, he asserted, were in a one-to-one correspondence with the Platonic solids, whose number (five) is a mathematical theorem (a polyhedron is "Platonic" if all its faces are congruent to some regular polygon). It is instructive to consider why Kepler's laws are impressive discoveries while this conjecture is rather quaint from our vantage point.

The first three laws embody patters hidden in the reams of Tycho's data. Kepler was not expecting ellipses; they imposed themselves. In contrast, his conjecture was supported by a single coincidence: the equality of two small numbers. This is not convincing, but reasonable as a hypothesis to be tested. Kepler might have reasoned as follows: Since the motion of every planet is governed by concepts related to solid geometry (intersections of cones and planes), and since the planets' orbits are mutually related by a universal ratio that suggests the ratio of a solid's volume to its surface area (up to a dimensional factor), the number of planets might also be determined by regularities to be found in solid geometry. The Platonic solids were the unique answer. Such thinking is commonplace in modern research, but hypotheses are subjected to rigorous experimental tests and rejected for failing a single one.

Ptolemy's and Kepler's models also bring out a philosophical question over which mathematicians disagree: Are mathematical theories "invented" or "discovered"? While these examples are not mathematics but physics, we may discuss them here because the boundary becomes fuzzier with each new paradigm. The pre-ptolemaic mythological models were clearly pure inventions: They brought some fanciful order into observed phenomena but predicted no future ones. Ptolemy's model is an invention for having been engineered, but since it correctly extends past motions of planets into the future, it mirrors an objective regularity — which makes it also a discovery. Similarly, ellipses were "discovered" in Tycho's data, not "invented into them".

The Newtonian revolution (late 17th century)

The puzzle of global laws was solved by a new paradigm which substituted local laws governing the motion of fictitious "mass points" for global ones that defined planetary orbits.

One can see without recourse to calculus how a local theory explains Kepler's laws by deriving them as theorems: Let τ denote a small time interval. The motion of a point is thus specified by its successive positions at the ticks of a "tau-clock". Question: How long is the memory of a mass point *not affected by external forces?*

If, at any instant t, the point remembers its positions at the previous two instants, $t - 2\tau$ and $t - \tau$, this makes three points. Since any three points uniquely define a circle, a mass point left to itself would move forever on the circle on which it was started. This being false, it's memory must be shorter than 2τ. If it had no memory, the point would remain at rest, which is false as well. The correct assumption is a memory of one τ, which gives us two point. Since two points define a straight line, Newton's law of inertia follows.

But planetary orbits are not straight lines: a circle of radius r (or of curvature $1/r$) is uniquely defined at every instant t. Referring to the external agent that causes this curvature as "force", solving the puzzle of global laws meant determining the force that reproduces Kepler's laws. To this end, Newton postulated universal gravitation and developed the infinitesimal calculus and classical mechanics. His theory yielded Kepler's laws and Galileo's law of free fall as theorems. Further developments led to conservation laws (for energy, momentum, and angular momentum) that provided the foundations for all subsequent physics.

An analysis of Kepler's conjecture is instructive. Motions based on differential equations are described by rigid "laws" and arbitrary "initial conditions". But there is no such separation if the differential equations are not known. Since this was the case in Kepler's time, he incorrectly guessed that the number of planets is a "law". We know it stems from initial conditions during the formation of the solar system. On the other hand, we are not in a better position than Kepler in 'guessing' where the dimensionless constants of the Standard Model belong. Do they stem from initial conditions at the Big Bang, or are they theorems (like the value of π in mathematics)? Opinions differ on this point.

Electromagnetism (2nd half of the 19th century)

By the mid-19th century, one hundred years of "normal research" had culminated in an apparently complete system of laws for electricity and magnetism. Thus, electric currents created magnetic fields and variable magnetic fields generated currents. But this systems contained an experimentally invisible but conceptually essential flaw that was identified and corrected by Maxwell (1873).

