Mathematical Physics as Topological Numbers, Types and Quanta

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Abstract

The development of mathematics used in physics is most likely to become concerned with finite elements that are measurable. This means that topology and the computation of topological charges and indices, quantum numbers, and connection to logical switching theory are likely to supplant concerns of geometry, metrics and infinitesimal structure of manifolds. This is examined, with a possible counter direction to this as well with super-Turing machines and second order λ-calculus. Mathematics and its deeper foundations may share a similar nature with physics in regards to quantum information.

1 Mathematics, civilization and physics

The relationship between physics and mathematics is complicated and has a long history. The earliest form of mathematics was counting and this was used to count cattle, for barter, or to find a bride price and so forth. Numbers existed at their start for very practical reasons. Given a set of objects of the same or similar property one can assign a number to these to order them and to understand in a quantitative way the number of these objects. Numbers most likely did not emerge as purely abstract objects, but as a way to order and find the size of a set of objects. We might then say the counting of cattle or the bargaining over a price was the earliest form of using mathematics to catalogue physical objects.

The connection between mathematics and the world became more nuanced once people started to measure distance. Distance is an additional relationship between objects. This was further done with the additional information of angular relationships. One may order objects according to an integer label, but alignment in space involves a new type of relationship. This relationship can be with respect to the size of an object or how different objects are laid down on the ground with respect to each other. This appears to be a way of thinking that started once people began to build shrines and Neolithic constructions. The arrangements of large stone structures are often in some henge-like system. The Giza pyramids are arranged in a north-south and east-west manner, and the three great pyramids of Giza are set in a manner that appears to represent the belt of Orion [1]. Other constructions from Stonehenge to the latter constructions on a similar technological level at Chaco Canyon have arrangements that are astronomical.

This transition into geometry began to reach maturity with Pythagoras and his famous theorem for right triangles. At this point mathematics became an axiomatic system that developed proofs of theorems. This approach to mathematics reflects the social and technological level of humans, which straddled the Neolithic, agricultural into the Iron Age. The term geometry means earth measure, and it is a mathematics that reflects our early civilizations. Ancient mathematics and geometry was developed further by Euclid, Archimedes and Apollonius. These developments increasingly brought methods of calculation into geometry, and with the inclusion of Indian mathematics by Aryabhata and Brahmagupta, this lead to the development of algebra by Al-Kwarizmi [2]. These developments accompanied a growing technological trend, which included astronomical measuring instruments such as the sextant and the earliest developments of clocks, and medieval mathematics developed into the next revolution.

The next mathematical revolution matured with the development of calculus and the mathematics used in physics. These developments paved the way for the industrial revolution and the new view of the universe that was continuous and which involved mathematics that lead to point-set topology of infinitesimals, continuity and differentiation. In this picture there is in addition to the arithmetic of numbers and the concept of distance the notion that space is absolutely continuous and that a process occurs smoothly. The development of classical physics accompanied the mathematics of differential equations that had continuous and smooth solutions. The classical picture of the universe is a continuum of flows [3].

The classical picture dominated the intellectual world for over two centuries. Towards the end of the 19^{th} century this picture began to suffer cracks. There were a number of developments which brought this about. One of them is quantum physics, which indicated that physical reality as understood classically is an illusion. Quantum states are manifested in waves that are not physically real in the standard sense, and that what is measured about the world happens by stochastic processes that are to this day subject of research and debate. The quantum wave function is not real in the usual sense, but is an epistemological gadget that contains information that might be measured. The other problem was with the foundations of mathematics. Bertrand Russell asked what would happen if you have a set of sets that does not include itself. A list, thought of as a set, of all possible titles and lists that does not list itself is not complete, but if it does list itself it must list that it lists itself, which means it must list that it lists that it lists, ... and so forth [4]. This seemed to point to a basic paradox in the underlying foundations of mathematics according to set theory. Along these lines came the results of Turing and Goedel which proved that mathematics could not be made absolutely complete.

