Putting the Elephants to Work

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Abstract

A conversation with Tevian Dray, on the last day of GR21, brought to light some of the 'elephants in the room' with quantum gravity researchers and highlighted the essential part played by non-associative Maths – in any robust formulation attempting to address those looming concerns. While this is seen as a complication by most researchers; I assert that it is instead a clear road to a solution or family of solutions to long-standing problems in Physics. This lends support to the author's idea that nature employs the totality of all Mathematics – discovered and undiscovered – in its handiwork, such that invariant realities in Math spell out their own importance to Physics, and give rise to the universe we see today. As those same mathematical invariants unfold in the universe over time, they also give rise to processes that make the appearance and evolution of life inevitable.

Introduction

Imagine having to calculate in Roman numerals. It makes many of our everyday calculational tasks insanely difficult or onerous. In contrast; working in binary makes pedestrian calculations a breeze – so it is easy to see its value in computing. However; most of us work with Arabic numerals in a base 10 system out of convenience, and because of their utility as a representational schema. Modern humans are mired in a range of assumptions and conventions in Math that were adopted because they work well within the areas of application where we have primarily tried to use them, but they are not set in stone. Mathematicians know that there is a larger palette, but also see a kind of hidden order that appears when various different areas of investigation touch on the same structure, which I call the mathematical invariants – that extend Plato's archetypes of form in various ways. The list includes beasts like the Monster (Fischer-Greiss) group, the Mandelbrot Set, E8, extended family like E7, E6, F4, G2, and even the humblest constructions like simplexes or simple numbers like pi. The idea is that when one considers the totality of all Mathematics, certain patterns and ideas stand out as recurrent themes that appear regardless of the context or application. In my view; the laws of nature arise largely because the Math has its own ideas about what is relevant to Physics, and also engenders the evolution of form that is capable of consciousness and volition. So in this essay; I will spell out, in some measure, what I think Mathematics is telling the universe to create.

Nature is not limited by our views about what in Math is relevant to physical law, nor is it partial to any specific patterning our Math conventions have introduced. Whether we calculate in Roman or binary is not relevant to how nature generates what we know as Physics. True physical law is not limited to what we find easy to solve formulaically either, nor does it make sense to assert that the real world is content to function within the space of Real variables. In my view; the Complex, Quaternion, and Octonion, algebras all play a part, but so does the reason why there are only four normed division algebras. I am not saying they are the only dimensional algebras of importance to Physics, as a broad variety of analytical tools and frameworks are used by physicists, and get employed constructively by nature in its handiwork. However; these algebras and other mathematical invariants exert influence early in the evolution of the Cosmos, and continue to shape every aspect of the universe we see today – especially when we extend our view into higher dimensions. So it can be said that the universe emerges because of Mathematics, where the totality of what Math says is considered – all at once [1]. One might add that Math makes the emergence of the universe and life inevitable. However; this emergence occurs largely because of some remarkable facts of higher Math that are often ignored by physicists.

A conversation with Tevian Dray, on the last day of GR21, confirmed my deep suspicions about the 'elephants in the room' with quantum gravity researchers, while giving me hope to address some long-standing problems in gravitational Physics – by putting the elephants to work! To be more precise; I assert that nature puts the looming realities of higher Math to work automatically, because they help set nature in motion, and keep it on course toward ideal or optimal goals. This remains hidden to most of us largely because we are preoccupied with the specificity of real numbers – simple quantities that do not change – while the complex and hyper-complex (quaternion and octonion) numbers encode degrees of variability or variation – the properties that engender dynamism – and are in fact more general number types. In my view; nonzero ordinary or simple numbers are the end of the process chain, and measurable quantities are evidence that variation has taken place to create a particular result. To restate this; for a specific entity or quantity to exist, there must be a process by which it arises, or it must be constructible in some manner. But in the current era; we are surrounded by a whole universe of form with well-defined properties. So it is easy to forget that every object is the product of a process by which it came to be, and that ongoing processes maintain its form – giving it specific properties.