To visualize the problem, consider a flat plate capacitor in the circuit of an arbitrary current. According to the physics inherited by Maxwell, this current produces a variable electric field between the plates and a variable magnetic field around the conductors, but the latter disappeared abruptly in the interval between the plates. This conclusion could not be verified with 19th century instrumentation. Maxwell took it to be false and conjectured that the variable electric field between the plates gives rise to a magnetic field not different from the field we would measure *if the current were not interrupted.* With the help of this fictitious "displacement current" he formulated a conceptually simple theory of electromagnetism that did not have to be modified to this day.

An unexpected consequence came to light in the process: Once created, an oscillating pair of fields would be self-sustaining and would propagate away as electromagnetic radiation. Hertz verified this phenomenon experimentally (1887).

Special relativity (1905)

Newton's mechanics and Maxwell's electromagnetism had reached textbook status by the end of the 19th century, but they were mutually inconsistent. To see how, we first observe that no medium is "absolutely transparent" to fields. Two constants, denoted by ε and μ, specify its opacity to electric and magnetic fields respectively. The measured values for vacuum are

$$\begin{aligned} \varepsilon &= 8.854187817 \times 10^{-12} \text{ F / m} \\ \mu &= 1.2566370614 \times 10^{-6} \text{ H / m} \end{aligned}$$

One notices that the number c, defined as

$$c = \frac{1}{\sqrt{\mu\varepsilon}} = 299,792,458 \text{ m/s}$$

numerically coincides with the speed of light, and that its dimension (not obvious but true) is that of speed: meters per second. This number appears in Maxwell's theory as the speed of propagation of electromagnetic waves. Being defined with respect to no particular reference frame, this speed is absolute — which is nonsensical in Newton's theory, where speed is relative. This mystery was solved in four stages:

(1) The physicists who insisted on preserving classical mechanics at any conceptual cost fruitlessly sought solutions in ever more complicated engineered models of a hypothetical space-filling fluid, the "aether". (2) Lorentz almost solved the problem by discovering the transformations that bear his name (1904), but their mechanical interpretations missed the point. (3) Einstein boldly modified

classical mechanics (1905) to make it consistent with electromagnetism. (4) Minkowski discovered the geometry within which Einstein's solution assumes its simplest and most natural form (1908), and also elegantly unifies electric and magnetic fields. This is the gist of special relativity.

General relativity (1916)

The conceptual simplicity of general relativity stands in contrast to its technical difficulty. Let us review the thinking that led to this theory.

After 1905, Einstein set out to extend special relativity to universal gravitation, but this cannot be done by modifying the axioms of Newton's theory of gravitation. Einstein bypassed this problem by taking from the latter only the equivalence of inertial and gravitational mass. According to this principle, the two concepts of mass (as gravitational charge and as inertia) are equivalent. Viewed by others since Newton as a curiosity, Einstein saw this property as the defining characteristic of gravitation. So streamlined, the unification problem becomes a strictly mathematical one: *Find a mathematical structure which incorporates the principle of special relativity ($c = const.$) and the principle of equivalence.*

This problem is strictly mathematical because experiments and known equations play no role in its solution. The latter is based on the observation that since the apparent curvature of the world line of a mass point in a gravitational field does not depend on its mass, it must be a property of the space: The space is curved, the world line is straight, where "straight" means auto-parallel (a curve is auto-parallel if its tangents at any two consecutive points are related by a parallel displacement). The solution is based on Riemannian geometry, a then obscure corner of mathematics brought into the limelight by Einstein's work.

Quantum mechanics

The theories discussed so far deal with concepts rather accessible to intuition, but a new class of counter-intuitive observations burst upon the stage at the turn of the 20th century. The corresponding theory was developed in two phases, the first being known as the "old quantum theory". Its key word, "quantization", refers to *ad hoc* rules that modify some "laws" of physics to account for the jumps observed where classical physics predicts smooth change. This theory was a stepping stone to the second phase, which is not a modification of classical physics but a structurally sound theory completed by 1930 and referred to as "quantum mechanics".

