What is Beyond Reckoning?

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Abstract

Questions arise both in the physical and abstract reality of what is decidable or undecidable, what is computable or uncomputable, and which events are predictable or unpredictable. This is connected, in some measure, with questions in Mathematics of what is solvable conventionally, what can be computed through extended procedures, and what remains entirely beyond solution or computation. But similar questions arise in Physics; in part because indeterminacy at the Planck scale and quantum uncertainty even in large macroscopic systems force us to redefine what is possible to know with absolute or reasonable certainty. So this essay examines how we may find our way through some barriers to knowing or solving and come to grips with the full extent of what is truly beyond reckoning.

Keywords – indeterminacy, uncertainty, undecidability, uncomputability, unpredictability

Introduction

When deeper questions of what can be known are considered; we see that while we can know much with certainty, and there is a lot we can know something about, there remains a wealth of information we can never know, and some things which cannot ever be known in a distinct and precise way even in principle. Sometimes the correct answer is "it depends…" because we face imprecise or misleading questions and variables or conditions that cannot be pinned down simultaneously. So it is very important to delineate what we can know with certainty and exactitude, what we can know in part or can find out by making certain assumptions or using extended procedures, and what can never be learned in full or is beyond determination. Such questions arise often in both Mathematics and Physics, so it is wise to understand there are things we can know, things we can yet learn, and things we can never know with certainty. Of course; new discoveries afford a perspective we did not have, which may expose things we thought unknowable and pave the way for further advancements, but in many cases discoveries require someone to have a new perspective already, for advances to be made at all. Things requiring proof or explanation by one person may be intuitively obvious to another, however. So varying degrees of proof are required and various evidence procedures serve to validate facts or ideas for a range of people trying to learn the same things. This is one of the reasons scientists look to experimental evidence and empirical data to find answers and validate facts or novel ideas.

Ignoring questions about mental quickness or perceptual acuity; we still find limits to what we can know. There will always be undecidable yes/no questions, uncomputable algebraic or numerical solutions, and unpredictable events or outcomes. This is true in both Mathematics and Physics. This means there will always be more to learn, no matter how much we find out about the universe. This gap between what is possible and what is strictly determined may also be what allowed life to emerge. So it is not entirely a bad thing. And it could be of even greater significance as the ultimate cause of existence, because the universe needed a certain amount of freedom to vary before familiar conditions could emerge. Fortunately; the conditions at the Planck scale afford such freedoms, but only in the smallest possible increments. Ambiguity and variability at the Planck scale are seen to become precisely determined quantities and definite qualities or forces at the common scale in the current era or present-day epoch of the cosmos. This is agreed on by most physicists, but there is some disagreement about the details of how it occurred, especially when discussing what happened in the earliest phases of cosmological evolution. I will present views in this paper largely in agreement with what I said in previous FQXi essays. But I will incorporate what I and others have learned in the meantime, wherever possible. My research into the Physics applications of the Mandelbrot Set affords many examples for explaining the limits of what is knowable both in Physics terms and as pure Maths. So I expect this paper to be an exciting exploration of what is possible to know, and what is beyond reckoning.

What Converges or Condenses?

When considering what is possible to know and what in Math is important to Physics, an answer to both questions is 'that which converges or condenses into congruent forms.' My research suggests this is a general pattern guiding physical law [1]. Many people in Physics take a limiting view of this notion, though, looking only at ideal conditions (like isolated systems at equilibrium) where the Physics reduces to easily-solvable linear equations. But when physicists or engineers ignore non-linear aspects of physical systems; it is at their own peril (and sometimes that of others). There are many systems in nature where non-linear elements play a vital part, and few real-world undergrad textbook examples, which makes those textbook cases misleading for average people, and leaves society ill-prepared for the level of uncertainty folks find in real life. But we live in a condition that could not exist except that a large part of what is real *is* regular or predictable. We reside in a middle ground between fixed realities and variable conditions, over which we have no control. This middle ground is like a shoreline or estuary, in some ways, because it is in the boundary regions where life flourishes in the greatest measure – so these regions form a 'Goldilocks zone' that is friendly for life. Physics has arguably been more concerned until recently with discerning fixed laws than with understanding how things vary or how variations in general lead to stable conditions, and there is much we have yet to explore. We may have been looking only in a tiny pool on the shore of a great vast ocean.

