It From Bit is Undecidable

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Abstract

The digital model of the universe or "It From Bit" is not decidable. A model of the physical universe encoded by algorithmic means will not compute reality. One unknown domain argued to be outside any computerized model based on current quantum field theory is quantum gravity. A change in axiomatic basis is proposed to address field nonlocality in quantum gravity.

1 Information and undecidability

Wheeler's "It from Bit" dictum states that all observed quantities in the universe are emergent from information bits, or in a quantum mechanical setting qubits [1]. This implies that not only tangible aspects of the world such as mass or the spectrum of elementary particles emerge from quantum information, but theoretical constructs as spacetime, locality and nonlocality as well as causality are emergent from information. This conjecture has transformed the character of theoretical physics. Information theory has become a growth industry, and increasingly the entire universe is being thought of as a grand computer.

There is a long history of this. During the renaissance the advanced technology was the pump, and it was common to see pumps in action in many aspects of the natural world. Newtonian mechanics brought about the ideology of the clockwork universe, by which the operations of systems was thought of as clock-like mechanisms. Similar analogues came with steam engines during the industrial age. Today in this grand tradition the hot topic in physics is to see the universe as a computer.

The universe has a mechanistic aspect to it. Technology is a way of constraining physical processes in a manner that permits what we call the "machine." Physics however tells another story with the principle of least action. A quantity defined as the product of momentum and position called the action may be expressed in a manner which minimizes this quantity. This principle of least action defines the Euler-Lagrange equation for the Lagrangian $L ~=~ K ~-~ V$. In this perspective dynamics is seen as due to all possible continuous variation of the action, or the path a particle takes, and using the calculus of variations to find the minimal path. Physical systems in some funny sense have a premonition about how to evolve. In quantum physics Lagrangian-Hamiltonian dynamics gives the path integral, where a quantum particle is in a superposition of all possible histories; see [2] for further discussion with respect to spacetime. There is no information transfer in this process, but in a nonlocal manner a quantum particle "knows" how to evolve by sampling all possible paths. So while the quantum information machine perspective is currently popular it can still be argued there is either an underlying or a dual perspective on nature which is continuous and not digital.

A quantum bit is simply a way of expressing a quantum state. A two state system $|\psi\rangle ~=~ \frac{1}{\sqrt{2}}\big(|0\rangle ~+~ e^{i\phi}|1\rangle\big)$ is a bit expressed as $\binom{0}{1}$. The system may be expressed with a density matrix ρ which defines entropy $S ~=~ -Tr\rho~log(\rho)$. The trace is a sum over the diagonal elements, which if they given equal probabilities $P_0 ~=~ P_1 ~=~ 1/2$ the entropy is $S ~=~ -\sum_{n=1}^{2} P_n log P_n ~=~ log(2)$. For N elements in the qu-Nit the entropy is $S ~=~ log(N)$. This construct the Khinchin-Shannon information theory. In this setting quantum states can be interpreted according to information theory; the qubit, or qutrite, or qudit.

The quantum computer perspective of the world describes the universe as some master quantum Turing machine that deterministically computes information states. This is really a modern version of the clockwork world. Conversely the path integral perspective tells a somewhat different story, for the evolution of a physical system or quantum states is due to the extremization of the action. In this setting the evolution is less due to the deterministic automata processes of a Turing machine than they are due to a continuous extremization.

Here it is argued that "It From Bit" is not decidable. The issue of determinism in a "clockwork" or computer fashion is equivalent to an ordering of states, as well as a statement about logical necessity, or modal logic [3]. Logical necessity $\Box p$, for p some proposition or information, is equivalent to $prove(gn_p)$ for gn_p the Godel number of p. The proposition p is then analogous to a physical state that is determined by prior physical states by deterministic automata, or the universe. This is modal logic with $\Box$ also includes possibility $\diamond$ with the relationship

$$\diamond(p) \leftrightarrow \neg\Box\neg p, \& \Box p \leftrightarrow \neg\diamond\neg p \tag{1}$$

These can be seen with informal logic. The first says it is not necessary that it will not $p = rain$, which means it is possible, and second says it is not possible it will not rain, meaning it is determined. Godel's second theorem is [4]

$$\diamond\top \rightarrow \neg\Box\diamond\top \tag{2}$$

where $\top$ means True, and conversely $\bot$ means false. Equation 2 tells us that if $\top$ is possible then it is false that it "ought" to be. The argument for this is in the footnote
[1]