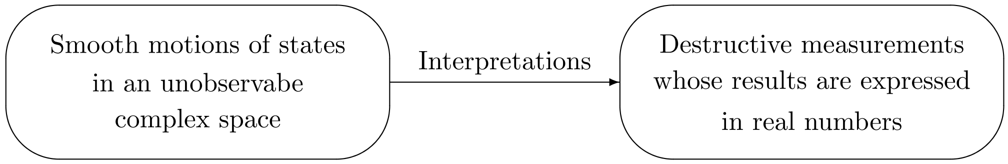
A discussion of the two extreme roles mathematics plays in physics will help clarify the essential difference between classical and quantum mechanics. Let's refer to these roles as "quantitative" and "structural".

The "quantitative" view of mathematics is the only one the proverbial 'man on the street' is aware of. It answers the question "How?" — as in "How to compute something?" The key word is "equation". From Ptolemy to Newton, the primary role of mathematics was the quantitative numerical modeling of the continuous time evolution of physical variables. Classical physics (in the sense of 'non-quantum') is the part of physics that admits a one-to-one correspondence between mathematical and physical variables. It is thus possible to follow the time evolution of the former by integrating differential equations, and of the latter by making measurements. Comparison may be made at any time without disturbing the system.

The "structural" view of mathematics — according to which mathematical objects are not defined ontologically but only by their mutual relations — was introduced for two special cases in the 1830's (Galois and Hamilton) and had become standard by the 1940's (Bourbaki). This is 'the mathematics of mathematicians'. It answers the question "What?" — as in "What is possible?" or "What exists?" The key word is "structure". In this conception of mathematics, equations not anchored in mathematical structures are meaningless.

Just as mathematics was becoming more and more structural, its role in physics was becoming more structural as well. In general relativity, its role is essentially structural. This theory was not "engineered" to solve a minor anomaly in Newton's theory, but "discovered" as the *unique* mathematical structure that unifies relativity with gravitation. It closes the book on non-quantum physics.

Quantum mechanics opens a new book, in which the one-to-one correspondence between observation and theory does not hold. The loss of this link made the structural role of mathematics even more pronounced: Equations of motion operate in an unobservable universe of concepts based on complex numbers, while observations, as usual, take place in a universe based on real numbers. Partial correspondences between the two universes are established by interpretations (the fundamental one being Born's, which introduces statistics).



This completes the review of the major paradigm shifts until 1930. Since then, fundamental research has been mostly about the approximate merging of relativity and quantum mechanics.

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Let us summarize these discussions in a sequence of idealized stages.

The empirical stage

The initial objectives of a rational approach to natural phenomena were both practical (like calendar design and navigation) and philosophical (understanding for the sake of understanding). As a first pass, Ptolemy's model was impressive on the first point. On the second point, it put scientific thinking in motion.

The descriptive stage

Engineered solutions that call for the fine-tuning of a large number of parameters can always be invented for any set of observations, but they are no longer satisfactory at this stage. Observations must fit into a minimal mathematical framework that seems 'natural'. Kepler's laws and the premaxwellian theories of electricity and magnetism belong to this category.

The explanatory stage

Systems that only describe or organize observations — like Kepler's laws, Mendeleev's periodic table, and the classification of particles by roots of Lie algebras — eventually call for explanation by an underlying theory. Once discovered, this theory does much more than provide economy of thought: The class of phenomena it explains exceeds the original expectations. Newton's and Maxwell's theories are of this type.

The finalization stage

A new theory is usually inelegant and hard to follow because it is initially formulated in the mathematical language of its predecessor. The mathematics which brings its essence to the fore and eliminates the chaff is developed later within "normal science". Maxwell's theory in the exterior calculus and classical mechanics in the canonical formalism are examples of such finalizations. They are not only works of art of great beauty, they provide the foundation for further research.

Observations

The following general research guidelines, stated as observations rather than rules, are suggested by the previous discussions.

(1) *There is no return to more naive intuitions.* Successive physical theories have always been built on ever more abstract mathematical structures. Attempts at reversing this trend (like aether theories and hidden variable theories) have been a waste of effort.