Alan Turing introduced the idea of computation in a mathematical model of a machine [5]. This machine could perform any possible computation of a certain nature. This nature was defined concisely by Alonzo Church and Stephen Kleene as the λ-calculus [6]. However, a Turing machine could not compute everything. Some computations halt with an output, while others do not halt and continue indefinitely. Turing demonstrated that no Turing machine can emulate all other possible Turing machines to determine if it halts. To do this it must emulate itself emulating all possible machines, which gets one into the same conundrum that Russell found. This fact is found in the fact that no matter how hard programmers work there continues to be problems with programs that cause computers to freeze up. There are similar uncomputable problems, such as there does not exist a universal algorithm that can determine if any algorithm is the minimal sized algorithm that computes an output. We are now in the era of the computer revolution beyond the industrial revolution and must embrace the mathematics this implies.

This was accompanied by Kurt Goedel′s proof that no mathematical system can ever prove all possible statements as theorems about itself. This enumerates a list of theorems and proofs coded as Goedel numbers. The Goedel numbers are numerical representations of prepositions of any logico-algebraic system. The diagonal elements in the list when increased or decreased by one can be formed into a string of numbers that does not exist in the list. This Cantor diagonalization procedure is at the root of Goedel′s first theorem that demonstrates the existence of unprovable propositions, and further that these statements effectively declare their unprovability. The second Goedel theorem is that these statements must be true, because their falsehood would contradict the statement declaring its own unprovability. These statements about a mathematical system are self-referential, unprovable and which state their own unprovability. This tore a hole in David Hilbert′s Entscheidungsproblem, which sought to put all of mathematics within a single logical program.

With all of this have come questions about what is meant by the continuum and infinity. Paul Cohen showed that the cardinality of the continuum, thought to be larger than countable infinity, is not decidable, but where one can construct models independent of the axioms of set theory [8]. This is a form of Goedel′s theorem. With this there are now questions about what is meant by infinitesimals. Other areas of mathematics are brought to question as well, such as Peano′s number theory. As a result standard mathematics has become increasingly shaken at its core. Conversely, what has grown is computer science and machines that are based on Turing′s analysis of algorithms.

We have an intuitive sense of numbers and the inductive reasoning for why if there exists the integer N then the integer $N + 1$ must exist. Goedel tells us that something goes wrong with this; there is something in basic arithmetic that is not computable. We might be best to think of this according to computers or Turing machines. The advantage of this is that we can consider computation as a physical process. We might then consider the number $10^{10^{10^{10}}}$. We have here a compact representation of this number. It involves a small number of binary units or bits even though the number is utterly enormous. Following Peano theory we can start adding numbers to this. We now imagine that we do this until we reach the number $10^{10^{10^{10^{10}}}}$. Is this physically possible? It is clear that between these two numbers there are numbers with an enormous amount of complexity. The description of these numbers is larger than all the atoms in the observable universe. There does not exist any Turing machine in the physical universe that can count between these two numbers. We might then ask the question: do all the numbers between these two large numbers exist?

The same holds for infinitesimals, where in point set topology there are an infinite number of points between any two points on the real number line with a finite distance between them. This means if they exist in some meaning according to computation there must be a machine that performs any calculation of points separated by any tiny finite set of intervals segmenting the distance between these points. In general this is just a reciprocal form of the arithmetic problem with huge numbers above.