Godel's second theorem indicates that any consistent theory is unable to prove its consistency. Given a set of falsehoods $\bot$ then $\Box(\Box\bot \rightarrow \bot) \rightarrow \Box\bot$, Lob's theorem [4], or for $q = \neg p$ a false statement about p $\Box(\Box \rightarrow q) \rightarrow \Box q$. Unprovable statements are those for which modal reflection $\Box p \rightarrow p$ holds, but where there is not logical necessity $A \rightarrow B$, A then B. Given a set of true statements S about an axiomatic system, the set $S\backslash S_p$, S_p being provable statements, are all those which are true but not provable. We may then appeal to Turing's proof there does not exist any Turing Machine (TM) which is capable of determining the halting status of all TMs [5]. This is in effect because this Universal TM (UTM) must emulate all possible TMs, including itself emulating TMs, and it must emulate itself emulating all TMs. This leads to an infinite recursion that is not enumerable. A computer program is a binary string equivalent to some natural number $n \in \mathbb{N}$. A TM T acts on n such that $T(n) = n'$. A TM is also specified by some number m, call it T_m. UTM is then a TM that acts on $T_m(n)$ as $U(m, n) = T_m(n)$ and emulates all possible TMs. The numerical output of a TM is a finite string if the TM halts, but infinite in length if it does not. The set of all possible finite numbers is a recursive set $\mathcal{R} \subset \mathbb{N}$. Can our UTM determine all halting programs, or determine $\mathcal{R}$? The answer is no, for if we have a recursive subset $\mathcal{R}$ we may apply the Cantor diagonalization argument to show that there exist Turing machines T_m which lies outside our set. The set of numbers m for all possible halting T_ms is not recursive, but is recursively enumerable. We are then not able to apply a UTM to determine all possible recursive numbers as outputs of TMs. This is equivalent to Godel's theorem in a digital machine context, where $\mathcal{R}$ is contains a set equivalent to $S \setminus S_p$.

2 Wallace, Modals and Topos

There are various models that employ modal logic in causality, such as by von Wright [3]. David F. Wallace employed modal logic to show that Richard Taylor's argument for fatalism is wrong [6] . Wallace employs a form of modal logic to illustrate this problem. The argument also uses "daughters," or branches in causal paths from "mothers," which have a remarkable similarity to the path integral and causal sets. The overturning of fatalism demonstrates how a prior action to a later event is not absent any choice by the actor.

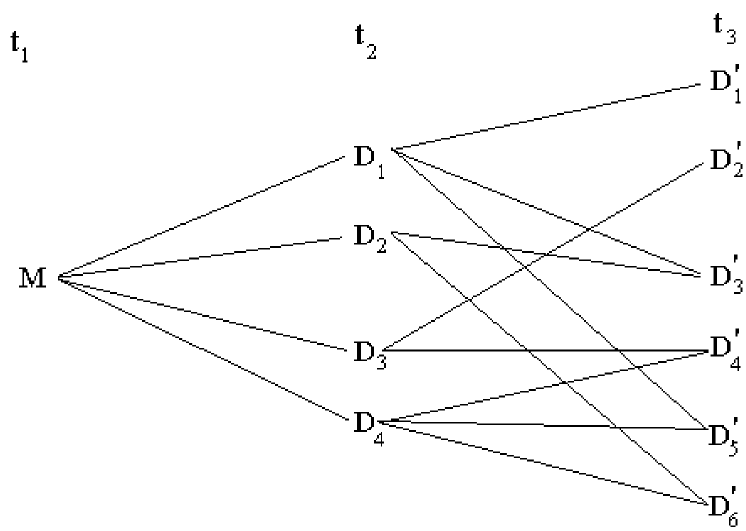
Taylor's argument may be seen as follows. Suppose event A is the sufficient condition for event B and $\neg A$ the sufficient condition for C. It follows that if C occurs there is no necessary condition for A, and if B occurs there is no necessary condition for $\neg A$. If B and C are inclusive of all possible outcomes the excluded middle tells us that B or C must be true, B and C is false. Hence the necessary condition either for A or $\neg A$ is false. If A and $\neg A$ are possible choices to be made by an actor, that actor does not have the power to make one of these choices. This leads to a conclusion for fatalism. This argument employs sufficient and necessary conditions in a tensed fashion, forwards and backwards in time, to give a causal chain. The future determines the past in this argument, though it is not hard to imagine a reversal of the conditionals for temporal meaning. If we assume the presence of a necessary condition is what determines an event exists, whether as in Taylor's argument in a time reversal or in a time forward manner, this argument is a sort of logico-algorithm for the ordering of events in the world.

[1] The naturalist fallacy is $p \rightarrow \Box\diamond p$, which says if p then it is necessary that p is possible, or p ought to be. The modus tolens of this fallacy is$\neg\Box\diamond p \rightarrow \neg p$, which we know is false. From $q \rightarrow \Box q$, then for $q = \neg p$ this syllogism is $\neg\Box\diamond p \rightarrow \neg p \rightarrow \Box\neg p$ which is equivalent to $\neg\Box\diamond p \rightarrow \neg\diamond p$. Yet this is a fallacy and so for general truth statements $\top$ we have $\diamond\top \rightarrow \neg\Box\diamond\top$. This connects Godel's theorem with Hume's "is-ought" or naturalist fallacy.