Researchers know that the rules change, when examining the smallest levels of scale, especially near the Planck scale. The laws of geometry become first non-commutative and then non-associative – as we approach the limits of small size – requiring us to redefine first size/distance, and then interiority/exteriority, as relative instead of absolute quantities. This also introduces a kind of dynamism. Alain Connes, one of non-commutative geometry's foremost experts, said "Noncommutative measure spaces evolve with time!" [2] and suggested this is a template for other similar statements about NCG. What do we know about

non-associative geometry, in this area? Paul C. Kainen said of the octonions "Of course, multiplication in the octaval arithmetic fails to be either commutative or associative, but that could be a blessing in disguise. If multiplication depends on the order of the elements being multiplied together and even on how they are grouped, then at one fell swoop, geometry enters the calculation in an organic way. The Principle of Indeterminacy could then arise in a natural fashion from relativistic considerations, making quantum theory a consequence of an underlying 8-dimensional hidden-variable process, very much in the flavor of the theories of de Broglie and Bohm." [3] So we see that octonion Math dictates emergence.

Once we arrive at the octonions; the mathematical laws are anything but aimless, and instead display a kind of directed sequential evolution – both in the calculations themselves and in the character of the spaces or physical laws they engender. However; this is not an isolated example, but instead it is a broad feature of higher-order algebras, which has some interesting implications for Physics. We are forced to give up our view of entities as fixed quantities, and to regard them constructively – where at its root or core, every thing is a process. To imagine the real world is dominated by the real numbers is to ignore the 'elephants in the room.' One can think of the reals as a special case of the complex numbers, which are a subset of the quaternions, which are a subset of the octonion number set – that is the most general regular algebra. But working in the most general case, one must throw away both the commutative 'law' and the associative 'law' that we all learn in elementary Algebra – because these rules do not apply – and we have to carry out all the individual terms longhand, in the correct order, until the last step. This takes all the fun out of Algebra for those who have learned to rely on every handy shortcut!

However; it will make things more fun for Physics, when physicists start to grapple with some of the larger problems – attempting to shed light on the 'elephants in the room' with researchers in quantum gravity and other topics. It is clear that nature plays with a larger palette of options than physicists typically explore. Nature has been using for eons Math we are only discovering today, and it employed higher-order algebras in its handiwork long before we discovered their properties. In my view; it is natural that the octonions would come into play before the reals, in the early cosmos. Octonionic Inflation avoids some of the problems with conventional Inflation cited by Steinhardt and colleagues [4], and it is a subject of my current research. But in the present era; it seems more natural to base Physics on the real number Calculus – until attempting to tackle some very advanced topics. However; this means those 'elephants in the room' could linger for quite a while, until physicists discover that they have been hard at work all along, invisibly creating the underlying reality upon which the visible universe rests. Of course; this would make us acknowledge that they are not mindless or aimless. To correctly employ the rules of non-associative Math requires one to be mindful and intentional, which suggests that a universe following those rules would give rise to forms with those qualities as well.

How big is the Playground?

We see the cosmos as 4-dimensional, with 3 dimensions of space and one of time, which in Special Relativity are unified and referred to as spacetime. But String Theory works best in 10 dimensions, where some are assumed to be curled up in a small space – and thus hidden from view. That higher dimensions should be small makes sense, when we consider the volume of *n*-dimensional spheres, because hypervolume declines for $n > 5$. [5] Adding extra dimensions past 5 makes them shrink. So in a 10-dimensional framework, some of the higher dimensions *must* be small, and can remain hidden from view at ordinary levels of scale. However; this information is lost, if one studies hypercubes instead of hyperspheres, because their volume increases with increasing *n*. The measure spaces are not the same, so we must be careful about what we are comparing. Exactly what is meant by large and small can be confusing; therefore, especially if the dimension of the space we are in is unknown. But nature works on all levels of scale and in all available dimensions at once, exploiting every niche. More properly; nature probes its boundaries, as though it is trying to determine the dimensionality of all the available spaces. The freedom to vary may be large in one dimension and small in another. So while *we* like to view reality as 3-dimensional, from the middle range of scale, *nature* is not limited or preferential to that point of view.

Richard Feynman once said "There's plenty of room at the bottom" [6] when speaking about the opportunity to create very small structures that exist in what is called the nanoscale. Indeed; inside and between every atom and molecule, there is plenty of empty space – and some room to fill with more structure. But scientists now think space itself must have a microscopic and quantum mechanical structure, like a fabric with an amazing weave that knits space and time together. The discussion is ongoing, about what the character of the fabric must be – to give the observed properties of physical phenomena – but the existence of a quantum mechanical microstructure in the spacetime fabric is widely accepted. So there is a limit to how much structure nature or scientists can fit into any space, and how much information it can contain. It appears that at the smallest level of scale, space behaves like it is 2-dimensional, which shows up in various quantum gravity formulations, including Causal Dynamical Triangulations [7] and Quantum Einstein Gravity [8]. This effect was first noted by Gerard 't Hooft [9], when exploring the connection between black hole event horizons and quantum gravity in 1993, giving birth to holographic universe theories. The reduction from 3-d to 2-d is such a defining feature of quantum gravity theories, though, it recently prompted Steven Carlip to write about "Spontaneous Dimensional Reduction" [10]. But this affects how much room there really is at the bottom of the size or distance scale.

