A fundamental space-time geometry: does it exist?

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Abstract

The search for a quantum theory of gravity must include the recovery of the classical space-time. We consider some of the difficulties that must be confronted in any such enterprise. These problems seem to go beyond the technical level, to the point of questioning the overall feasibility of the project. The main issue is related to the fact that, in the quantum theory, it is not possible to assign a trajectory to a physical object and, on the other hand, according to the basic tenants of the geometrization of gravity, it is precisely the trajectories of localized objects that define the space-time geometry. This indicates that we should revise the standard geometrical concepts and explore the corresponding notions that could have, at least in principle, operational meaning and that would be at the heart of the above mentioned recovery. The insights gained in this analysis should be relevant to the quest for a quantum theory of gravity even before such theory is completely developed, and might help refocus some of its goals.

1 Introduction

The recovery of classical space-time from theories involving some fundamental space-time discreteness, like Loop Quantum Gravity [1], Causal Sets [2], Causal Dynamical Triangulations [3], *etc.*, is a highly non-trivial task. This issue is in principle distinct, but perhaps not completely independent, from the so-called problem of time, afflicting approaches that involve canonical quantization, and is present even in covariant theories like Spin Foams [4]. The problem we want to consider is tied to the recovery, at least at an effective level, of the standard geometrical notions associated with classical General Relativity (GR) from such discrete structures. Naturally, the space-time metric that would emerge from a generic fundamental theory of space-time (FTS) must play its standard role, namely, it should encode the manner in which free physical objects move, without the need to postulate that free particles follow the geodesics of a space-time metric which is obtained from the FTS through some independent procedure. That is, even if one learns to build the space of physical states of FTS, and to construct the operators associated with geometrical quantities, such as areas or lengths, it is essential that such notions play the role they have in the classical theory. In other words, space-time geometry should be at the basis of the behavior of the physical objects, and it is this feature, which should be a direct consequence of the theory, what should give empirical meaning to the space-time metric.

It is not generally recognized that it is very complicated to fully address the issue of the operational determination of the space-time geometry in a certain region. This is because, to start with, one would need to label the space-time events using some sort of coordinate chart[1], and then use free point-like particles to trace the space-time geodesics. This procedure allows the reconstruction of geometry by integrating, say, the Christoffel symbols. Mathematically this is a non-trivial task, and integrability is expected to be ensured by certain conditions which would be among the first predictions of the FTS. That is, in case these resulting equations cannot be integrated to produce a space-time metric, one would have to say that the theory is incorrect. Thus, at the level where we accept the hypothesis of availability of point-like free particles, the notion of geometry would be, in principle, well defined. From this point of view, the question we must face in the quantum context is: What would define the space-time geometry in the event we replace the hypothesis of the existence of point-like free objects by more realistic objects that take into the account the quantum nature of our world?

To earnestly address the question above seems daunting to the point of practical impossibility. This is why we limit ourselves to an initial exploration of related issues, making along the way simplifying assumptions where most technical problems are not an obstacle. We warn the reader that nonetheless, we shall find various unexpected problems that should also appear, perhaps in a more problematic fashion, in the program we outline above. In fact, we will point out some serious obstacles facing this program that seem to indicate that the issue of interpretation in any FTS is by far much more problematic than one might have anticipated.

The first thing we do is to rephrase the question in a much less fundamental way, which, we hope, can still shed some light into the issues encountered in the exploration of the full problem. That is, we consider the question: Assuming that a classical underlying space-time geometry exists, how does the quantum nature of the available probing objects modify the way in which we perceive the geometry? This question suggests the possibility of trying to define an effective space-time geometry that would be extracted when exploring the world with slightly more realistic objects, assuming, for the moment, a classical underlying space-time geometry.