(2) *Every new physical theory requires some new mathematics which is not an evident extension of the previous one.* This is easily justified: If the mathematics of a theory could also support the next theory, the latter would be a continuation of the former (part of "normal research"), not a new theory (a "paradigm shift"). Thus, Newton's mechanics required the infinitesimal calculus (developed for this purpose); special relativity required an indefinite Pythagorean metric (introduced for this purpose); general relativity required Riemannian geometry ('revived' for this purpose); quantum mechanics required Hilbert space (developed primarily for this purpose).

(3) *The goals of physicists are very different from those of mathematicians.* Generally speaking, mathematicians seek powerful generalizations while physicists aim at the final theory — which is a very specific mathematical structure. It cannot be a special case of a more general one, for if it were, external information would still be needed to specify the rule by which it has been selected. Thus, general relativity may be the final non-quantum theory, but not the final physical theory because the space-time dimension $3 + 1$ must be put in by hand, and the gravitational constant must be measured. Both would be theorems in an absolutely final theory.

(4) *New theories are best developed as answers to pressing questions.* Einstein's life work is most instructive in this respect. His special and general theories of relativity were answers to specific conceptual problems that could not be left unsolved. In contrast, the futile search for a theory unifying electromagnetism and gravitation was motivated only by a personal conviction that all classical fields must be geometrized. We know that this cannot be true because electromagnetism is rooted in a quantum theory (it is the macroscopically observable part of the electroweak theory).

(5) *The unification of theories takes place automatically once the correct mathematical structure has been identified.* This is most evident in both theories of relativity. Conversely, if merging two theories is a very laborious enterprise to which many physicists additively contribute to the total knowledge, one may conclude one of two things: Either the theories in question are not candidates for unification (like gravitation and electromagnetism), or the unifying mathematical structure is still unknown.

(6) *Two theories cannot be unified until they have been finalized and characterized by principles.* The reason seems evident: The unification of technically defined theories — meaning theories based on axioms — would require the modifications of the axioms. But this is virtually impossible because axioms are inflexible.

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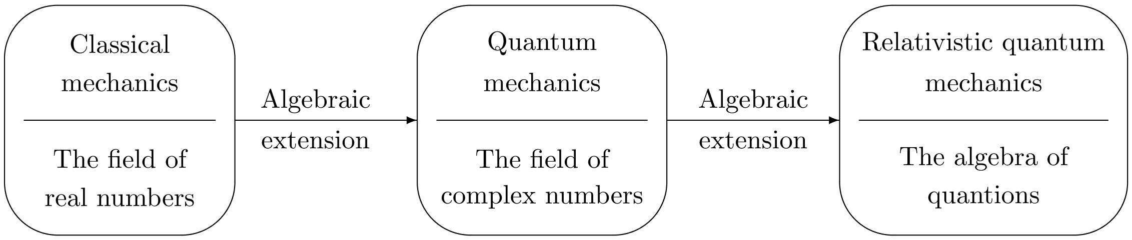
Some new ideas concerning the question of what's ultimately possible in physics will be discussed after a brief statement of the problem that brought them to light.

Setting aside cosmology, research in fundamental physics since 1930 has been driven by experiments in particle physics and by the need to overcome the structural incompatibility of quantum mechanics and relativity. The results are condensed in quantum field theory (meant to include the Standard Model). This theory agrees with experiments to more decimal places than any theory ever has, but it is an intermediate solution to the unification problem. Thus, "quantization" (of fields) is still the key word — which has not lost its century-old connotation of "afterthought to classical thinking". By observation (4), the time is more than ripe for a structurally sound theory that would yield quantum field theory as an approximation, just as quantum mechanics contains the old quantum theory as an approximation.

It is not evident, however, that a mathematical structure which is both relativistic and quantum mechanical exists. If it doesn't, we might already have reached — details aside — what is ultimately possible in physics. Observation (6) suggests a constructive question: Are the theories to be unified characterized by principles? Special and general relativity are, but quantum mechanics is defined only by technical axioms. It is therefore not ready for unification. Postponing unification, we must first search for the "principle of quantum mechanics" which implies the axioms of the latter.

