

# Does gravitational collapse lead to singularities?

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## Abstract

According to conventional modelling by general relativity the collapse of radially symmetric gravitating objects may end in a singular state. But by inclusion of potential energy into the energy tensor, which is required to guarantee global energy conservation, the occurrence of singularities is avoided. Instead the final states of the collapse of mass concentrations of arbitrary size are nuclear matter objects, from which jets of matter can be recycled into space. The mysterious dark energy, supposed as the main constituent of the universe, may even be the potential energy of matter itself.

## 1 Introduction

Einstein's theory of general relativity (GRT) is regarded today as the best description of gravitational interaction. The concept to interpret motion under the influence of gravitation as geodesic motion in a non-Euclidean space-time continuum has successfully passed all observational tests. The aberration of light passing close to the sun, the perihelion shift of Mercury or other planets and the frequency shift of light by gravitational interaction have impressively confirmed the theoretical model.

But all these observations only test the range of weak gravitational fields. That means that interactions of test bodies or quanta can be described by geodesic motions in a geometry, which is set up by a given matter distribution. In the strong field range, where every element of matter contributes to geometry as well as it is object of changes, observations are less conclusive. Though the gravitational collapse of stars or galaxies into objects of extreme density appears as observationally confirmed, the details of these processes are far from being well understood. According to the textbooks of general relativity (see e.g. Wald [1] or Hawking & Ellis [2]) massive objects may contract into a final state, from which an escape of matter is impossible, and finally even into a singularity, a state of infinite matter density. But to all our experience in other parts of physics singularities do not exist in nature as physical entities. The

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occurrence of singularities in the mathematical description is always the consequence of an inaccurate modelling or unallowed extrapolations. Thus we must rise the question, if the singular states occurring in GRT should be regarded as a purely mathematical approximation. There is no observational evidence that the collapse to a singular state really happens.

Thus it appears reasonable to look somewhat closer to the physics of the collapsed objects, normally denoted as black holes. Is it possible to understand their physics within the geometrical concept of general relativity, but without creating mathematical singularities? In this article we will show that this is possible, if we only apply the concept of energy conservation more strictly than in conventional models. We will concentrate on the description of spherically symmetric objects in static equilibrium, as the correct description of these equilibria must be regarded also as the basis of all dynamical developments.

## 2 Spherically symmetric solutions

Spherically symmetric objects in GRT are defined by the condition that in the field equation

$$R_{ij} - \frac{1}{2}R g_{ij} = \kappa T_{ij} \quad (1)$$

as well the Ricci tensor  $R_{ij}$ , derived from the metric  $g_{ij}$ , as the elements of the energy tensor  $T_{ij}$ , which contains the sources of gravity, depend only on one spatial parameter. This radial parameter can be defined in different ways. In Euclidean geometry it is normally identified with the length  $r$  of the shortest line connecting some point with the center of symmetry. This is identical with the definition that  $r$  is the length of a closed line at constant parameter  $r$ , divided by  $2\pi$ .

If space is curved, these two definitions do no longer agree. If we write the line element in the form

$$ds^2 = -f(r, t)dt^2 + h(r, t)dr^2 + r^2 d\Omega^2 \quad (2)$$

where  $t$  is the time parameter,  $r$  a radial parameter and  $d\Omega^2 = d\vartheta^2 + \cos^2\vartheta d\varphi^2$  defines the element of solid angle  $d\Omega$ ,  $r$  is defined by the length of a circle divided by  $2\pi$ . This fact we should keep in mind, when we try to do calculations in curved space.

One well known example of a radially symmetric problem is the so called vacuum solution of the field equations, first derived by Schwarzschild [3], which describes a system, in which the geometry is dominated by a central gravitating object, so that the influence of masses outside of this object on the metric can be neglected. In this case the field equations are reduced to the vanishing of the Ricci tensor  $R_{ij} = 0$ . Of course, the denomination 'vacuum solution' is somewhat misleading, as normally it is not used to describe a vacuum, but to describe motions or interactions of matter outside the central object, only that the influence of this matter or radiation on the geometry is negligible compared to that of the central gravitating object.