The modal operators $\Box$ and $\diamond$ have the meanings of necessary and possible. The above argument leaves us with $\Box A \vee \Box\neg A$, and $p \rightarrow \Box p$ demands that we have $\Box(\Box A \vee \Box\neg A)$. Between A and $\neg A$ a choice is necessary and the alternative is excluded. Wallace then constructs a set of possible causal paths from a single event. This is illustrated in the diagram below. The one event at time t_1 is the "mother," to the "daughter" events D_i at time t_2, which in turn are the mothers for various events at t_3. At time t_3 suppose that event D'_4 occurs. We inquire as to whether D_1 could have occurred. We conclude that $\neg\diamond D_1$,and in general we have $\neg\diamond D_1 \vee \neg\diamond D_2$. However we have two possibilities at t_2 which are D_3 and D_4 and so we have $\diamond D_3 \vee \diamond D_4$. We clearly have from $p \rightarrow \Box p = \neg\diamond\neg p$. From this we conclude that

$$\neg\diamond D_1 \vee \neg\diamond D_2 = \Box\neg D_1 \vee \Box\neg D_2, \tag{3}$$

which means the nonoccurrence of these events is necessary given D'_4. However, there is a necessity for their non occurrence as well as the possibility for two other events. We then have the inference at t_3 of possible events, but we do not have an inference of the sort $\neg\diamond\neg E$, of event E at a prior time. This then removes fatalism. An in depth discussion can be had with Wallace[6].



Wallace's construction is worth comparing to causal set theory. We also make note that these graphs and networks of causal relationship are computed; they exist through various modal-logical relationships. For a chain of events through a sequence of times $\{t_n,\ n = 1,\ 2,\ \dots,\ m\}$ we have at a time t_n an inference of the sort $E(t_n) \rightarrow \Box E(t_n) = \neg\diamond\neg E(t_n)$. The stronger fatalistic statement $t_n\ \neg\diamond E(t_n)$ is also refuted, for some event at t_n prior to the final event. Therefore we have $\diamond E(t_n)$ and $\neg\Box E(t_n) = \diamond\neg E(t_n)$. It is clear that given the entire set of events that happen we have $\bigvee_i E_i(t_n) \vee \neg E_i(t_n) = \top$, and these statements together define $\diamond\top$ with $\diamond\top \rightarrow \neg\Box\diamond\top$. We then have as a consequence of this the second theorem of Godel. This is equivalent to saying the "computer," whether the universe is an actual computer, or the computer as just a model, is not able to compute all events that occur in the universe.

A modal type of theory may have homotopy structure, such as seen in the supplementary section. Such construction have contact with category theory. Such modal models are monads with Grothendieck topology on sheafs and presheafs [7]. The connection to homotopies discussed connects a subset of $S'_p \subset S \setminus S_p$ that are provable within a new axiomatic structure with the physics of geometric nonlocality in quantum gravity. Further discussion along these lines is beyond the scope of this paper, and at this point this connection is somewhat heuristic.

3 Physical Theory

What does this imply for physics? Most of the above is really philosophy of physics and not physics. Godel's theorem tells us that any mathematical system that is consistent is incomplete. This implies there are certain propositions about any axiomatic system that is true but not provable. The famous case of this is Euclid's fifth axiom that no two parallel lines intersect. This axiom can be turned on or off, so to speak, and this has lead to a rich variety of geometries. This also changed physics, for Newtonian physics assumed Euclidean geometry, but general relativity (GR) later on removed this assumption. An unprovable proposition in mathematics can serve as an unnecessary constraint which prevents greater generalization. Similarly in theoretical physics there may exist assumptions that act as excess baggage that prevent workers from addressing fundamental problems.

We do not have an explicit proof of what sort of causal proposition is undecidable. We may though make a reasonable guess. It most likely has to do with the physical nature of locality and nonlocality. GR is a geometric theory of spacetime, which means that quantum gravity is quantization of spacetime itself. It is not entirely clear what this means. A number of questions have to be answered, and currently there are obstacles in our current theories which do not permit us to address these issues well. Standard quantum field theory is local, but the fundamental physical observables of quantum gravity, meaning diffeomorphism-invariant, are necessarily nonlocal [8]. Quantum mechanics is nonlocal, but the wave function is defined by the action of field operators that act on a Fock space so as to define amplitudes locally. In a related manner quantum field theory takes causality as a fundamental postulate, but in quantum gravity the spacetime geometry, and thus light cones and causal structure, are subject to quantum fluctuations. This has the curious meaning that a quantum field is propagating on spacetime, but where spacetime is the quantum field. This heuristically appears self-referential, and the physical ansatz is this nonlocality is an undecidable proposition of the above modal theory of causality.