Forty years ago at Yeshiva University, Aage Petersen and myself undertook this search. Our collaboration was interrupted half-way, but it had also hit an apparent dead-end — due only to an oversight I noticed thirty years later. Following the correction, quantum mechanics was easily generalized to an abstract algebra. This algebra has exactly two concrete realizations (in the sense in which, by Cartan's classification, the abstract simple Lie algebra has nine concrete realizations).

As expected, one of these is nonrelativistic quantum mechanics. The other one is relativistic. According to observation (2), its 'new mathematics' is a number system referred to as the "algebra of quantions". The following graph puts it in perspective.



We observe that the development of relativity is based on a similar sequence of geometric extensions:

$$\text{Euclidian space} \rightarrow \text{Minkowski space} \rightarrow \text{Riemannian space.}$$

Definition:

A quantion is a quadruple of complex numbers, and the algebra of quantions, denoted by $\mathcal{L}$, is the unique extension of the field of complex numbers that enjoys the following properties:

(1) *Quantions are not degenerate.* "Degeneracy" refers to the collapse of two conceptually different objects into a single formal one. For a complex number, the formal object is the norm $\|z\|^2$. It has an algebraic origin as the denominator in the inverse ($z^{-1} = \frac{z^*}{z^* z} = \frac{z^*}{\|z\|^2}$) and a geometric origin as the Pythagorean distance in the Gaussian plane ($\|z\|^2 = x^2 + y^2$). We refer to the first concept as the "algebraic norm", and to the second as the "metric norm". For a quantion Q, these two norms do not coalesce. The algebraic norm $A(Q)$ plays the role of $\psi^*\psi$ in quantum mechanics; the metric norm $M(Q)$ is the norm defined by the Minkowski metric. The fundamental concepts of relativity and quantum mechanics are thus anchored in the number system. According to observation (5), the unification is automatic.

(2) *The algebra of quantions admits a derivation operator.* Such n operator, $\mathcal{D}$, is needed in equations of motion. Its two main properties are linearity an the Leibniz identity $\mathcal{D}(FG) \equiv (\mathcal{D}F)G + F(\mathcal{D}G)$, where F and G are quantionic fields. Formally, $\mathcal{D}$ must be a quantion, but since the product of quantions is not commutative, the Leibniz identity cannot be satisfied if F and $\mathcal{D}$ belong to the same algebra. This problem does not arise, however, because the algebra $\mathcal{L}$ is paired with a dual algebra, $\mathcal{R}$, which commutes with it. Fields belong to $\mathcal{L}$ while $\mathcal{D}$ belongs to $\mathcal{R}$. This is a theorem, not a matter of choice. It is analogous to the geometric duality of contravariant vectors v^μ and the covariant derivation operator ∂_μ.

Implications:

The investigation of the mathematical properties of quantions is currently in progress and far from completion. The following partial list of theorems will suffice bring out some new physical insights.

(1) If interpreted as a number (meaning that addition and multiplication are both relevant), a quantion Q belongs to $\mathcal{L}$ or $\mathcal{R}$, depending on the role it plays.

(2) As a number system, $\mathcal{L}$ structurally falls between the complex numbers and the quaternions. It differs from the complex numbers in only two properties: It is not commutative, and it is not a division algebra. Both facts are desirable because division is superfluous (ψ^{-1} is never needed) and commutativity is trivializing. In contrast, quaternions admit division, do not admit a derivation operator, and are degenerate. This explains why quaternionic quantum mechanics does not support unification with relativity.

(3) When multiplication is irrelevant, Q belongs to a 4-dimensional Hilbert space, $\mathcal{H}_q$. Quantionic fields are of this type.

(4) Denoting by $\mathcal{L}_h$ the Hermitian subspace of $\mathcal{L}$, an element Q of $\mathcal{L}_h$ is a vector in the Linear Minkowski space M_0^4. Its metric norm $M(Q)$ is the Minkowski norm. Thus, the linear Minkowski space and the Lorentz group are quantionic theorems. In other words, the algebra of quantions is *inherently relativistic.* Unification is thus inherent — it does not have to be done separately.