I propose we examine how that which converges or condenses – in general – shapes what is studied in Math and what becomes Physics in the natural world. A simple example will suffice to start. We will see what happens when certain mathematical operations are repeated. We note that 1 is an identity under multiplication, which means that multiplying 1 by itself (squaring) yields 1 as a result. It is stable at this value when the process is repeated. If we start with a number larger than 1; the result is a larger number still, and if we multiply this number by itself the result is still larger. So we say the result diverges under iteration when squaring any positive real number larger than 1 and it converges for positive real values less than 1. If the value starts at 0; it will remain 0. So iterated squaring separates the number line into convergent, stable, and divergent initial values. The result shrinks toward 0 for smaller values, remains the same at 1, or increases in magnitude with each iteration for larger values. And we find that the sign does not matter, because an initial value of –1 is also stable (after the first inversion) when squared repeatedly, so that all the negative reals with a magnitude less than 1 also converge, while those more than 1 unit from the origin grow to infinity when iterated. Points on the number line between –1 and 1 converge, while points further from 0 diverge. This example can be extended by continuation, if we use complex numbers instead of reals, and the boundary of convergence at a distance of 1 then becomes a circle of radius 1 around the origin.

The expression for the above example is $z \to z^2$ where $z \in \mathbb{R}$ in the first case and $z \in \mathbb{C}$ in the second case. As stated; this formula converges for all values $-1 \le z \le 1$ and diverges outside this range of real values. In the complex-valued case; a circle $r = 1$ defines the range of convergence for the same iteration formula, $z \to z^2$. We note for future reference that $r = 1$ also defines a unit sphere and a range of higher-dimensional spheres or hyperspheres – as well as a circle and a pair of bounding points on a line mentioned above. We note for the present discussion that $z \to z^2$ is a minimal formula involving multiplication alone or only. This is almost the formula for the Mandelbrot Set – a minimal formula using both multiplication and addition written $z \to z^2 + z$. In this case; we take the initial value as any location on the complex plane ($z \in \mathbb{C}$), multiply that number by itself, then add back the initial value for that location. This process of multiplying a complex number by itself and adding back the initial value is repeated over and over, to find out where the function converges. But in the case of the Mandelbrot Set (i.e. – for $z \to z^2 + z$); the outline of what converges is not a circle but a very complex figure indeed, one of the most complex objects known in Math. So for starters; let us look at what happens along the real axis, or on the real number line, for $z \to z^2 + z$. We find –2 is the bounding point for the negative reals, because any negative number with a magnitude over 2 goes to infinity. But for positive reals; initial values larger than 0.25 go to infinity rapidly.

This marked difference between the behavior for positive and negative reals is one of the most striking features of the Mandelbrot Set (or $\mathcal{M}$). It is as asymmetrical as it could possibly be, where both ends point toward the extremum at (–2,0*i*). But while it is globally asymmetric, $\mathcal{M}$ contains a host of structures with perfect symmetry in every possible variety. And it exhibits perfect mirror symmetry across the real axis, as well. So we find an interplay of local symmetry and global asymmetry in $\mathcal{M}$, which offers many insights into various aspects of Physics – especially the nature of symmetry breaking. One area of Physics for which the Mandelbrot Set provides useful insights is the study of gravity. Though it is ubiquitous; gravity remains poorly understood, when we attempt to unify our understanding of gravitation with the Standard Model of particle Physics and our knowledge of the other fundamental forces. Multiple theories of gravitation are simultaneously represented in $\mathcal{M}$, [2] so it offers researchers new ways to knit the theoretical structure together.