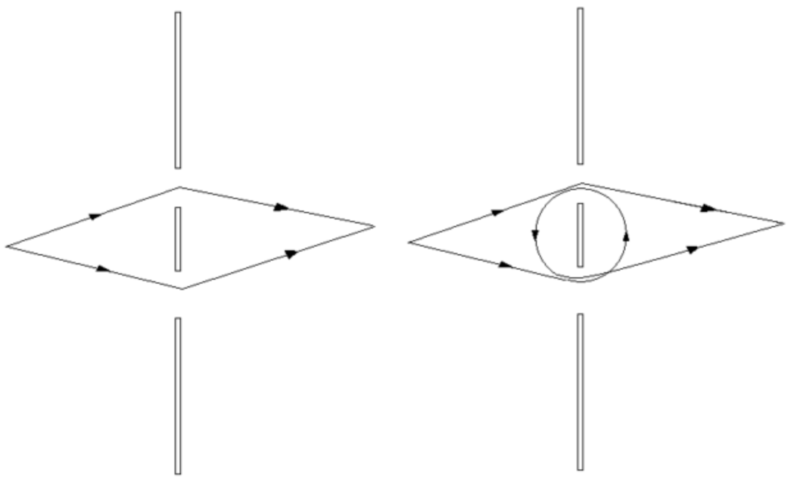
The question is one of the Kolmogorov entropy, due to Andrey Kolmogorov, which is an information theoretic approach to symbol strings or numbers [9]. The occurrence of a symbol in a string has some probability, such as defined by its frequency or estimated in a Bayesian prior, and a repeating sequence of 1 and 0 has probability $\frac{1}{2}$ for each of these and this is summed by the number of their occurrences. A highly random symbol string may have a large Shannon entropy $S = -\sum_n p(n) log\ p(n)$. Further description in Komogorov entropy can be enormously complex and the entropy very large. The entropy can be larger than what is possible in the entire universe. We can require a computer to produce an output that is smaller than a particular Kolmogorov complexity. A computer program that produces a list of symbol strings with fewer than 10^9 bits can be written so it is smaller than this bound, which contradicts the occurrence of this program with that bound. This is related to something called the Berry paradox. This is related to size optimization problems, trying to find the smallest possible program, or the optimal data compression algorithm. This is not computably possible, and we are prevented from knowing the smallest large number that can be represented in this universe.

This means we face a double problem. We know that between $10^{10^{10^{10}}}$ and $10^{10^{10^{10^{10}}}}$ there are numbers that have enormous complexity, but we cannot know what is the smallest of these numbers that has no such description. We are then in effect lost in a mathematical space that has this fundamental level of undecidablility and uncomputability.

2 Quantum mechanics and type theory

We are then left with finite methods, and uncertainty about what upper bounds exist on them. However, this is in many ways our friend. We return to physics beyond just counting atoms or possible information registers in the universe. We have mentioned quantum mechanics, and that this implies a stochastic and a discrete type of structure. Quantum mechanics is a system that can map a continuous variable, such as the epistemological wave function, onto a discrete set of real numbers or eigenvalues. We also know that it is a system that has no dynamic description of the measurement process. The outcome is not determined, even though modern research has come to understand the einselection of a basis and the decoherence of a wave function. All this does is to tell us what outcomes are probable as the wave is reduced to classical-like probabilities, but not which one obtains. Quantum mechanics then has one foot in traditional mathematics, but another foot in discrete mathematics.

One of the first systems studied by quantum physicists was the two slit system [10]. In this experiment a photon or electron is sent to a screen with two openings. The electron wave vector upon passing through the two slits has two parts $|a\rangle = e^{ikx}|r\rangle$ and $|b\rangle = e^{ikx'}|\ell\rangle$ for the left and right slits. The wave vector $|\psi\rangle = 2^{-1/2}(|a\rangle + |b\rangle)$. The occurrence of the wave on the final photoplate is then given by the modulus square of the wave vector $\langle\psi|\psi\rangle$ and the overlap terms $\langle a|b\rangle$ and $\langle b|a\rangle$ results in a term of the form $cos(k(x - x^{'}))$. The quantum wave vector describes a set of paths that loops through the two slits. The system is a topological system, in fact it is a fundamental form $\pi_1(C) = \mathbb{Z}$ of possible curves C. The integers describe the possible winding of paths, which in a path integral perspective can wind an arbitrary number of times. The only way this system can be violated is if a needle state is used to determine which slit the system passed through. If we think of the needle states as given by a spin system, such as described with σ_z, then this entangles with the wave function. The two spin states $|+\rangle$ and $|-\rangle$ are orthogonal which removes the periodic wave signature on the screen. The orthogonal needle states remove the superposition of the wave through the two slits, and this is converted into an entanglement. The topology of the system has been transferred into the entanglement.



Topological winding numbers in two slit experiment

The conversion of the topological obstruction of paths through the two slits into entanglement means that the homotopy of this system is equivalent to a type of logic gate. Of course quantum mechanics does not predict what the outcome is, whether that be $|r\rangle$ or $|\ell\rangle$, but the system is reduced to a binary output. This is in part an aspect of quantum computers [11], where an octal register of bytes can hold $2^8 = 256$ possible numbers, and a quantum octal system of bytes can hold all of those numbers in a quantum superposition. The teleportation of states has a similar structure, where the Hadamard matrix transformation converts an EPR pair into a new entangled state, but Alice who performs this operation must send a classical signal to her partner Bob [12]. The structure of quantum mechanics is remarkably similar to logic gates and their operations.