The action for GR is $S(g) = \int d^3xdt\sqrt{-g}R$, for R the Ricci curvature scalar. The Lagrangian may be decomposed into the Ricci scalar for a 3-spatial surface $\Sigma^{(3)}$ plus a kinetic term in the ADM form of GR. The amplitude computed in a path integral is a summation over 3-metrics g

$$Z = \int \mathcal{D}[g]e^{-iS(g)}. \tag{4}$$

A standard method is to Wick rotate the phase $e^{-iS(g)} \rightarrow e^{-S(g)}$. This attenuates high frequency modes, and it is a "bit of a cheat" though at the end one must recover the $i = \sqrt{-1}$ and "undo the damage" for the most part. This amplitude becomes $\simeq e^{-GM^2}$, which interchangeability between the action and entropy. The integration measure $\mathcal{D}[g] = D[g]/diffeo$ is a quotient with the volume of the symmetry group. Consider the measure with volume $ln(\mathcal{D}[g]) \simeq S \simeq ln\,dim(\mathcal{H})$, for $\mathcal{H}$ the Hilbert space of size $dim(\mathcal{H}) \simeq e^S \simeq e^{GM^2}$. Consequently $\mathcal{D}[g] \simeq e^S$ and the path integral or partition is then on the order

$$Z \simeq e^Se^{-S} = 1 \tag{5}$$

This holds universally, where for black holes of any size this quantitu is small [9].

The entropy is corresponds the number of microstates. If we let M be equal to the number N of Planck mass units then

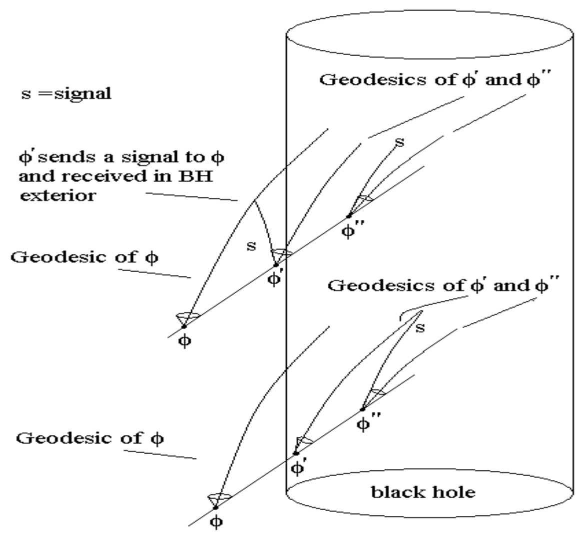
$$ln\,dim(\mathcal{H}) \simeq S \simeq GN^2M_p^2, \tag{6}$$

and further $N^2 = n = \#$ of Planck area units on the horizon . GM^2 in appropriate units is equal to the number of Planck unit areas on the horizon, $ln\,dim(\mathcal{H}) \simeq S \sim n$, and the number of states $= D^n$. There is a holographic element to this argument. The integration measure is over all possible $\Sigma^{(3)}$s, while the information which defines the action is contained on a two dimensional surface, the horizon. The matching of the two is due to an underlying nonlocality in quantum gravity that is not understood.

The partition function for a black hole with $S(g) = GM^2$ is over $e^{N^2\beta}e^{\sqrt{N}}$[10]. This partition function is formulated by an approximate counting of states. The degeneracy of these states is the cardinality of the sent of elements $\{n_1,\ n_2,\ \dots,\ n_m\}$ such that the number of Planck horizon areas is $N = \sum_{i=1}^m n_i$. The degeneracy then derives a complete partition function that is formally equivalent to the partition function for the integers. This parallels a basic development in string theory where the density of states for the boson string is equivalent to the Hardy-Ramanujan circle approximation for the integer partition [11]. The recent proof of a complete formula for the partition of integers by Ono, Bruinier, Folsom and Kent [12] provides a possible algorithm for computing the black hole entropy that corresponds with string theory.

An underlying mathematics are the Jacobi θ-functions, and the integer partition function is related to the Mock θ-function of Ramanujan [13]. Central is the Dedekind η-function as well. These are modular function realizations of the octonions or the E_8 exceptional group. This is at least suggestive of a role for the octonions in physics. The role of octonions has not been clear, and they are generally avoided by most physicists. Field operators on the other hand are expressed according to Lie algebras which obey commutators of the form $[A_i,\ A_j] = {C_{ij}}^kA_k$, where the A_i is meant to mean a gauge potential and ${C_{ij}}^k$ is a structure constant. Physically however this commutator is written according to $[A_i(\mathbf{r}),\ A_j(\mathbf{r}')] = {C_{ij}}^kA_k(\mathbf{r})\theta(\mathbf{r} - \mathbf{r}')$ such that the Heaviside function is one when $\mathbf{r} - \mathbf{r}'$ is a null or timelike separation. This Wightman distribution permits commutators of boson fields, and equivalently anticommutators $\{\psi(\mathbf{r}),\ \psi(\mathbf{r}')\} = 0$ for fermion with spacelike separation. In this manner initial data for fields may be specified on a spatial surface. This condition however is not possible if spatial surfaces are configurations for amplitudes in a quantum superposition. A measurement of any quantum field on a spatial surface will be subject to the fluctuations of spacetime. There is not any frame where initial data can be specified in a proper manner. This means some other means are required to specify data. One possible choice is associativity instead of commutivity.