Our point of view is that when obtaining the general relativistic characterization of gravitation from FTS, one must also recover the characterization of matter in terms of particles, which, when free, should follow the geodesics of the emerging space-time. Clearly,

[1]One can use a coordinate system build with physical objects relevant to the problem such as the null rays of the geometry corresponding to the paths of point-like photons. Thus, the issue of coordinates might be incorporated, in a self consistent way, as part of the general recipe to define the space-time geometry.

such recovery is possible only in the limit in which the resulting geometrical operators have large expectation values when compared with the corresponding uncertainties. Therefore, we face the issue of extracting from FTS the objects that ought to play the role of particles in the classical regime. We thus assume that the matter degrees of freedom (MDF) are incorporated in FTS and are associated with quantum fields defined on suitable space-time regions. One can hope to identify a particle if, for example, one is given something like the energy-momentum tensor for the MDF as a suitable operator $\hat{T}_{ab}$. In that case we can focus on the expectation value $\langle \hat{T}_{ab} \rangle$ in the corresponding state and we define the space-time region $\mathcal{U}$ where $\langle \hat{T}_{ab} \rangle$ is non-zero as the region occupied by the object. We recognize a "free particle" using the notion that any interaction other than gravity involves matter (gauge) fields which have support in the region between the interacting particles. Thus if we can identify a region where the support of $\langle \hat{T}_{ab} \rangle$ vanishes outside a world-tube $\mathcal{U}$, we could say, that we have, an isolated object. Furthermore, if the support can be taken as sufficiently localized we could identify that object as a particle.

In order to proceed, we need to extract a world-line in $\mathcal{U}$, characterizing the particle's evolution, which is to be identified as a geodesic of the space-time. It is clear that the world-line has to be defined in a canonical and background independent way, otherwise we might introduce ad hoc structures in conflict with the underlying diffeomorphism invariance of the whole construction, thus ruining any chance of recovering GR. The most natural option available corresponds to choosing the covariant center of mass of $\langle \hat{T}_{ab} \rangle$. Note that we are also simplifying the problem by assuming that the underlying space-time (classical) geometry is given *a priori*. The covariant definition of the center of mass in curved space-times was originally formulated by Dixon [5] and it is presented in the technical section at the end of the manuscript for the particular case of an extended particle that is a collection of N free, test, massive point-like particles. In order to be certain that the center of mass can be defined, we assume the hypothesis of Ref. [6] which essentially implies that space-time curvature in the region where the object is located is small in comparison with the size of the extended object, in such a way that $\mathcal{U}$ is a normal convex hull.

2 Effective geometry

The situation we have in mind is one where we do not know the space-time geometry in some region and we want to extract it by studying the trajectory of the center of mass of some extended test objects. As it was pointed out long ago [7], such center of mass world-lines generically do not correspond to geodesics of the underlying metric. This observation implies that with this procedure one cannot extract the background space-time geometry. Nonetheless, we can still ask if, for a given extended object, there is an effective geometry such that its geodesics coincide with the trajectory of the center of mass and which depends on the extended objects' characteristics.

Space-time curvature can be empirically inferred through the relative acceleration of infinitesimally close geodesics. We thus define an effective space-time curvature as the entity that describes the relative acceleration of the center of mass world-lines X of neighboring free extended objects. More concretely, if we have a uniparametric family of extended objects where we select the object with a parameter κ and denote by T the unit tangent of the center of mass world-line, then the deviation vector of neighboring geodesics can be written as

$$Z = \frac{\partial X}{\partial \kappa} + bT, \tag{1}$$

b being a scalar, fixed by the condition that Z be orthogonal to T. Then, the relative acceleration, a, is defined as the second covariant derivative of Z along T, while the effective curvature tensor in the probed region, $\widetilde{R}^d_{abc}$, is such that

$$a^a = T^c \nabla_c (T^b \nabla_b Z^a) = -\widetilde{R}^a_{bcd} T^b Z^c T^d \tag{2}$$

holds. We have carried out a series of detailed studies using this approach for the case of underlying space-times of constant curvature and simple extended probes and found that the resulting effective curvature can deviate substantially from the underlying one [8].