Some would construe that the world we live in is actually 2-d at the root, where we reside in a 3-d hologram projected upon the 2-d screen of the true reality. But the 2-d reality is no more real than the 3-d one. 2-d and 3-d reality coexist, along with higher-d reality as well – so it is wiser to see dimensionality as a spectrum, rather than being fixed across all levels of scale. It appears that; at the smallest scale, we must imagine space to be both higher

and lower-dimensional – rather than having just one dimension. Nor are whole numbered values required, but dimension can vary smoothly from one level of scale to the next instead. Nature is not limited to what is convenient for human beings, because it resides in or is connected to all dimensions at once – without a fixation for a single dimensional description. But if a goal is to create forms that hold together over time, 3-d space is ideal because knots do not untie themselves. We see that nature takes advantage of this property, by populating the universe with a variety of 3-dimensional forms. However; the character of space has likely changed since the inception of the universe, and we have reason to imagine the cosmos' dimension continues to change or could evolve further. String Theory asserts that this spacetime fabric emerges from excitations in 10-d space [11], which still play a part today. So we must include higher-order Maths and spaces in our investigation.

This introduces some intriguing complications. Using the octonions as an example, we see evidence of both wandering and goal seeking in the algebra itself. Some experts state there are as many as 480 distinct multiplication tables for the octonions [12], all of which work so long as you follow one entirely throughout all calculations. Rick Lockyer points out that there is a lot of duplication in that list, but 16 fully distinct variations remain [13]. Only four tables suffice [14], or are in general use today, and most mathematicians whose work involves the octonions are content to work with one or two. But the fact there are options for how to do multiplication makes things very interesting. Once the calculation commences, however, one must follow the same pattern through to the end. We deal with three imaginary elements at a time in a specific order, and then work through seven stages of this in the proper sequence, before we get to combine any terms at the finish. So one must be patient and methodical – doing every step in the correct order and sequence – in order to make the calculation come out right when one is done. Working in higher-order algebras therefore requires one to work for a reward, and certainly teaches the value of delayed gratification. But since nature has been using higher-order algebras all along, it is reasonable to say that it understands this aspect of things, and teaches it as well.

Playing by the Rules of Evolution

While simple numbers are fixed entities, and simple algebras describe collections of things that just sit there quite well; systems that vary are a different matter entirely. The Calculus of Variations allows us to optimize for various qualities, and to find the maxima and minima of various quantities – within a spectrum of variations or possibilities. This allows us and nature to find ranges where stable forms can be constructed. But there are different orders of evolutive activity, where some actions have short cycles yielding more immediate rewards, while other actions create outcomes with longer periods or more stages toward completion required – before a creative cycle is complete. This longer view is a hallmark of higher-order algebras, because once higher-dimensional properties of a system are considered; non-linear dynamics are rolled in, which require us to address evolutive and sequentially

evolutive properties arising naturally, already at dimension four. More specifically; as soon as we correctly address the subtleties of Math that emerge because of non-commutative and non-associative properties – evolution will no longer be a mystery. We will see the Math requires it. And especially; evolution of consciousness or intelligence will be seen as a natural outgrowth of higher-order Math. However; this insight has already been foreseen or established by others over the course of time.

In conversations with Leo KoGuan and Brian Ji (participants in previous FQXi contests) [15,16]; it came to light that Leibniz wrote about the I Ching bagua and their relation to the binary numbers [17]. Closed and open lines in the trigrams translate into ones and zeroes. This proposes a link between numerical properties and the modes or moods of change, and their representation. We discussed that perhaps ancient Chinese mathematicians tried to incorporate secrets of Math into philosophical teachings, to convey a longer view of reality than is conventionally pursued – which is inspired by higher-order Maths. In the context of civilizations and their survival; the long view inspired by non-associative Mathematics teaches us how nature views evolution – and what is required to survive. While the earthly or human order is typically fueled by short-term gain, the divine creation or heavenly order has a longer period or greater number of steps leading to fulfillment, according to this philosophy. But people can learn; we can create in a way similar to the universal order using higher-order reasoning, which provides greater rewards in the long term, over lower-order reasoning for short-term gain. This is a more mature or adult view, but it requires either the wisdom of elders or knowledge of higher-order Math for most people to gain that insight.