Under these conditions the field equations are reduced to

$$\frac{1}{hr^2} \frac{dh}{dr} + \frac{1}{r^2} \left(1 - \frac{1}{h}\right) = 0, \quad \frac{1}{f} \frac{df}{dr} = -\frac{1}{h} \frac{dh}{dr} \quad (3)$$

with the solution  $f(r) = 1/h(r) = 1 - C/r$ . To derive this formula use has been made of the special form of the coordinate system, in which the angular part of the metric is defined by the condition that the length of a closed line with  $r = \text{const.}$  is  $2\pi r$ . From comparison of the weak field limit with Newtonian gravity then the parameter  $r$  is identical with the radial coordinate and the constant  $C$  is related to the gravitating mass at the center. This leads to the formula

$$f(r) = 1 - \frac{r_s}{r} \quad h(r) = \left(1 - \frac{r_s}{r}\right)^{-1} \quad (4)$$

where  $r_s$  is the well known Schwarzschild radius  $r_s = 2GM/c^2$  ( $G$  is the gravitational constant and  $M$  the mass of the central object). But already with this simple solution we get into conceptual problems, if we try to extend it to processes which take place in regions close to the Schwarzschild radius. Assuming that space outside  $r_s$  contains matter of density  $\varrho(r)$ , the amount of matter in a spherical shell of thickness  $dr$  at  $r = r_s$ , expressed by our coordinates, is given by

$$dm = \sqrt{h}\varrho(r)r^2 dr d\Omega = 4\pi \frac{\varrho(r)r^2}{\sqrt{1 - r_s/r}} dr \quad (5)$$

But even if the density is arbitrarily small but not exactly zero at  $r = r_s$ , this yields an infinite value, in contradiction to the assumption that the mass is negligibly small. For  $r < r_s$  a real value of  $dm$  does not even exist. This singular behavior is caused by the fact that the size of the spatial volume element  $dV = 4\pi r^2 \sqrt{h(r)} dr$  becomes infinite at  $r = r_s$ .

There are two possible ways out of this dilemma. Either the definition of the radial parameter is no longer valid in the strong field regime or the definition of energy or matter density breaks down and must be modified. But the definition of the radial parameter leaves no room for a recalibration, as on one hand in the limiting case of weak gravitation it should agree with the radial distance and on the other hand by the definition of the angular part of the line element by eq.(2) the parameter  $r$  is uniquely defined. There is no physical reason, why a closed line with length  $s \leq 2\pi r_s$  should not exist. So the only way out is that the parameter  $\varrho$  is not adequate to describe the matter or energy distribution and we have to redefine the density parameter.

Anyway a unique definition of densities appears as a problem in a system, in which the size of the spatial volume element varies, when the amount of matter in the neighborhood changes. If we propose that matter consists of individual particles with their rest mass as an invariant property and that the number of those particles in some volume, which is defined by fixed limits, cannot change, when the matter distribution outside this volume is changed, local densities, defined as the amount of some quantity per volume, cannot be regarded as invariant properties of the spatial distribution.

The proposition of Einstein, to use the densities as defined in the local Euclidean tangent space, can be regarded as a suitable approximation in the weak field regime, but it appears invalid in strong fields. Incorporating gravitation into the geometry of space requires incorporating curvature into the definition of density as well. The matter density as defined in the Euclidean tangent space must be regarded as a local parameter. But if we want to determine the total matter content in some volume, defined by limiting values of the radial parameter, we cannot simply do it by summing up the locally defined tangent space densities, that means, by integration of  $dm$  as given by eq.(5). To define a conserved integral property like the total mass, we must add a correction term to the integrand, accounting for the change of volume by curvature. In the definition of density we have to replace the size of the tangent space volume element by the corresponding element in curved space. The volume element  $dV_0 = 4\pi r^2 dr$  has to be replaced by  $dV = 4\pi r^2 / \sqrt{h} dr$ .

What this means becomes clear immediately, considering again the weak field limit. According to the definition of the line element eq.(2) the mass in a spherical shell between radii  $r_1$  and  $r_2$  with density  $\varrho$  is

$$\Delta m = 4\pi \int_{r_1}^{r_2} \sqrt{h(r)} \varrho(r) r^2 dr = 4\pi \int_{r_1}^{r_2} \frac{\varrho(r) r^2}{\sqrt{1 - r_s/r}} dr. \quad (6)$$

With  $r_1, r_2 \gg r_s$  it can be approximated by

$$\Delta m = 4\pi \int_{r_1}^{r_2} \varrho(r) r^2 dr + 4\pi \int_{r_1}^{r_2} \frac{G\varrho(r)}{rc^2} dr. \quad (7)$$

The second term is just the mass equivalent of the gravitational binding energy or of the negative of the potential energy of the system. Integration of the density, as defined in Euclidean tangent space results in an overestimate of the total mass. Thus it appears logical to consider the summed density of mass energy and potential energy as the equivalent to mass energy in Euclidean space. With this redefinition at the Schwarzschild radius the matter energy density  $\varrho c^2$ , defined in the local tangent space, just balances its own potential energy  $\varrho c^2 (\sqrt{1 - r_s/r} - 1)$ , so that the effective contribution to the total energy remains negligibly small.