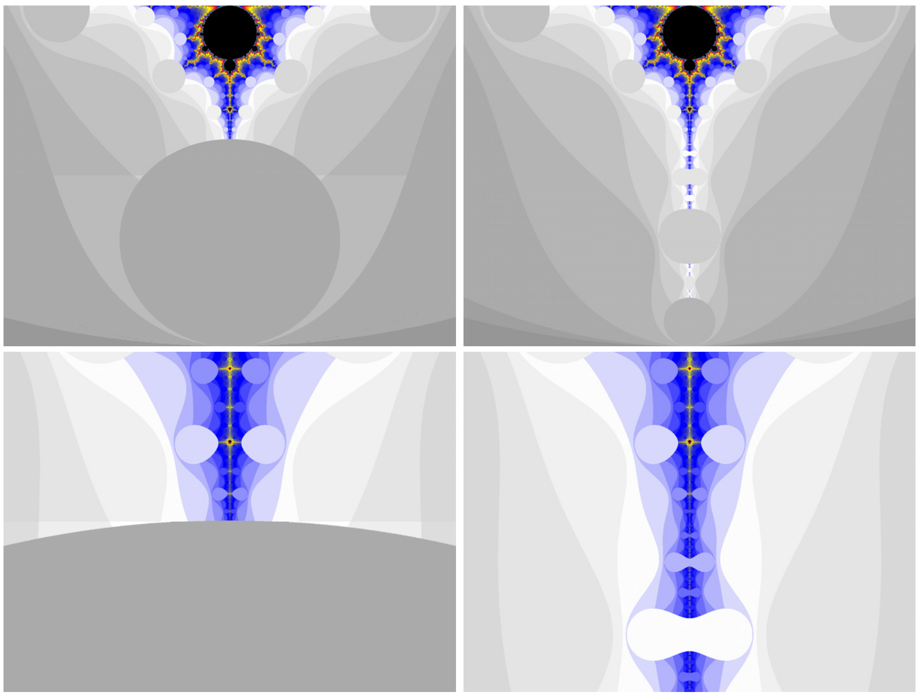

Fig. 1. Misiurewicz point $M_{3,1}$ illustrates a Schwarzschild Event Horizon (on left) and Bose-Einstein Condensation (on right) in the Mandelbrot Butterfly figure.

A location in $\mathcal{M}$ of special interest for gravitation research is a Misiurewicz point about halfway down the spike or antenna in the figure's tail, at about (–1.543689,0*i*). It is designated $M_{3,1}$, because its pre-period is 3 (the delay before repeating) then it repeats every time. This point is seen in my research to represent both a Schwarzschild event horizon and the onset or quantum critical point of Bose-Einstein condensation. The visual evidence is striking. When I first saw these phenomena visually connected in $\mathcal{M}$ and its family of associated figures; I was quizzical because it was hard to imagine that the action of gravity is like BEC formation. But I later found out that this idea has been extensively explored by a large number of researchers [3], going all the way back to Sakharov [4] in 1967. More recently; experiments using BECs as a black hole analog [5] have demonstrated similar properties to what we expect real black holes to have. So finding this analogy represented in $\mathcal{M}$ and in the Mandelbrot Butterfly figure (seen by coloring in where the iterand magnitude monotonically diminishes) at $M_{3,1}$ suggests this point is worthy of further study. But even though it is one of the lowest-order Misiurewicz points in $\mathcal{M}$; $M_{3,1}$ is one of the very few such locations we can analytically calculate solutions for. This point satisfies the equation $((c^2+c)^2+c)^2+c=(c^2+c)^2+c$, where 3 iterations of the Mandelbrot formula on the left hand side are set equal to 2 iterations on the right. We can obtain exact algebraic solutions and precise numerical values for this equation (see endnotes), to any degree of accuracy.