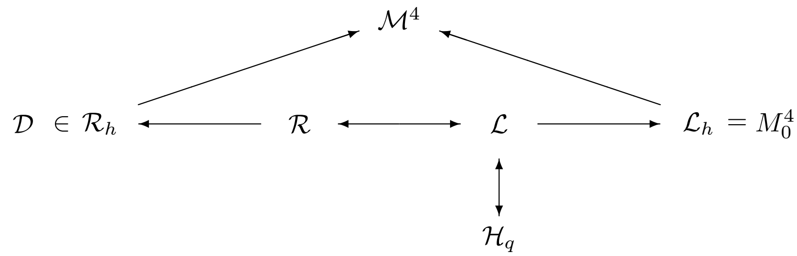
(5) The algebraic norm $A(Q)$ of an arbitrary quantion Q is Hermitian. It is thus a Minkowski vector, but of a particular type: It belongs to the future-oriented cone. For the unit I in the quantionic algebra, $A(I)$ is the unit vector that defines the direction of time. It follows that the Minkowski space generated by the algebra of quantions is more structured than the Minkowski space of relativity: It has an *intrinsically distinguished arrow of time.*

(6) The property (5) calls for a physical interpretation. Nikola Zovko suggested one: that $A(Q)$ be viewed as a current. This is a constructive hypothesis because it can be tested. The results are most encouraging: In the non-relativistic limit — where the quantionic field $Q(\vec{r}, t)$ differs only infinitesimally from a complex field $\psi(\vec{r}, t)$ — one obtains the Schrödinger equation with an arbitrary potential. In the general case, one obtains the Klein-Gordon equation and the Dirac equation with a potential that satisfies Maxwell's equations. There are also three other possibilities reminiscent of the three vector bosons, but they are still being investigated.

(7) The group $U(1)$ of quantum mechanical gauge transformation $\psi \to e^{i\chi}\psi$ generalizes to the group $U_q(1)$ of quantionic gauge transformations. Structurally, $U_q(1) = U(1) \times SU(2)$, which is also the gauge group of the electroweak theory. This is consistent with the partial result (6).

(8) The Cartesian product of the two linear spaces $\mathcal{L}_h$ and $\mathcal{R}_h$ of Hermitian quantions naturally splits into the direct sum of four Minkowski spaces. One of these is distinguished as the space of electromagnetic vectors.

The following graph illustrates the mutual relationships of all quantionic concepts mentioned so far



Conclusions:

The great amount of relativistic quantum physics condensed in the Standard Model ought to suggest what to expect in the future, but it does not. It generates more questions than insights. For example: Where does the mathematically unbelievable but physically fundamental gauge group $U(1) \times SU(2) \times SU(3)$ come from? Two extreme answers have been suggested: (1) It's a property of the particular universe which happens to be hospitable to *homo sapiens* (the anthropic view); (2) it is a 'slice' of the group $SU(5)$ (the supersymmetric view). In the absence of a structural unification of quantum mechanics and relativity, we have no rational basis for looking into the future.

Let an "ideal final theory" be defined as a mathematical theory which is: (1) based on a single "fundamental structure", and (2) related to observations by physical interpretations. Given such a theory, no other mathematical structure would have to be postulated in order to derive all known physics in the form of theorems (including the numerical values of all constants).

The fundamental structure of the quantionic approach is a new number system (the algebra of quantions) and the basic interpretation is Zovko's (which extends Born's interpretation to quantions). We see that the eight theorems listed above proceed in the direction of the ideal final theory. Speculatively extending these concrete results and other partial results under development, it is conceivable that the Standard Model is just a big quantionic theorem. And since the affine Minkowski space plays no role in the quantionic approach, this approach is compatible with curvature in an underlying manifold. This leaves open a path to general relativity.

It thus appears that what is ultimately possible in observable physics (let's say down to 10^{-18} m) coincides with how deep one can go in mathematics and still find a fundamental structure which is both very rich (to support all of physics) and very specific (to guarantee finality). In the quantionic approach the fundamental structure is a number system. I cannot think of something deeper that would not be too general (like set theory).

All references to works that led to the algebra of quantions can be found in [2].

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