The Wightman distribution is necessary for causality conditions. Physics demands that fields be ordered in causal sequences, within light cone conditions. One example of causality condition is the common calculation of time ordered product of fields in a path integral. The classical spatial surface of relativity provides an unambiguous way to specify initial field data with equal time commutators. The rapid fluctuation of a spatial surface disrupts the ability to do this. This can be particularly of important for field information near the event horizon of a black hole. Consider three quantum fields ϕ, ϕ', ϕ" , where ϕ is exterior a black hole, ϕ' is very near a black hole horizon, and ϕ" is interior. The field ϕ' is associated with either of these according to whether it is in or outside the BH, $(\phi\phi')\phi$" and $\phi(\phi'\phi$"). Consider these three fields coincident on a null ray which is a causal fiducial on these fields, thought of as quantum computers, to send signals or qubits. If these three fields are on a connected null ray entering the BH the association is interpreted as saying: $(\phi\phi')\phi$" = the field ϕ' can communicate to ϕ a signal from outside, but ϕ" can't send a signal which ϕ' receives on the outside. $\phi(\phi'\phi$") =the field ϕ" can communicate to ϕ' inside the BH, but ϕ' can't communicate to ϕ while is is outside. The association refers to what set of states (in or out) ϕ' can transmit to a receiver that share the same state space.



If these fields are not causally connected we then have $(\phi\phi')\phi" - \phi(\phi'\phi") = 0$. This is zero because there is no initial transmission which can connect all three fields, such as the null field above, to initiate the signaling or causal process between the three. Consider these fields in a causal set or connected by a null ray propagating in the BH. If spacetime is fluctuating we may have these two define the associator $(\phi\phi')\phi" - \phi(\phi'\phi")$ that is nonzero. There are superposed amplitude for ϕ' in the BH and outside. If the black hole is classical the associator is again zero. In the semi-classical to classical case of a BH the wavelength of metric fluctuations approaches zero. For a nonzero associator the field must be incredibly close to the horizon, but this requires a measurement of time $\Delta t \simeq \hbar/GM^2$ and requires a clock on the order of the mass of the BH. Any real observer outside or inside the BH would not be able to distinguish between the two cases.

With regards to quantum information this tells us that in a teleportation experiment if ϕ' were an element an EPR entanglement that Alice holding this state can send a classical piece of information to either Bob holding ϕ or to Bill holding ϕ" to perform a teleporation of a quantum state. If spacetime is purely classical she can then teleport a state to either Bob or Bill. If spacetime is fluctuating and Alice sends this signal we then have Alice close to the horizon able to send a signal that has geodesic-amplitudes leaving and entering the black hole. Bob and Bill must then perform a quantum measurement to access the signal necessary to teleport the state. There is then an uncertainty whether the teleported quantum state is obtained interior or exterior to the black hole. This is a novel concept in the theory of quantum information, one which is not contained in the "rule book," but is added to the rule book, as the associator, which is something not "provable" within a purely associative quantum physics. This inclusion of nonassociativity is a new form of nonlocality, the nonlocal property of qubits with respect to their spacetime configuration with a black hole.

4 S-Matrix of Black Holes

The S-matrix is thought by some to be the most fundamental to physics. A whole direction in physics was oriented this way in the 1960s. The intended bootstrap theory did not live up to expectations for hadron physics. Because a spin = $2\hbar$ particle was predicted by these development it managed to survive as string theory. This was reincarnated in a different form in the 1980s. At the core of this is the physics of unitarity, and the holographic principle is a stringy physics that demonstrates how unitarity and the conservation of information is compatible with black hole physics.

This nonassociative symmetry may be applied to the S-matrix for black holes and quantum gravity. A more detailed analysis may be seen in the supplement section. The S-matrix is defined for a domain $\{-\infty,\ \infty\}$, where a black hole has the domain $r\ \in\ \{0,\ \infty\}$. A coordinate change $2m/r\ =\ e^u$ permits one to place the singularity at $r\ =\ 0$ at $u\ =\ \infty$ and $r\ =\ \infty$ becomes $u\ =\ -\infty$. Quantum fields in these coordinates are computable with the S-matrix. There is also an interesting nonlocal implication. The singularity in these coordinates remains at ∞ throughout the Hawking quantum decay. Nonlocality is a necessary aspect of the holographic principle; the creation of a particle removed from a black hole is entangled with a quantum state of the black hole.

A black hole changes the topology of a data set [14]. Consider a sphere just inside the black hole that emits two spherical wave fronts of radiation in the outwards and inwards directions. Due to the fact this is in a trapped region the two wave fronts converge to the center of the black hole, but along different null geodesics. The two wave fronts converge on the singularity and disappear. The two wave fronts are absorbed by the same singularity. Hence the topology of the two wave fronts is a sphere with two points removed, and the remaining surface is identified at those points. This topology is then a torus. The entropy of the wave fronts $S_{front}\ \simeq\ dim\mathcal{H}\propto\ \#$ photons has then a contribution from the singularity $S\ =\ S_{front}\ +\ S_{sing}$. The nonlocal connection between states interior and exterior to the BH means this entropy is from topological states that construct the entanglement entropy of the BH.