To purse this tie between discrete mathematics, topology and computation, we now take this into the less familiar ground of quantum gravity. The holographic principle as initially outlined by $^{'}$tHooft illustrates how the event horizon of a black hole is composed of Planck units of area, which are digital like storage sites. The Schwarzschild black hole with the horizon area $A = 4\pi r^2$ for $r = 2m$ has entropy $S = k~A/4L_p$, and the area is an integer multiple of Planck units of area L_p^2. The black hole is then a vast information storage system from the perspective of the exterior observer. The information though appears lost because the gravitational red shift is so large that it is hard to observe the horizon. However, on the stretched horizon, a surface a Planck length unit or string length above the horizon the quantum field information that composes the black hole does exist.

We can examine the horizon as a case of an N-slit experiment. A quantum which interacts with the black hole interacts with each possible slot on the horizon. We may then simplify this to a two slit experiment with the screen at a radius R from the the source and pixel size Δx. The resolution of two slits is then set by the pixel size and further the size of the screen $2\pi R$ sets the angular uncertainty of the wave $\Delta\theta = \Delta x/2\pi R$. The square uncertainty in distance is then much smaller than the Planck area and the resolution of the screen, or the reduction in angular uncertainty, becomes sharper with increased R. The event horizon as a type of N-slit system becomes more classical-like as the size increases and the number of slots increase. This is a way of seeing that the wave function of a system interacting with the black hole spreads across the horizon, such as with the radius R, and the appearance of the system becomes more classical.

The measurement of the occurrence of a quantum bit on the horizon has a similar problem with resolution. If you attempt to measure a qubit on the horizon this requires an enormous amount of energy. This is the Heisenberg microscope problem, for by performing such a measurement there is a complete loss of information pertaining to the position of the qubit. This effectively reduces the observed distance, such as for the screen $2\pi R \rightarrow \theta R$ for $\theta << 2\pi$. The resolution breakdown is due to the Heisenberg microscope. The interaction of the qubit with the black hole is then a form of entanglement, and this mean the superposition of units of area the wave may exist in is equivalent to an entanglement of the qubit with the black hole. The smaller the units of Planck area observed the more quantum mechanical the system appears, but as N increases the horizon grows black, cold and this superposition appears as an entanglement. In fact the existence of what we interpret as spacetime is a signature of entanglement. Spacetime is built up from entanglements [13].

This means this is a system that is invariant in some way with respect to the size of the black hole. The small black hole and the large black hole are fundamentally the same, the small ones appearing more quantum mechanical and the large as classical-like systems that entangle with qubits. The quantum state of the system is a large $SU(N)$ gauge type of system, and where by the mathematics of Bott periodicity there is an eight fold equivalency with homotopy groups and equivalencies as the size of the groups increase and an equivalency of this periodicity as $N \rightarrow \infty$ [14]. This large N limit corresponds to holography of the AdS/CFT correspondence. The mathematics of Bott periodicity can be found in Milnor′s classic book.

The physics is largely contained in the homotopy groups, and these are the indices which correspond to the entanglement of qubits in the large N limit. This means that the fundamental description of reality is not with space, spacetime or anything geometric. Geometry or metric space is something which is a measure of entanglement of quantum bits with black holes and the inability to follow the entanglement phase. Geometry is then not fundamental. What is fundamental are topological quantum numbers, such as those here associated with the two slit experiment or black hole horizon units of area.