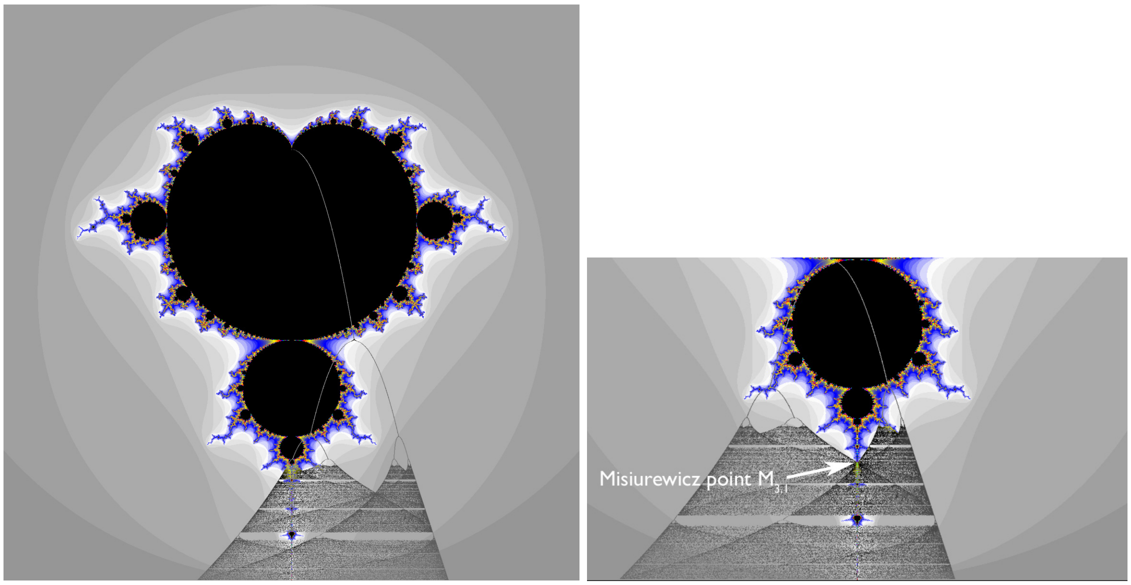

Fig.2. The Mandelbrot Set with its bifurcation diagram overlaid is magnified and shifted in the inset, showing that $M_{3,1}$ is the band-merging point.

This location is especially interesting in the study of complexity and unpredictability because it is maximally dense in probabilities, when the bifurcation diagram for the quadratic Mandelbrot function ($z \rightarrow z^2 + z$) is plotted. Peitgen and Richter [6] cite Grossmann and Thomae [7] who describe it as a 'band-merging point.' Overlaying the bifurcation diagram on the image of $\mathcal{M}$; we find that the trajectories split everywhere the Mandelbrot Set folds back on itself, along the real axis. But at the Misiurewicz point $M_{3,1}$; trajectories from all of the separated branches come together, all in one spot, at this location in $\mathcal{M}$ near (–1.543689,0*i*). This makes that feature easy to identify. But this location in $\mathcal{M}$ is otherwise innocuous and easily missed. Like all Misiurewicz points; the scale factor goes to 0. We see two rows of 'telephone poles' shrinking to extinction along a straight stretch, then appearing and growing in opposite sequence afterward. We can see $M_{3,1}$ as a 0^{th} order branching point, with only one incoming and one outgoing branch, while all others have several or many outgoing branches. And it is the lowest-order inflection point, where the phase or sense of the form beforehand reverses after the point of extinction is passed. The third type of Misiurewicz point is a terminal point – the end of a branch or tendril. The scale factor goes to zero, and there that thread simply stops. The extremum of $\mathcal{M}$ at (–2,0*i*) is the lowest-order example of this. And as it turns out; $((c^2+c)^2+c)^2+c=(c^2+c)^2+c$ has two complex-valued solutions that are the extrema of $\mathcal{M}$ in the $\pm$ imaginary direction – the terminal points at about (–0.2281555,1.1151425*i*) and (–0.2281555,–1.1151425*i*).