The S-matrix is an operator that acts upon a linear chain of fields or channel $|\phi\rangle\ =\ |p_1,\ p_2,\ \dots,\ p_n\rangle$ and $\langle\phi'|\ =\ \langle p'_1,\ p'_2,\ \dots,\ p'_n|$ so that $\langle\phi'|S|\phi\rangle$ describes the evolution of one into the other. The distinction between the two channels may be as simple as the commutation or permutation of two fields. The entries in the channel are associative so there is no application of this structure. The channel as a sequence of fields may be written as

$$|p_1,\ p_2,\ \dots,\ p_n\rangle\ =\ a^\dagger_1(p_1)a^\dagger_2(p_2)\dots a^\dagger_n(p_n)|0\rangle, \tag{7}$$

or in greater generality instead of the operators $a^\dagger_i(p_i)$ we have $\overline{\Phi}_i(p_i)$. The vacuum state however for a black hole is partitioned into two parts $|0\rangle\ \rightarrow\ |\Omega_{in}\Omega_{out}\rangle$ for modes inside and outside the black hole. As argued above if there is a vacuum mode 0_k that is close enough to the horizon that fluctuations of spacetime make it uncertain whether this is in or out of the black hole we then replace our vacuum state with $|(\Omega 0_k)_{in}\Omega_{out}\rangle\ -\ |\Omega_{in}(0_k\Omega)_{out}\rangle$. The fields are associative even though the operators are elements of a Lie group of elements that are noncommuting, or $[\phi_a,\ \phi_b]\ =\ C_{abc}\phi_c$. However, the vacuum modes obey $(Q_aQ_b)Q_c\ -\ Q_a(Q_bQ_c)\ = [Q_a,\ Q_b,\ Q_c]\ \neq=\ 0$, for Q_i representing a set of vacuum modes.

This leaves us with an interesting possibility. Classical physics is symplectic or pseudo-complex, with conjugate relationships between momentum and position and similar relationships between other conjugate variables. By extension quantum mechanics is complex valued. A complex number multiplies with another according to rules which must account for the $i\ =\ \sqrt{-1}$. This rule may be emulated by a symplectic matrix form $M\ =\ \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I are $n\times n$ matrices [15]. For n position variables and n momentum variables a vector composed of n position and n momentum variables is transformed by M in a manner that emulates complex structure. Quantum mechanics emerges with this replaced by a complex valued system of operators and state vectors. Complex numbers $z\ =\ x+iy$ are then a pairing of real numbers with multiplication rule that takes i into account. We may of course then pair up complex numbers into quaternions. It is possible to express gauge theory according to quaternions. In fact Maxwell's original formulation of electromagnetism was according to quaternions. The quaternions can generate transformations of a quantum wave function $\psi\ \rightarrow\ \psi e^{-i\int\mathbf{A}}$, where $\mathbf{A}$ is a one-form that may be either vector or quaternion valued. This is similar to the Aharanov-Bohm effect. The octonions are nonassociative, and quantum gravity fluctuations of spacetime near an event horizon "associate" a vacuum mode at a point in spacetime to both sides of the event horizon. Equivalently the event horizon is uncertain or blurred in such as way that it is not possible to assign the vacuum state close to the horizon to the interior or exterior region. In this sense there is a deeper level to the universe, where octonions contain a set of quaternions. Consequently if there is an underlying nonassociative structure to quantum gravity it does not change the logic of quantum mechanics.

The associator $[Q_a,\ Q_b,\ Q_c]\ =\ C_{abcd}Q_d$ is a vacuum representation of a quantum fluctuation of spacetime.

The entries in a channel with three slots $|[0(p_1),\ 0(p_2),\ 0(p_3)]\rangle$, with the associator, defines a zero point energy for uncertainty fluctuation of the horizon. The structure constant $|C_{abcd}| \sim \frac{\hbar c^2}{G}$, which requires a noncommutative uncertainty $[p_\mu,\ p_\nu] = \sqrt{\frac{\hbar c^2}{G}}\epsilon_{\mu\nu\sigma}p^\sigma$. This connects nonassociative systems with braid groups, which is discussed further in the supplemental section. Quantum gravity is then a quaternion theory, or a system of quaternions in the octonions. There is then a hierarchy $\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O}$, where classical mechanics is real valued, quantum mechanics complex valued, and underneath are quaternion and octonion valued fields and vacuum structures.

5 Qubits, Causality and Undecidability

The undecidability of "It From Bit" is a blessing. Just as Euclid's 5^{th} axiom is undecidable lead to a much richer variety of geometries, so too the undecidability of "It From Bit" can lead to more general theories of quantum information. In this case we see that "bit" is not able to account for additional nonlocal properties, but with the inclusion of the associator we can now address this. Any model of causality is necessarily incomplete, and this may serve to enlarge our concepts in physics. The axiom that is being removed from the domain of physics here is associativity of all quantum fields, or any underlying structure to such fields. The proposal here is there exists an associator with respect to the underlying vacuum of field theory.