This then leads us into a new paradigm of mathematics, and it is interesting that physics may pave the way. One foundation that could replace set theory is HOmotopy Type Theory (HOTT) which is a new method for the foundations of mathematics [16]. This shares commonality with other aspects of mathematics of magma and motives, which bases mathematics on monoids and groupoids. HOTT is a form of category theory, in that it constructs mathematics according to types of terms that have an equivalency according to paths. HOTT emerged from the category theory of groupoids of one dimension, and there are developments in HOTT towards groupoids of higher dimensions. Current approaches work to show there are categorical equivalencies with ω-groupoids. These are systems of homotopy of paths which map into each other as a single category. HOTT replaces spaces in homotopy theory with types, and points on these spaces with terms, which we might think of as analogous to bits or bytes. A fibration is type dependency, and path space replaced with an identifier to two terms, and a homotopy as a set of such in an equivalency. The HOTT then removes the vestiges of geometric constructions and reduces the entire mathematics to a logical system of terms and types.

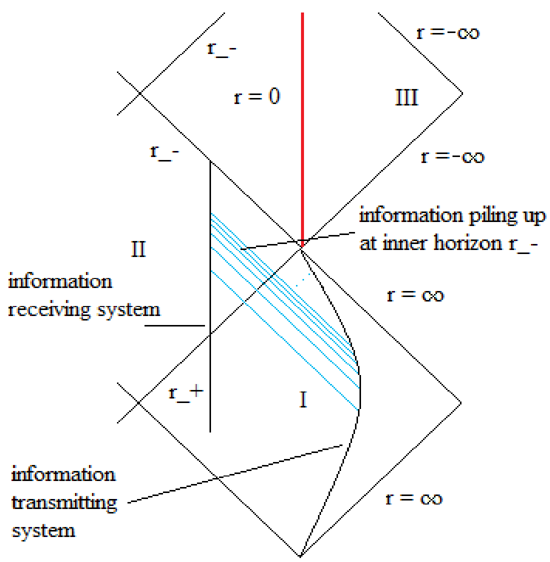
HOTT may not exist entirely on its own, but it may play the role of the foundations for category theory based on topology in the way set theory has served as a mathematical foundation. The movement in this direction will remove more of the continuum concepts from the foundations of mathematics, or mathematics that is used in physical science. This movement in mathematics accompanies the digital basis of technology in our age. This is in line with previous developments in mathematics, which in general follows agrarian civilization, industrial civilization and now mathematics is likely to change according to the recent emergence of a digital information age.

3 Continuous mathematics and super-Turing machines

Does this mean that older forms of mathematics will disappear? This of course is not going to happen. It can always be maintained that numbers in their pure abstract form, without explicit computation, still exist. This would be in line with mathematical objectivity, which in its purest sense can be called Platonism. Standard mathematics will continue to exist, and it will have contact with more finite and discrete methods of types, categories and groupoids.

There may in fact be a reversal of this finite and digital form of mathematics. This requires that computation transcend the limitations of Turing machines or λ-calculus. This means that noncomputable problems, those not Turing computable or undecidable by Goedel$'$s theorem, are decidable by other means. This is sometimes referred to as a second order λ-calculus. A simple example of this is the problem of whether a light switch that is turned on in the first second, then off in the next half second, then on again in the next quarter second, then ..., and whether the light switch is on or off at the end. This is an elementary Boolean problem that has no computable solution. Uncomputable problems that require an infinite number of steps are of this nature, and an answer to them requires some sort of asymptotic speed up of the processor. Because the sequence of switching is compressed by a time asymptote in principle this will have an answer. This of course ignores the practical problem that any physical switch subjected to this would fall apart before the problem is completed.

The physical aspects of these super-Turing machines are important, and this may prevent their reality. We may think of the switch as a rotating system, similar to a commutator on a motor. In order for the switch to rapidly increase its angular momentum must diverge. This system thought of as a quantum system, say a rotating string, obeys the Regge trajectory $\ell \sim E^2$ and the angular momentum $\ell = n\hbar$ as $n \rightarrow \infty$. The energy E^2 will diverge until it is equal to mass of a black hole of that radius. This will prevent the system from returning an answer, for it will become a black hole that locks away the answer [17].



This appears to suggest that super-Turing machines are associated with black holes. This is the basis for the Malament-Hogarth spacetime [18]. This is a spacetime where the state changes in a machine accumulate, or are observed to accumulate, at some point. This means an observer which reaches this point may witness the output of an infinite sequence in a finite time. This can occur in a Kerr metric. The inner horizon $r_- = m - \sqrt{m^2 - a^2}$, for m and a the mass and angular momentum parameters, is continuous with $r = \infty$ in the exterior region. This means the observer passing r_- can receive all information from the outside incident on the black hole. A computation, which can be interpreted as all quantum transitions involved with the black hole, can be observed to its infinite endpoint.