Including potential energy into the energy balance in this way, a basic property of Newtonian gravity is recovered, the conservation of energy. Without inclusion of potential energy into the balance, gravitational collapse would be accompanied by a continuous gain of energy from the gravitational field. But in the geometrical concept of general relativity there exists no gravitational field which might possess energy. Gravitation is only a consequence of curvature. So this energy is created from nothing. Only if we include potential energy of matter itself into the balance, conservation of energy is guaranteed also in systems of strong curvature.

The question remains, how to define a general expression of potential energy within the formalism of GRT. It is a basic concept of GRT that it is a strictly local theory. That means that the complete information, which describes the

behavior of matter under the influence of gravitation is, besides of locally defined parameters, contained in the metric tensor.

That Einstein used the field equation (1) instead of simply setting  $R_{ij} = T_{ij}$  was motivated by the fact that the divergence of the left hand side should be zero and thus represents a conserved quantity, as he took for sure that the energy tensor on the right hand side was a conserved quantity, too. But the fact that the tensor divergence vanishes locally is not sufficient to guarantee conservation also on integral scale. Exact conservation of the total energy is possible only, if we include potential energy into the definition of the energy tensor.

In the case of a radially symmetric solution the only reasonable way to describe potential energy is a term in the energy tensor of the form  $\lambda(r)g_{ij}$ , where the scalar function  $\lambda$  depends only on local parameters, on the tangent space matter density  $\varrho$  and the local pressure  $P$  (leaving out of consideration radiation and other possible minor contributions to the energy field for simplicity).

In curved space we must distinguish between two types of local parameters, those which represent intensive properties of particles resp. their local mean and densities of extensive properties, defined as the amount of some property per volume. When, as in general relativity, geometry depends on the matter distribution, the latter group of parameters must be adjusted for the variability of volume, if we want to make a transition to macroscopic quantities by integration.

A typical quantity of this latter kind is the particle density. If a fixed number of particles is contained in some volume defined by its limits, this number will not change, when the geometry changes by adding matter somewhere outside this volume. The same holds for the matter content, if we assume that every particle has a conserved property, its rest mass. Thus the effective matter density must change with the geometry of space.

To determine macroscopic data like the total mass of a body from local quantities by integration, we have to replace the local matter density, as defined in Euclidean tangent space, by the sum of this density and the matter equivalent of its potential energy. We have to replace the quantity  $\varrho$  by  $\varrho/\sqrt{h(r)}$ , where  $h(r)$  is the quantity defined by eq.(2), describing the deviation from Euclidean geometry.

### 3 Interior solutions

By now we have only addressed the problems occurring outside the Schwarzschild radius. But the question put in the beginning was, what happens in the interior of a matter distribution in the case of strong gravitation. Inclusion of potential energy into the balance equations does not only influence the exterior of the Schwarzschild solution, but also the balance in the interior of any spherically symmetric matter distribution. The field equation for the static spherically symmetric case is well known as the Tolman-Oppenheimer-Volkoff (TOV) equation

[4]

$$\frac{dP}{dr} = -(\varrho + P) \frac{m + 4\pi r^3 P}{r^2(1 - 2m/r)}, \quad (8)$$

(in units, where  $G=c=1$ ), which relates the local equilibrium pressure  $P$  to the matter distribution. In the conventional description the quantity  $m(r)$  results from integration of the density  $\varrho$  over a sphere of radius  $r$ .

$$m(r) = 4\pi \int_0^r \varrho r^2 dr. \quad (9)$$

The component  $h(r)$  of the metric tensor is related to the density by

$$\frac{1}{rh^2} \frac{dh}{dr} + \frac{1}{r^2} \left(1 - \frac{1}{h}\right) = 8\pi\varrho \quad (10)$$

which is connected to  $m(r)$  by  $h(r) = 1/(1 - 2m(r)/r)$ .

But when space is curved and we want to determine the influence of matter inside a sphere of radius  $r$  on the metric by integration of eq.(9), inserting the density as defined in Euclidean tangent space will not give the correct result. As has been discussed in the last section, we have to include the potential energy term into this equation, replacing the Euclidean matter density  $\varrho$  by  $\varrho + \lambda(r)$  with

$$\lambda(r) = \varrho(1/\sqrt{h} - 1) \quad (11)$$

Thus the integrand of eq.(9) implicitly depends on the metric and by this on  $m(r)$ , so that the function  $m(r)$  now has to be determined from the differential equation

$$\frac{dm}{dr} = 4\pi r^2 \varrho \sqrt{1 - 2m/r} \quad (12)$$

with the boundary condition  $m(0) = 0$ . This leads to a completely different form of the solution.