The prior discussion shows there is a Math context for containers and things that are contained. Similarly; there are definers and things that are defined. These concepts are useful in Physics, and they help us get a handle on what it is possible to know with certainty. Our bodies are containers, as are atoms and sub-atomic particles, but not everything in nature can be contained. There are rules or laws that shape nature, which are the definers of reality. But not everything theoretically possible can be strictly defined, nor can all that is physically realizable be localized in space and time. And yet; what is familiar in the condensed matter universe we live in is the part of reality that *can* be contained or *is* strictly defined. So many people have come to believe anything that exists can be known, and anything known can be verified true. Unfortunately; reality is at its root more fuzzy or uncertain. Heisenberg discovered that total measurement uncertainty cannot be reduced beyond a certain limit, and that attempting to probe beyond this limit only results in a trade off between one kind of knowledge or information and another. A more precise measure of a particle's energy, beyond that point, results in what he called an 'unsharp' measure of its position, where the total sharpness of the combined measurements is limited by Planck's constant [8]. It is a hallmark of Quantum Mechanics that uncertainty is a property of all physical systems, when measured at a scale where the inherent graininess of the universe (and so its quantum mechanical nature) becomes apparent.

This may in fact be true *because* variability is a necessary precursor to fixed natures, where variation and the capacity to vary give rise to fixity. This variability appears to be woven into the very fabric of spacetime, because quantum uncertainty at the Planck scale gives rise to spacetime quantization (of some nature) in virtually all Quantum Gravity theories. So all that exists in space and time partakes of what we call quantum-mechanical nature, and is thus constrained to vary in discrete amounts – when viewed from a localized framework. Thus matter and energy appear to operate in a quantum-mechanical arena of space – when the flow of time is treated as constant. The main difficulty theoretical physicists face when crafting a unifying theory of Physics is that in Relativity time is flexible, while in QM it is absolute. The need to reconcile these theories is the main incentive for quantum gravity theorists and the reason so many worthy approaches to the problem have been spawned. The amazing thing is that there are similar findings and common elements [9] among those various approaches. In the realm of the very small, near the Planck scale, it is seen in several cases [10] that spacetime reduces from a 4-d expanse to a 2-d surface with one space-like and one time-like extent. This may explain why time and space are equally-weighted or indistinguishable at the common scale. But the Planck domain is mainly comprised of uncertainty or indeterminacy, which is the freedom to vary.

The freedom to vary is encoded in Math by the imaginary numbers, which represent a specific amount of variational freedom, instead of a specific fixed value. So a value of i or $-i$ represents the freedom to vary by $\pm$ one unit from the origin, in a particular direction, while a value of $2i$ or $-2i$ means the freedom to vary by 2 units, and so on. The complex numbers are constructed of one imaginary part and one real part, so they reduce to the real numbers when variation (the imaginary part) is constrained to 0. As we saw in the earlier example (with $z \to z^2$) a value of 1 continued into the complex domain defines an axis of rotation, so paired with a real each imaginary is seen as rotation in a particular direction. This is important because just as the reals are a subset of the complex numbers, the complex are a subset of the (4-d) quaternions, which are a subset of the (8-d) octonions. We can write that out like this:

$$\mathbb{O} \supset \mathbb{H} \supset \mathbb{C} \supset \mathbb{R}$$

This means that the octonions, with 7 imaginaries and 1 real part, reduce to the quaternions (with 3 imaginaries *i*, *j*, and *k*) when we fix 4 of 7 axes of rotation, they reduce to the complex numbers (with 1 imaginary part) when we fix 2 of the remaining 3, which reduce to ordinary real numbers when we constrain the final axis.

The reason it is important to examine these higher-dimensional cases is that we know things are more unconstrained at the Planck scale, even while what is determined is lower-dimensional or simpler in structure. The explanation is found in a brief conversation I had with Tevian Dray at GR21 [11], where he confirmed my wild suspicion that it is unavoidable, where below perhaps 10^{-12} cm. we must assume geometry is non-commutative, and as we approach the Planck scale we need to use non-associative geometry. In the last example; the octonions are non-associative, and the quaternions are non-commutative, while only the complex and real numbers follow all of the familiar laws of algebra. So we can think of the familiar territory in both Math and Physics as a subset of a larger vocabulary of natural variables and phenomena. The idea of a restricted subset resulting in forms that are more well-defined is seen often in pure Maths as well as in Physics. With the caveat that we are looking only at those objects and spaces of relevance to Physics; we can write:

$$Smooth \supset Top \supset Meas$$

Smooth forms are a larger category. When objects acquire a surface they become topological, and with specific attributes like size they are measurable. This explains the earlier discussion because associativity pertains to what is contained, or to containers, and commutativity is what makes size and distance measures absolute. So if we are connecting this back to the realm of natural objects in Physics; we can write:

$$Gas \supseteq Liquid \supseteq Solid$$

What is presented above is meant to show that what is familiar in nature might arise from conditions that are very unfamiliar, because in the earliest phases of cosmology near the Planck epoch, the things we are familiar with had not solidified yet. It is arguable that the laws of associativity and commutativity that rule familiar objects and spaces are in fact emergent [12]. If String Theory or one of several alternatives is true; we must presume that familiar Physics occurs in a higher-d embedding space. Theories like DGP gravity [13] or Cascading DGP [14] suggest our cosmos is higher-dimensional at great distances, where farther implies an earlier era, and some suggest a black hole in a higher-d cosmos [15] created the 4-d bubble we live in. So if our present day cosmos is the product of a higher-d precursor, or lives in a higher-d space; perhaps the universe is a literal embodiment of 'that which converges or condenses into congruent forms.' This makes cosmology a bit like a process of fractional distillation, where the entirety of the condensed matter universe is only the denser portion of reality with fixed attributes, the lowest fraction.

A Vast Landscape with Predictable Islands

While it is uncomfortable to imagine that uncertainty and unpredictability are the rule, or to see all fixed attributes as the settled product of variations that assumed a particular value; the reality in both Math and Physics is that what is relevant or real arose from a larger spectrum of what is possible. And it arose by a process where what is indefinite gives rise to well-defined things. It is only natural that the freedom to vary is captured in real-world forms, as quantum uncertainty, because this attribute is a necessary piece of what allows the universe of extended forms to exist at all. With all that is predictable or certain in life; it is hard for some people to accept that much of what happens in our lives cannot be pinned down or foretold with accuracy. In Evan Pritchard's book "No Word for Time" [16] we are told that in Native American culture, preoccupation with the clock and calendars is seen as part of "white man's foolishness" because things take as long as they take, and you can't go on to what's next until you are done. But that age-old wisdom is lost in modern culture, at least on people who build their lives around a tight schedule. And yet; Turing's work on the 'halting problem' [17] proves that the indigenous sensibility is more true to fact than what modern people believe. And it has more general implications, because in any process; as we add complexity we increase the likelihood it will eventually fail. So the fact complex systems afford more possibilities comes at a cost of our losing predictability.

This means that now there is a growing knowledge gap because as things become more complex; more things cannot be known, more facts cannot be proven, and more proofs can never be tested. There are hard limits even in the pure abstract, to what can be computed, and I hinted at one example in the earlier discussion. Mario Livio's book "The Equation that couldn't be solved" [18] tells about how Abel and Galois separately showed that there is no general solution to the quintic equation [19], but in so doing elucidated the mathematical language of symmetry and the general theory of how algebraic equations are solved. If one continues adding terms, to calculate Misiurewicz points beyond $M_{3,1}$; one quickly runs into this limitation (by obtaining equations of high degree), to discover that only a handful of locations in the Mandelbrot Set can be computed exactly. Numerical approximation is still possible, so long as you can obtain sufficient precision by zooming in with that spot centered on the screen. But this method breaks down too, because when the pixel size is less than the smallest number that can be represented in the computer's memory; the image becomes blotchy or blocky due to 'binary decomposition.' So even though we might know about places of interest in $\mathcal{M}$, visible only at extreme magnification, the calculation breaks down at some point. While we can then use special hardware or software to utilize extended-precision variables throughout; we still run into a hard limit set by the memory capacity of the machine, or a practical limit because the calculation takes too long.