Models of causality, whether they are aspects of standard physical theory or metaphysical models in the philosophical sense as with von Wright or Wallace, are necessarily incomplete. The incompleteness of metaphysical models illustrates the general nature of incompleteness that translate to standard physics. The incompleteness of $\top$ of all causal propositions is undecidable, but the existence of physical states implied by this set means that certain physical states can exist for reasons not computed by the "rule book." The heuristic invoked here is that this concerns the nonlocality of quantum gravity and the existence of a new structure. The physical axiom that is proposed to be removed is associativity. This introduces a different level of quantum nonlocality which may be important for quantum gravity.

The incompleteness of modal causal models is argued to justify nonassociativity as a means towards nonlocality of the quantum gravity field. This is a "bottom-up" type of argument, where an incompleteness of a higher level physics requires a more fundamental physics "further down." It is entirely possible this could be used to argue for a "top-down" physics with the emergence of higher level properties. There is a prospect this may play a role in the emergence of biology and even conciousness.

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6 Supplementary Note on S-Matrix and Nonassociativity

A toy model of the S-matrix can be proposed. The S-matrix is meant for a scattering region $\{-\infty,\ \infty\}$, which can work for a BH in certain coordinates. The final evaporation of a BH brings the singularity close to the horizon in Schwarzschild coordinates. Let $2m/r\ =\ e^u$ and with $dr\ =\ -2me^{-u}du$ the metric is

$$ds^2\ =\ (1\ -\ e^u)\,dt^2\ -\ 4m^2e^{-2u}\left[\frac{1}{(1\ -\ e^u)}du^2\ -\ d\Omega^2\right]. \tag{8}$$

The singularity is at $u\ =\ \infty$, and he singularity removed to infinity vanishes once the black hole evaporates. This suggests nonlocal properties of the singularity. This also permits a black hole to be considered according to the S-matrix

The black hole singularity introduces a topology change [14]. Outward and inward fronts of light emitted by a sphere just inside the horizon converge so their areas are less than light emitting sphere. If these light fronts were absorbed at two different points the two null sheets would define a 2-sphere. However, they both terminate on a 3-d spatial surface, and this null surface is instead a 2-torus. The source is the singularity where the Weyl curvature diverges. Exterior states exist on a different topology than interior states. The entropy is then $S\ =\ S_{local}\ +\ S_{top}$, where the S_{top} is a quantum property of the singularity.

The ordered S-matrix defines each vertex, or particle, and its neighbor. In a linear chain a general state is an S-matrix channel of the form,

$$|\phi\rangle\ =\ |p_1,\ \dots,\ p_i,\ \dots,\ p_j,\ \dots,\ p_n\rangle \tag{9}$$

that is distinct from the channel

$$|\phi'\rangle\ =\ |p_1,\ \dots,\ p_j,\ \dots,\ p_i,\ \dots,\ p_n\rangle. \tag{10}$$

The S-matrix is written according to $S\ =\ 1\ -\ 2\pi iT$, so two states or channels $|\phi\rangle$ and $|\phi'\rangle$ are related to each other by the S-matrix as

$$\langle\phi'|S|\phi\rangle\ =\ \langle\phi'|1\ -\ 2\pi iT|\phi\rangle\ =\ \langle\phi'|\phi\rangle\ -\ 2\pi i\langle\phi'|T|\phi\rangle \tag{11}$$

The in and out channels p_1 and p_n are neighbors through the T-matrix. This eliminates an open vertex in the chain. The vertices or particles p_1 and p_n are the open elements in the chain and defines the ”anchor” for the chain, and are thus defined as neighbors or connected for form a loop. This is a time sequence of fields or equivalently a cycle of gauge fields on a compactified space, similar to a necklace of groups on a wrapped string. Each element p_i defines a particle or vertex according to a set of quantum numbers and is defined in a vector space V_i, which is some Hilbert space. The linear chain here is an ordering on a total Hilbert space $\mathcal{H} = \hat{\text{A}} \otimes_i V_i$. This construction is based upon relationships between p_i and p_{i+1} by bilinear operation of the form $[-,\ -] : V \times V \ \rightarrow\ V$, as a product structure for position exchange. To define physical states this bilinear operation must obey the Jacobi identity. This requires the vector space be k-equipped so the bilinear operation is an isomorphism on the vector space $\mathcal{H}\ =\ k \times V$, where the modulus $|k|$ is the number of elements in the chain [17]. This gives the isomorphism,

$$Y : \mathcal{H} \times \mathcal{H}\ \rightarrow\ \mathcal{H} \times \mathcal{H},\ Y((x,\ p) \otimes (y,\ q))\ =\ (x,\ p) \otimes (y,\ q)\ +\ (1,\ 0) \otimes (0,\ [p,\ q]) \tag{12}$$