This can be best demonstrated assuming as an example a system of constant density  $\varrho_0$  and radial extension  $R_0$ . It is reasonable to assume that in every real isolated matter distribution the condition  $d\varrho/dr < 0$  holds, so that with  $\varrho = \varrho_0$  the maximum deviation of  $h(r)$  from unity will be obtained. In this case introducing a normalized coordinate  $x = r/R$  with  $8\pi\varrho_0 R^2 = 1$  and the new variable  $y = x/h$  eq.(10) can be written in the form

$$\frac{dy}{dx} = 1 - x^{3/2} y^{1/2}, \quad (13)$$

while without potential energy we have

$$\frac{dy}{dx} = 1 - x^2 \quad (14)$$

Fig.1. shows the resulting function  $h(x)$  for both equations. Without potential energy the solution exhibits a singularity at  $x = \sqrt{3}$ , while with potential energy  $h(x)$  is finite for all  $x$ . The solution of eq.(13) with  $y(0) = 0$  exhibits an

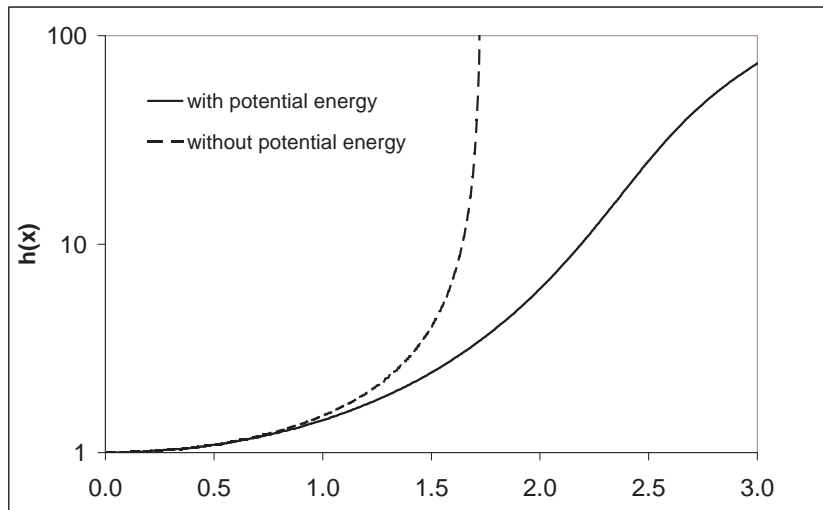


Figure 1: Radial metric parameter  $h(x)$  as a function of the normalized coordinate  $x = r/\sqrt{8\pi\rho_0}$

extremum near  $x = 1.2$  and then decreases monotonically towards zero with increasing  $x$ . That means that the quantity  $2m(x)/x = 1 - 1/h(x)$  can never reach the value one for any reasonable matter distribution. This value of  $2m/x$  must be regarded as an upper bound also for other density distributions. With other words: The Schwarzschild limit, which requires a value of one, can never be reached for any reasonable matter distribution, if only the condition  $d\rho/dr < 0$  is satisfied. The Schwarzschild radius must be regarded as a purely mathematical quantity. Under the assumption of a monotonic density profile a central object of arbitrary mass cannot exist in a volume limited by  $r_s$ .

The fact that the quantity  $(1 - 2m(r)/r)$  can never reach zero is essential also for the solution of the pressure balance eq.(8). The denominator of the TOV equation is always positive. No infinite pressure is ever necessary to balance gravitational force. Due to the form of the potential energy term  $\lambda(r) g_{ij}$  there is also a negative contribution of curvature to the pressure, but this does not change the general form of the solution.

The solution of the system of eqs.(8) and (9) requires the knowledge of an equation of state, some functional relation between  $\rho$  and  $P$ , just like in conventional modelling of stars, where potential energy is neglected. But there is no situation, in which a static balance is impossible so that collapse would proceed into a singular state. The only proposition for a stable equilibrium configuration is that  $\rho$  and  $P$  decrease monotonically from the center to the surface of the matter distribution.