The work of Arthur M. Young is a distillation of insights from ancient wisdom and higher-order Math. His book "The Reflexive Universe – the evolution of consciousness" [18] describes an evolutionary order including every aspect of the physical universe – from the sub-atomic to the mineral and then to the biological. The story begins the same as in Physics, where potential is bound into particles, which form atoms, which combine into molecules as the expanding universe cools. Young asserts that these material components need to reach a level of fixity in the mineral phase first, before life which allows the property of growth we see in plants and then the mobility we see in animals to be emergent properties – each building on the structure of previous generations. The subtitle "the evolution of consciousness," reflects his view that the process giving rise to the universe also requires the emergence of conscious entities capable of intelligent action – including intentions, aims, and long-term goals. According to Young; it is inevitable that living beings will evolve in consciousness to develop first knowledge, then wisdom, on the way to an enlightened state of being. His book depicts a process-theoretic universal order where natural systems evolve through seven stages, in each level of organization. While Young does not make that explicit; I see this pattern has deep connections with the octonions, which have seven imaginary dimensions that encode their sequentially evolutive properties. This is significant evidence that higher-order algebra drives evolutionary processes, in my opinion.

It makes sense that the secret of heavenly creation can be found in the Math of higher dimensions. But too many people – even in Physics – do not have the inclination to become well-grounded in the concepts that define higher-dimensional objects and spaces, though they do put them to use in theoretical constructions. Problems like the close-packing of spheres, one of the oldest problems considered by mathematicians in ancient times, are still relevant today – especially when extended into higher dimensions [19,20]. Now we have proofs showing why the E8 or Gosset lattice is optimal in 8-d and the Leech lattice is the closest arrangement of spheres in 24-d, as part of a growing list of exact values for mathematical invariants that affect Physics. We ignore facts like these at our own peril. Basic relations of geometry are part of the fundamental underpinning of universal law, because they don't go away if you can cast the problem in a way that hides their existence, or makes them appear less relevant. My conversation with Tevian Dray at GR21 confirmed my suspicions that the effect of non-associative Math is more pervasive, and the role of higher-order algebras like the octonions is more universal, than some researchers would care to admit. It is worthy of study, to examine how when we extend our purview into higher dimensions; we must consider that higher-order algebras contain non-linear elements that engender evolutive behaviors and sequential evolution, which can have short or long periods of cyclical development, and can build in stages toward a specific result.

Once a certain stage in a process has been completed, one can work through the challenges of the next stage, and only when various things have all been put in place can one embark on the final phase of a project. The challenge is to survive and to continue building upon what has been created, to create the next phase of evolution. As mankind progressed through the agricultural and industrial ages, we gained in knowledge but the pace of life and progress quickened. In the information age; life and progress are moving at a furious pace, and we face increasing challenges to survive. The historical record shows that many ancient cultures met their demise when they were at a similar stage of cultural development – because they were not able to deal with those challenges. If humanity is going to live long enough to progress to our next stage of cultural evolution; it will be by taking a longer view rather than pursuing short-term gain. The increasing pace of life encourages some to do the opposite, by favoring goals with immediate return on investments over a more prudent course that maximizes long-term reward. But this only increases the risk for everyone else, and we can't afford to broker away the future of humanity on false promises. Surviving the challenges of the current age, while preserving conditions that allow for life and prosperity in the future, remains the only way humanity can progress to the next stage. However, higher-order Math apparently offers a roadmap to engineering that evolution.

The Road Home

The notion that the laws of Mathematics are mindless is mainly due to the fact that we started by learning Maths of the lowest order, and on the low end of the number type

spectrum, while nature likely employed complex and hyper-complex number types right from the start. The idea is that if we make no preference and let the Math decide, quantities that encode dynamism come into play long before fixed quantities make an appearance. If we understand that imaginary numbers encode a precise quantity of variability; the intriguing properties of Complex, Quaternion, and Octonion, algebras are easy to explain or understand. As we go up the ladder, with 1, 3, and 7 imaginary dimensions respectively; these algebras encode increasing levels of dynamism, from simple variation to evolution and sequential evolution. As Connes pointed out; non-commuting geometric spaces possess evolutive properties also. Furthermore, as Kainen stated; the complications born of non-associative Math's imposed order might be a blessing in disguise.