There is much we can learn by inference and analogy, despite our inability to find exact solutions for some problems. In the Mandelbrot Set; there is a lot one can say about any given location, in analogy to other familiar features or points of reference, but there are surprises too – where unexpected features will pop up if you zoom in far enough. This is true in nature as well, and it is one of the reasons we continue to build and upgrade particle accelerators – to achieve higher and higher energies. It is not the case that we only want to see those particles which we expect to find, or are looking for as confirmation of various theories. Instead; we are hoping to see some surprises that will help settle the case between competing theories, narrow down the field, or point in new directions for Physics. In his colorful anecdote of a meeting with Margaret Thatcher at CERN; theorist John Ellis told us at FFP11 what he told her [20], that we won't learn anything interesting if we find only what we expect to see, in the search for new particles. Here too; there are energies our best equipment will never allow us to probe – no matter how large an accelerator is built. We cannot hope to get near the Planck scale, using current technology, because it is orders of magnitude beyond what we could achieve even with an accelerator around the equator of the moon. But we know nature does create such extreme conditions near the rim of black holes, and very high energies in other astrophysical events, which can be studied.

I think we should continue to build better accelerators and find other ways to explore the high-energy regime. But we should not narrow our search so much, by looking for specific particle candidates, that we exclude evidence of new Physics. A tremendous volume of data is generated, both by accelerators and in astrophysical observations, and much of it is pre-sorted by necessity. However; at this point the sheer volume of collected data is also a barrier to knowledge, because it must be stored and then searched through later to find the bits of information (buried deep within) that contain what we want to know, or allow us to spot a pattern in the data signifying something interesting. So even if that data was kept during the pre-sort; it may still be difficult to obtain because there is so much to sift through and limited time is available. This is perhaps the most vexing problem of all, that we know there is something out there – or in there – waiting to be discovered, but to get the answer would require more waiting time than we have. So time is the ultimate barrier to knowledge, since it is the one limitation we can never hope to overcome. This drives the search for more efficient algorithms or methods to sort things. The search is 'always on' in Mathematics for both maximal and minimal cases, and for proofs that those examples are indeed the largest or smallest. In Computation; there is always a hunt for a quicker or simpler way to achieve a given result with repeatability, and sometimes there are indeed easier or quicker ways to know things. But there is always a limit to how quick or easy it can be to learn something, and we have only so much time to learn.

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Note: the Endnotes on 'Solutions for $M_{3,1}$' previously appeared in *Prespacetime Journal* **10**, 8, 2019

Endnotes

The remarkable formula $r = 1$

The utility of the formula $r = 1$ is under-appreciated. It is most familiar as the equation of the unit circle centered at the origin. But it also encodes the unit sphere, when applied to a 3-d volume instead of a flat plane. Furthermore; it represents the entire family of unit spheres in higher dimensions we call hyperspheres. The lowest-order hypersphere is the 4-d sphere which corresponds to the quaternions, and it has a Möbius-like surface that is seamlessly folded back on itself (or into itself). The unit spheres increase in content or contained volume, going from lower to higher-d, until we reach a 5-d hypervolume, and they shrink when we add yet more dimensions. However; the surface or hypersurface continues to grow until we reach 8-d, and shrinks thereafter. The common sphere in 3-d space is called a 2-sphere, because it has a 2-d surface. A circle is also known as a 1-sphere, because it has a 1-d boundary. So a unit 0-sphere is the pair of points at 1 and –1 we talked about earlier. And the 7-sphere is a round figure in 8-d space, corresponding to the octonions, which like the 3-sphere, has a Möbius-like surface. We note that in all cases, the surface of a sphere is one dimension lower than its enclosed (hyper-) volume. So a 4-d sphere has a 3-d surface, and so on for yet higher dimensions.