The application of $Y \otimes id$ on $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ then gives

$$Y \otimes id((x,\ p) \otimes (y,\ q) \otimes (z,\ r))\ =\ (x,\ p) \otimes (y,\ q) \otimes (z,\ r)$$

$$+\ (1,\ 0) \otimes (0,\ [[p,\ q],\ r]\ +\ [[q,\ r];\ p]\ +\ [[r,\ p],\ q]), \tag{13}$$

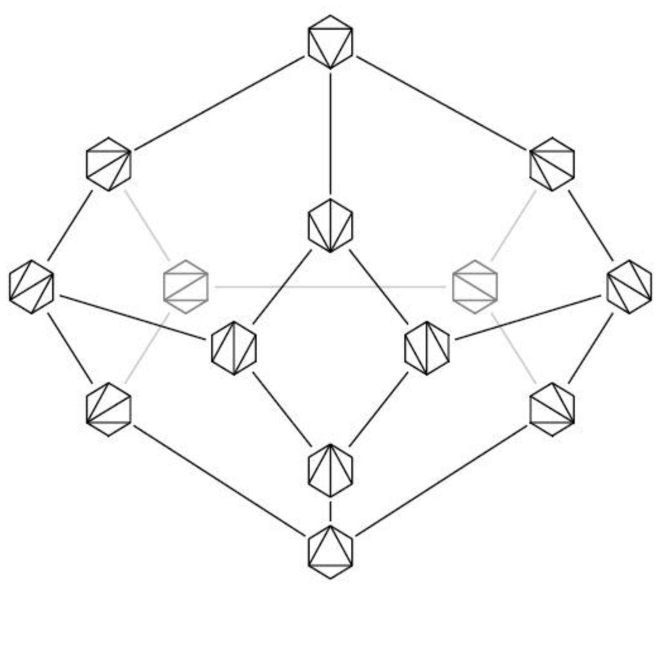
which is the Yang-Baxter equation. If the permuted double commutator sum vanishes as the Jacobi equation. The elements $p,\ q,\ r$ as momentum operators $D\ =\ \partial\ +\ iA$, defines Jacobi identity the conservation law $[[D_a,\ D_b],\ D_c]\ =\ \epsilon_{abcd} D_e F^{de} = 0$. This has the structure of a Yangian, a conformal dual type of gauge theory [18]. The Yang-Baxter equation describes braids, which are compositions of paths. In general this theory must be extended to compositions of loops. The S-matrix acts upon a loop composed of $\langle \ell|$ and $|\ell'\rangle$ to define the composition of two loops with $2\pi i \langle \ell | T | \ell' \rangle$. Homotopy is the mathematical theory for loop topology. For a topological space $(X;\ p)$ the loop space ΩX is defined by the continuous map

$$\phi : [0,\ 1]\ \rightarrow\ X, \tag{14}$$

with the compact open-set topology on the endpoints $\phi(0)\ =\ \phi(1)\ =\ p$. Here the vertex or particle p is considered to be the base point of the map. The composition of loops obeys the rule,

$$\pi_1\ \rightarrow\ \Omega X \times \Omega X\ \rightarrow\ \Omega X, \tag{15}$$

Higher homotopies exist for spaces with larger dimensions [17], where the ordering of homotopies determines the vertices of associahedra. A braid is a (ab)–(ba) edge-link, and an associator is $(ab)c$–$a(bc)$ for fields defined on the vertices . We have now introduced the association of fields in a channel. The associators with three elements define two hexagons, which link vertices in associator by commutation of the elements in parentheses. Braid links between the commuted vertices defines the general system of associators plus commutators. The associahedra K_4 for four elements is a pentagon, with hexagonal links at each vertex. In three dimensions the Stasheff polytope K_5 or associahedra[19]. This is within the theory of operads for the topology of configuration spaces.



The S-matrix as presented here however differs from the standard definition. The entries in the S-matrix $|\phi_1,\ \dots,\ \phi_i,\ \dots,\ \phi_j,\ \dots,\ \phi_n\rangle$ represent fields in some order. These fields emerge from the action of operators $\bar\Phi_i$ on Fock space entries, which above we have labeled as p_i. The nonassociativity is then not on the level of the fields, but on the level of the Fock space. Quantum mechanics is purely complex valued, not quaternion valued. Gauge fields may be represented as quaternions, but the quantum wave which result from the application of such operators is still complex valued. By the same reasoning the underlying state space may be nonassociative, but the application of the field operator produces the field in the matrix slot. This means the above S-matrix operator of the form $\mathcal{S}\ =\ \prod_m \bar\Phi_m S \prod_n \Phi_n$ that acts upon a channel with potentially nonassociative structure. For further physics oriented presentation of Yangians and self-dual conformal theories see [20]. This type of structure should then be associative on the level of particles, real fields and excited states of oscillator states, but where underneath on the vacuum level a nonassociative level exists.