This may however be an ideal. This relies upon an interpretation of an eternal black hole, which may not exist. Black holes quantum mechanically radiate quanta, and they decay away. Further, in a universe that appears to be exponentially expanding it is unlikely a black hole can exist in a situation where it is eternally replenished. This presents a further barrier to the idea of super-Turing machines. However, the idea does lead to how quantum gravity may compute the string landscape, which is NP-complete, where the black hole converts this to a P-algorithm [19].

It might be that super-Turing machines exist with shadow states in black hole interiors. These are an old idea of quantum states that have no probability or Born rule interpretation, but which still play a role in dynamics. It might also be the case that super-Turing computations exist with anti-de Sitter spacetimes. There are also less exotic ideas for super-Turing computation with plastic neural networks. The existence of super-Turing machines is not certain at this time. There are people who regard these as completely unrealistic [20].

4 Philosophical concerns

What I wrote does not particularly address the question of posed by Eugene Wigner′s observation about the unreasonable effectiveness of mathematics with respect to physics. We might never come to an answer to that question. There have been various proposals along these lines, such as the Mathematical Universe Hypothesis (MUH) of Max Tegmark. This was found to run into problems with Goedel′s theorem so the word ′′mathematical′′ is replaced with ′′computational′′ in CUH. Of what are all the computable functions is not a decidable problem. My observation about the MUH or CUH is that this might be an attempt to prove too much. The prospect of HOTT as the mathematical basis of physics does though have some similarity to the CUH.

Roger Penrose has proposed a triality system of physics, mathematics and mind. This has a certain Platonic flavor to it. Platonism is something which many people, particularly those in physics, regard as too mystical. Platonism, with its duality between physical and ideal forms, is thought by many to be a form of mysticism. The application of mind into the picture is akin to the idea of logos as bridging the two forms. This philosophy has been the basis of mysticism, such as in the opening of the Gospel of John. Mathematical objectivity is a weaker form of Platonism. Many mathematicians are objectivists who state there is some independent truth value to mathematics. Mathematics in this viewpoint is something out there and independent of us that is discovered.

The foundations of mathematics has been shaken by incompleteness results and further by what appears to be a fragmentation of conjectures on these foundations. Hilbert advanced that mathematics is just a formal sort of game in his idea of formalism. In this perspective mathematics has much the same existential level as the rules of chess, along with all possible games. All possible games represent the set of possible theorems that can be proven. Brouwer advanced intuitionism that claimed that mathematics was a pure creation of the human mind. This is in contrast to the objective stance on mathematics. Others such as Russell and Goedel maintained ideas about the logical and set theoretic foundations of mathematics. There are other set theoretic models, Polish set theory, Constructive set theory, Tarski-Grothendieck set theory, Monk-Tarski cylindric algebra, Morse-Kelley set theory, Von Neumann-Bernays-Goedel set theory, Internal set theory and so forth. As a result there is no unanimity on the front. As a result we have no particular foundations to mathematics, but rather we have systems of interpretations of mathematics.

Wigner′s question might be set on a back drop of how mathematics is at all effective within itself.

Chaitan has advanced ideas that mathematics is not something that exists in any sort of coherent wholeness. It is more a sort of archipelago of logically consistent systems that sit in an ocean of chaos [21]. This chaos is a set of statements that are purely self-referential and have truth or falsehood by no logical reason.

Possibly the quantum vacuum is similar. It may be a tangle of self-referential quantum bits, where some sets of these exist in logical coherent forms. These zones of logical coherence might form a type of universe. These logical coherent forms are then accidents similar to Chaitan′s philosophy of mathematics. It is very difficult to understand how this could be scientifically demonstrated, yet maybe regularities in physics described by mathematics exist for no reason at all. Mathematics and physics have this curious relationship to each other for purely stochastic or accidental reasons; there ultimately is no reason for this. At best we can only say that this state of affairs exists, but we may never understand any underlying reason why this must be so.

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