# An Undecidable Universe

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A central component in discussions of computability theory is the halting problem. Given an arbitrary program and an input, can one tell in advance whether the program will continue running for ever, or halt and deliver the answer? George Church and Alan Turing proved that there is no general solution to this problem. Turing analyzed it by proposing an idealized device, usually called a Turing machine, that could execute a repertoire of logical operations, and either eventually halt and output the answer, or chug away for all eternity. Applied to arithmetic, if a Turing machine halts after a finite number of steps, the number is deemed computable. If the machine never halts, the number is uncomputable. Roughly speaking, uncomputability does for the foundations of mathematics what Gödel's incompleteness theorem does for the foundations of logic.

The above considerations have been standard in the theory of mathematics and computing for many decades. What is omitted in the definition of computability, however, is any consideration of the *length* of the computation. To be sure, the total number of steps is of great practical significance for the computer industry, but for the purposes of the formal halting problem it matters not whether the Turing machine halts in a million years, a billion years, or a trillion years, so long as it is a finite time.

What is the relevance of the halting problem to physics and cosmology? It is very basic. The laws of physics are expressed as mathematical relationships, so the consistency and knowability of the laws of physics depends on computability, and its logical antecedent, decidability.

What manner of entity are the laws of physics? Where do they come from? What is their ontological status? Physicists divide into two camps. One camp regards the laws as convenient formalizations that humans have invented to organize observational and experimental data[i]. The other camp regards the laws as 'fundamental', by which I mean they constitute the ontological ground on which physical reality rests. Of course, the laws that we know and love from today's textbooks may not be the true and final version of the laws of the universe, merely successively improving approximations thereto. However, second campers assume that the true laws really are 'out there,' and that scientists inexorably grope towards them.

The ontological status of the laws matters little for almost all of science. Where it does have relevance is in cosmology. In the traditional big bang model, the universe springs into existence from nothing. Fifty years ago, the big bang itself was regarded as an event without a cause and therefore beyond the scope of science. The laws of physics were assumed to be imprinted on the universe from the get go, implying that the package of marvels 'universe + laws' popped into being spontaneously in a singular process, something that simply has to be accepted as a brute fact. Quantum cosmology attempts to go beyond this bald assertion by subjecting the ultimate origin of the universe to mathematical modelling, which is to say that the originating event, or process, is treated as lawlike, and not as an unexplained initial condition. In this approach, therefore, the laws must in some sense 'already exist' before the universe. Since in these cosmological models there is no spacetime before the big bang, and so no meaning can be attached to temporally prior causes, a more rigorous statement is that the laws of physics *transcend* spacetime and matter, and should be regarded as *ontologically* prior to the physical universe rather than physically prior.

Assigning ontological primacy to mathematical laws mirrors the philosophy of Platonism in the foundations of mathematics, according to which mathematical relationships exist in a realm outside of space and time and are true whether there is a physical universe or not. The relationships are discovered, not invented. Second camp physicists regard (sometimes only tacitly) the laws of physics as enjoying a Platonic existence. Roger Penrose has written eloquently on this topic.[ii]

Physical Platonism comes with subsidiary assumptions. The laws (again, I mean the ultimate fundamental laws, possibly still beyond our ken) are thought of as universal, immutable and eternal, labelling our universe like an indelible maker's mark. In addition, the (ultimate, fundamental) laws are taken to be *infinitely precise* mathematical relationships. The string theory Lagrangian, to take a popular example of a putative fundamental law, is supposed to be *exact*, and not just 'okay up to a degree'.

Now it is clear that infinitely precise, universal, immutable, eternal laws are an idealization that cannot even in principle be tested. For example, most laws of physics make use of infinitesimal quantities and real numbers. But the set of all real numbers is an idealization: all measurements and observations yield only rational numbers.[iii] If you believe the laws of physics exist in some Platonic heaven then this idealization isn't troubling. But what if this godlike status of the laws is a misconception?

Thirty years ago, the physicist John Wheeler challenged the very notion of idealized laws. 'There is no law except the law that there is no law… Everything comes out of higgledy-piggledy,' was the way he expressed it in characteristic style.[iv] Wheeler drew on a tradition going back to the philosopher William James that the laws of physics are not fixed and immutable, but evolve into their current form. Wheeler regarded the laws as somehow 'congealing' out of the ferment and extreme conditions of the big bang, and so not infinitely precise: 'The laws… could not have been always a hundred percent accurate.'[v]

Wheeler's rather vague deliberations were given a more concrete expression by Rolf Landauer, who viewed the laws of physics through the eyes of a computer scientist trying to model the physical world on a machine, with all the problems of rounding errors and imprecision. Wheeler espoused the notion that physical existence is 'an information-theoretic entity' because everything we discover about the world ultimately boils down to bits of information, a position summarized in his famous description, 'it from bit.' Landauer proceeded from an equally famous pronouncement, 'information is physical,' and in merging the two dictums arrived at a profound hypothesis:

'The calculative process, just like the measurement process, is subject to some limitations. A sensible theory of physics must respect these limitations, and should not invoke calculative routines that in fact cannot be carried out.'[vi]

In a nutshell, the laws of physics determine what can be computed, and computability determines what can be laws. Landauer's hypothesis provides a bridge linking computation and cosmology, for at issue here is the computational power of the real universe, with its finite age and resources.[vii]

Returning to the halting problem, it is important to note that a Turing machine is an abstract concept. A real machine, however, has physical limitations.[viii] Unlike in Turing's original vision, a real machine doesn't have an infinite tape and an infinite duration to execute its functions. Those limitations change the status of the halting problem, for they imply that a number might be in principle computable in a transcendent Platonic realm, but in the physical universe even the most efficient possible machine may not be able to return an answer in the age of the universe. Computability would then become a *time-dependent* property: the best machine might remain undecided today but halt tomorrow.