It makes sense to put the imposed order of non-associative Math to work in Physics and elsewhere, rather than seeing it as an unnecessary complication. This may be the quickest road to understanding evolutive processes in nature. Young's process-theoretic model of the universe is somewhat imperfect, and he remarked in later interviews that he would frame portions differently – if he could do it over – but the "Reflexive Universe" book is a testament to a single evolutive pattern working across all categories in nature. The seven stage pattern documented by Young is clearly connected to the octonions, where calculations for multiplication also proceed through seven stages – each using three elements – in order to produce a result. There is of course a connection with the 7-color map problem, and with projective geometry, both of which Young mentions. But aside from explaining that these are limiting cases in the spectrum of what is possible; I will make no attempt to explain why evolution should prefer seven stages. Instead; I note that the Math of non-associative spaces appears to have the general property of fostering sequential evolution. I am sure nature is perfectly happy with processes taking more or fewer stages to complete, but the Math itself dictates that certain patterns are more stable than others.

One might ask "Why is this relevant?" since non-associative Math should only be a factor in subjects like quantum gravity, or in extreme conditions involving processes at a scale smaller than sub-atomic. Since it upends both size/distance and interiority/exteriority; non-associative geometry permits us to exist within a higher-dimensional space that appears to reside in the realm of the exceedingly small – from our point of view. So our perspective, that higher dimensions are curled up inside a tiny space, does not prevent them from being paradoxically more spacious or immanent. Even at ordinary levels of scale, many real-world physical systems are not well-modeled by linear equations. Higher-order terms creep in quickly, if you try to include everything that can vary. Of course; it is nice to make a few limiting assumptions so one has easy-to-solve equations, to plug in numerical values and have a testable model. However; nature imposes fewer limits and must include every variable at once, so the realities of everyday Physics require some heavy Math at times. The challenge is to harness the dynamism we find in higher-order frameworks, in service of our goals. Perhaps subjects like weather forecasting could benefit from adaptive modeling and evolutive

programming that is made possible by working in higher dimensional algebras. But studying the way higher-order Maths fuel evolutive processes could help even more with Cosmology, where we study the long-term evolution of the universe.

We would not be here at all, if sequential evolution was not a part of the universe's master plan. But we observe that sequentially evolutive behavior arises due to subtle factors contained in higher-order Math, and by properties of the geometric spaces spawned thereby, which make variation directional instead of aimless. One does not need to go very high up the scale of mathematical complexity, for directional properties inherent in the algebra to be observed, nor as far up the chain as the hyper-complex numbers. A simple quadratic function, iterated in the Complex domain, yields the Mandelbrot Set – a highly directional figure illustrating sequentially evolutive behavior in the forms around its periphery. My research suggests it can teach us about Cosmology [21], and aid our understanding of the fundamental forces. But the work of Kricker and Joshi, back in 1995 [22], reveals that the Mandelbrot Set in the Complex plane also informs us about non-associative behaviors in the octonionic quadratic functions. So again we see a coexistence of the 2-d and higher-dimensional realms, because the same Math operates in both domains. Perhaps we should view the 2-d Mandelbrot Set in the Complex plane as a projection or cross-section of the same figure in the Quaternion and Octonion domains, in order to understand how nature makes use of it. But mainly; we must acknowledge that $\mathcal{M}$ illustrates an evolutive patterning arising in pure Math, which the universe *can* put to use. What started with i, the unit of the imaginary numbers, does not end there – but instead it finds its full expression in the octonions.

It seems silly to ask how aimless Math can give life and the universe a sense of direction, when Mathematics is anything but aimless. I have spoken mainly of higher-order Math where the dynamism is more obvious and emergent properties are easier to observe, but simpler examples like the Mandelbrot Set have been studied for years without people getting the message that Math is at its root dynamic and evolutive. Roman numerals carved in stone represent unchanging quantities well, but viewing Mathematics in that way is a mistake, because Math is about how unchanging attributes and quantities come to be that way through a process. Seeing Math as dry – as though it was mindless and lifeless – is the real problem, and the mystery of where evolution comes from will disappear when we realize what Math is at its root, a systematic exploration of features characterizing the laws by which form evolves. The unchanging quantities of Math itself include figures like the Mandelbrot Set, E8 and the other exceptional groups, as well as other mathematical invariants. While one could argue that humans constructed these things; it can also be said they were only discovered or always existed – even before the universe had its birth. So it is with even higher orders and levels of Mathematics we have not discovered yet, which the universe is already putting to use. But we are fortunate, at this juncture, to be equipped to learn how Math gives rise to life, in order to foster the evolution of consciousness.

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