Solutions for $M_{3,1}$

The generating formula for the Mandelbrot Set is $z \to z^2 + z$ where $z \in \mathbb{C}$. We expect that formulae for Misiurewicz points will contain similar terms, and the formula for $M_{3,1}$ is $((c^2+c)^2+c)^2+c=(c^2+c)^2+c$ * which has 3 iterations of the Mandelbrot formula on the left hand side and 2 iterations of that formula on the right. Expanding this out; we see it is an equation of high degree $c^8 + 4c^7 + 6c^6 + 6c^5 + 5c^4 + 2c^3 + c^2 + c = c^4 + 2c^3 + c^2 + c$. Solving for c, we find a trivial solution at $c = 0$, and another solution at the terminus or tip of the main antenna $c = -2$. There are two non-trivial results with imaginary components:

$$c = \left(\frac{\sqrt{11}}{3^{\frac{3}{2}}} - \frac{17}{27}\right)^{\frac{1}{3}} \cdot \left(\frac{\sqrt{3}i}{2} - \frac{1}{2}\right) - \frac{2\left(-\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)}{9\left(\frac{\sqrt{11}}{3^{\frac{3}{2}}} - \frac{17}{27}\right)^{\frac{1}{3}}} - \frac{2}{3} \quad \text{and}$$

$$c = \left(\frac{\sqrt{11}}{3^{\frac{3}{2}}} - \frac{17}{27}\right)^{\frac{1}{3}} \cdot \left(-\frac{\sqrt{3}i}{2} - \frac{1}{2}\right) - \frac{2\left(\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)}{9\left(\frac{\sqrt{11}}{3^{\frac{3}{2}}} - \frac{17}{27}\right)^{\frac{1}{3}}} - \frac{2}{3}.$$

These solutions are on the 'ears' of the cardioid, at the tip of a tendril, and they are the maxima of $\mathcal{M}$ in the plus or minus imaginary direction – at about (-0.22815549365396, 1.115142508039938*i*) and (-0.22815549365396,-1.115142508039938*i*) – the points farthest from the real axis (which have the greatest imaginary extent). But the solution we seek is the point about halfway out on the main antenna or spike, a non-trivial result that is purely real (i.e. - $c \in \mathbb{R}$ a point on the real axis). This solution is:

$$c = \left(\frac{\sqrt{11}}{3^{\frac{3}{2}}} - \frac{17}{27}\right)^{\frac{1}{3}} - \frac{2}{9\left(\frac{\sqrt{11}}{3^{\frac{3}{2}}} - \frac{17}{27}\right)^{\frac{1}{3}}} - \frac{2}{3}$$

and we note this can be simplified. Since $3^3 = 27$, the first segment becomes $\frac{\sqrt{11}}{\sqrt{27}}$ and using the fractions of roots rule this becomes $\sqrt{\frac{11}{27}}$, which makes the solution a bit neater and easier to calculate numerical results from. So then:

$$c = \left(\sqrt{\frac{11}{27}} - \frac{17}{27}\right)^{\frac{1}{3}} - \frac{2}{9\left(\sqrt{\frac{11}{27}} - \frac{17}{27}\right)^{\frac{1}{3}}} - \frac{2}{3},$$

and from this we can obtain numerical values of any desired precision, such as the 600+ digit answer below.

-1.5436890126920763615708559718017479865252032976509839352408
0403783116867392797386648515791457605912546212082922636706018
9278756463322141011522909218905292722992781697157582950422
1700895185634107003852001280002823664772617799869089968841046
1497697692708684796961197892018144612899134858559214530952415
3482664595734117098522176434389326263543393937637861596321
991246778909541806274987275657314799920191066093455727443186
4097999115541846773175410068470881770675967449333915590370244
900727438420939476242994768580218142366808917653874600291775
6360822043580399278877782055605036750966843145258100066047992
3719409341669347

The coastline paradox

A paper by Mandelbrot in 1967 (*Science* **156**, pp. 636-638) asked "How long is the coast of Britain?" and explored observations by Richardson (1961) that the measured length grows when using a smaller measuring rod, depending on the roughness of the terrain. This work shows that a coastline has a fractional dimension and its measure *D* allows us to gauge how rough or smooth the terrain is. This illustrates how fractal boundaries are a supportive environment for living things, by offering a blend of fixed and varying natures.

* - see Peitgen and Richter "The Beauty of Fractals" pg. 59 section 4.22 for more details.