Adopting Landauer's philosophy, then, means assigning an intrinsic level of uncertainty to the mathematical relationships we call the laws of physics, a Wheeler-type looseness or fuzziness of a magnitude that depends explicitly on the actual computational power of the universe at the current epoch.

To flesh out this picture, one needs to work out the computational upper bound of the universe. This has been done by Seth Lloyd.[ix] The answer is that the informational content of the observable universe in bits, and the total number of bit flips since the big bang, are both roughly $10^{120}$ in total. The Lloyd limit thus provides a measure of the degree of imprecision in the laws of physics, according to Landauer's hypothesis.

Now $10^{120}$ is a big number. In almost all cases we would never notice if there was any fuzziness at the level of one part in $10^{120}$. However, in cases where exponentiation is invoked by theory, the Lloyd limit is easily hit. For example, in the collapse of a star to a black hole, the red-shift rises exponentially on a time-scale of a fraction of a second. In deriving his well-known black hole radiance result, Hawking used this property in a Bogoliubov transformation, integrating over products of 'in' and 'out' modes of a field, all the way to infinity in frequency.[x] If the integral is truncated at the Lloyd limit, there is no steady Hawking flux, just a puff of transient radiation. Moreover, the number $10^{120}$ applies to the current cosmological epoch. In the early universe, the bound was much smaller. If there was significant fuzziness in the laws of physics during the inflationary epoch, for example, the downstream effects might show up in the CMB. More generally, there will be an additional source of uncertainty in nature – computational uncertainty – to go alongside quantum uncertainty and deterministic chaos.

Lloyd's limit was derived by combining cosmology with quantum mechanics. The number that emerges refers to bits, i.e. it is a classical information bound. Physical Platonists will not be perturbed by this line of reasoning, because for them ultimate reality is vested, not in classical bits in physical space, but by qubits in Hilbert space, which gives them exponentially more room for manoeuvre. We thus confront ontology head-on. If you think reality is the sum total of what sentient beings can actually measure or observe (e.g. classical bits of information and rational numbers), we have to take Landauer's hypothesis seriously. If you adopt an abstract view of reality – that it involves relationships and structures that lie far beyond anything that can in principle ever be observed – then belief in arbitrarily precise laws are immune from my argument.

There is much unfinished business in the foregoing thesis. How does it play out in cosmological models other than the traditional big bang; for example, in an eternal multiverse? What other physical processes involving exponentiation might serve as a test of the cosmic information bound? What happens in a quantum computer when the total number of branches of the wavefunction exceeds that bound? And if one accepts the idea of 'loose laws,' can that be made quantitative, such as by appealing to meta-laws formulated as superpositions of laws? Does my prescription open the way to a unification, not just of physics, but of physics and mathematics?

Gödel and Turing gave us undecidability and uncomputability, Wheeler gave us mutability and it-from-bit, and Landauer gave us physics-as-computation. By combining these deep insights, one arrives at a startling eschatological conclusion: not only is the fate of the universe undecided, it is actually undecidable.

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[i] Nancy Cartwright, *How the Laws of Physics Lie* (Oxford University Press, 2003)

[ii] Roger Penrose, 'The roots of science,' in *The Road to Reality: A Complete Guide to the Laws of the Universe* (A.A. Knopf, 2005)

[iii] Gregory Chaitin, *Meta Math! The Quest for Omega* (Pantheon Books, 2005), p. 115

[iv] John Wheeler, 'On recognizing "law without law"', *American Journal of Physics* **51** (1983), 398-404

[v] John Wheeler, 'Frontiers of time,' in *Problems in the Foundations of Physics*, edited by G. Toraldo di Francia (North-Holland, 1979), 395-497

[vi] Rolf Landauer, 'Computation and physics: Wheeler's meaning circuit?' *Foundations of Physics* **16** (1986), 551-564. See also *IEEE Spectrum* **4** (1967), 105-109

[vii] Paul Benioff, 'Towards a coherent theory of physics and mathematics,' *Foundations of Physics*, **32** (2002), 989-1029

[viii] David Deutsch, 'Quantum theory, the Church-Turing principle and the universal quantum computer,' *Proc. R. Soc. Lond.* A **400** (1985), 97-117

[ix] Seth Lloyd, *Programming the Universe: A Quantum Computer Scientist Takes on the Cosmos* (Random House, 2005).

[x] S.W. Hawking, 'Particle creation by black holes,' *Comm. Math. Phys.* **43** (1975) 199-220