In normal stars, where gravitational attraction is balanced by thermal pressure produced by thermonuclear reactions, pressure and density are always monotonic functions of the radial coordinate and besides that, the influence

of potential energy on the balance is negligibly small. This remains true also when the thermonuclear fuel is consumed and the star cools off by radiation. In this case the degeneracy pressure of electrons or neutrons takes over, to stabilize the star against further gravitational collapse. There may occur unstable situations, as e.g. in white dwarfs, where the degeneracy pressure decreases with increasing density, when electrons and protons recombine into neutrons. This leads to instabilities and stellar explosions, as we know them as supernovas. But finally every collapsing object can end in an equilibrium state, where degeneracy pressure of the constituting particles balances gravitation. There is no upper mass limit beyond which infinite pressure would be required. The occurrence of an horizon and of singularities in GRT is only the consequence of an improper definition of the energy tensor.

## 4 Discussion

In the last section we have demonstrated that an equilibrium state of collapsed matter will always be outside the singular conditions assumed in the Schwarzschild geometry. There is no horizon, from which no escape of matter or radiation is possible and black holes as singular points of infinite density cannot exist. Instead every spherically symmetric gravitational collapse can find a final equilibrium state of finite density. No matter is inevitably lost from the surrounding space.

Of course, by now we have only discussed equilibrium configurations. In nature such idealized equilibria are scarcely reached. Most collapsing objects continuously accrete matter from surrounding space. Besides in most cases there is no spherical symmetry. Accreting objects will accumulate some angular momentum and the inflow of matter may be concentrated to the rotational plane. Magnetic fields can also influence the balance, if the particles are electrically charged.

Thus during the formation of collapsing objects unstable situations may occur, leading to expulsion of the outer matter shell or to the nearly complete disruption of a star. But in a final state, when degeneracy pressure dominates the dynamics, kinetic pressure of incoming matter is negligible. It is only rotation of the complete system, which may influence the dynamical equilibrium. By principle rotation velocities may be close to the velocity of light, so that the rotational energy is comparable to the rest energy of particles, just like the degeneracy energy. In this case pressure is no longer isotropic. Inflow of matter near the rotational plane will cause an outflow along the axis of rotation. The observed formation of cosmic matter jets from collapsed stars or active galaxy cores can be understood more easily, if we take as a fact that the 'black hole' in the center is not a matter concentration on the other side of some semi-permeable horizon, but an accumulation of nuclear matter, which can be recycled into the universe under suitable conditions. This is of essential importance to understand the physics of the matter jets emerging from the supermassive 'black holes' in the cores of active galaxies. Detailed modelling of



these cores is still missing. But to understand them, the first requirement is to start from the correct balance equations.

Neglecting potential energy in the balance equations appears as a general problem in the conventional methods of general relativistic modelling. In the description of the global dynamics of the universe, in addition to the search for dark matter, people are looking for the so called dark energy, which is necessary to bring the theoretical model into agreement with observations. This dark energy should be present throughout the universe and exhibit a negative pressure and an energy density comparable in order of magnitude to that of matter. Potential energy of matter itself just fulfills all these requirements. In a homogeneous solution of the Einstein equation it would look just like a cosmological constant, with the only difference that it is not a true constant, but varies with the matter density. No mysterious dark energy is necessary to fulfil the balance. Potential energy of matter itself can do the job.

It should be mentioned in this context that with this additional term instead of a cosmological constant the static universe proposed by Einstein would be stable, as with varying radius  $a$  of the universe the potential energy varies as  $\rho/a$ , so that a virtual increase of  $a$  would produce a negative  $da/dt$ . The entire universe would be the only system which exists at its Schwarzschild radius, the state at which the total matter energy is balanced by its own potential energy.

By now we have no secure confirmation that general relativity delivers the correct description of the universe, but with the corrections discussed in this paper some of its shortcomings, the existence of singularities and the missing explanation of the mysterious dark energy are automatically resolved. Also the order of magnitude agreement between dark energy and matter density can be understood. The only thing we have to do is, to accept the principle of energy conservation not only locally but also on macroscopic and global scale, independent of the geometry of space.

## References

- [1] Wald, R.M., General Relativity, 1984, Univ. of Chicago Press, Chicago, USA
- [2] Hawking, S.W. and Ellis, G.F.R., The large scale structure of space-time, 1973, Cambridge Univ. Press, Cambridge, UK
- [3] Schwarzschild, K., 1916, Sitzungsberichte der Koeniglich-Preussischen Akademie der Wissenschaften. Reimer, Berlin, S. 189-196.
- [4] Oppenheimer, J.R. and Volkoff, G.M., 1939, Phys. Rev. 55, 374 .