Observe that, in general, the effective geometry depends on the details of the extended object playing the role of the free particle. This, by itself, seems to cast serious doubts on the possibility of recovering, in this way, an object-independent space-time geometry. In addition, in Ref. [8] it is shown that further complications arise when attempting to fulfill the program outlined above. First, the effective sectional curvature depends not on one, but on two extended objects, those whose center of mass world-lines define it, via their relative acceleration. One might consider assigning a different effective geometry to each type of object, but defining when two extended objects in different regions of space-time are "identical ", seems an impossible task in general space-times. Second, the effective curvature can be read off only in the plane defined by the vectors Z and T; the other components of $\widetilde{R}^d_{abc}$ cannot be defined without considering additional objects, and of course, that brings back the issue of defining "identical extended objects" in curved space-times. As we comment in more detail below, this is a serious drawback for the program described here. Remarkably, all these complications do not appear in flat space-time because, in that case, the center of mass always follows a geodesic [9].

An important difficulty with Dixon's covariant center of mass, which, as far as we know, has not been noticed before is the issue of associativity. Concretely, the problem is that when calculating the center of mass of an extended object, one cannot simply proceed to obtain first the center of mass of a part of the extended object, and then use it to represent this part when computing the center of mass of the entire extended object. This is essentially because the simultaneity hypersurface on which the various pieces of the extended objects are "summed up" is that of an observer that would see the extended object at rest (see the technical section at the end for the details) and, on the other hand, the velocity of this observer depends on the details of the extended object itself. In other words, when shifting our attention from an object to a sub-object, we have to use different spatial slices along which the relevant integrations are to be carried out. The fact is that, in calculating, say, the center of mass of three point particles A, B, C, by first "composing" two of them, and then composing the result with the third, one obtains different results depending on the order of the compositions. It seems that there is no way of defining an observer independent center of mass world-line which is associative, in the above sense, thus forcing us to relinquish one of the fundamental properties of the Newtonian center of mass. In order to recover associativity, one would have to sacrifice observer independence, and take a relational point of view, as is done in Ref. [8]. In this case one always sums along the simultaneity hypersurface associated with some particular observer, regardless of the object under study.

3 Quantum aspects

There are well known arguments describing the limits of localizability of objects in the Planck scale regime which are associated with black hole formation once sufficient energy is confined in a small enough region [10]. In this regard we would like to put forward a rather simple argument, suggesting that the issue ought to be revised. Suppose we manage to concentrate some amount of energy in a region that is not too small, say, it is a couple of orders of magnitude away from the black hole forming regime. Now, let us consider that situation as described by an observer who is boosted with respect to the frame of the description above. Let us assume that the boost factor γ is also a couple of powers of ten. Then the size of the region where the energy is confined is shrank by a factor γ while the energy concentrated in that region increases by the same factor γ. For the second observer, the black hole formation bound has been exceeded by a couple of orders of magnitude. Then we have a puzzle: the two observers disagree in their expectations of black hole formation. This illustrates the need to investigate further this kind of issues that lie in the interface between gravity and quantum theory, with more rigorous and secure methods, and that relying on heuristic arguments entails substantial risks.

Having said that, we embark on the analysis of a rather different issue which is connected with our general approach. We want to inquire about the way the discussions in the previous sections might impinge on quantum aspects of the description of the MDF. Here we concentrate, for simplicity, on a particle description, but we believe the characterization of the MDF in terms of fields defined on the space-time manifold would have to face similar issues. Let us recall that the quantization is designed to reproduce the symplectic structure, with the replacement of Poisson brackets by quantum mechanical commutators. Thus, modifications of the Poisson brackets would have repercussions in the construction of the quantum theory for the MDF. We now consider the Poisson structure of the variables characterizing the center of mass of the extended objects.

The position operator plays an important role in quantum mechanics. However, up to now there is no satisfactory definition of this operator in quantum field theories (QFT). On the other hand, as the center of mass can be constructed in terms of the energy-momentum tensor of the (classical) fields in space-time (see Ref. [6]), it is tempting to attempt to generalize the recipe to quantum field theories, where the energy-momentum tensor is an operator, and try to define a position operator following the center of mass construction[2].

We have discussed some interesting features of the notion of the center of mass and is natural to expect these would be shared by the position operator in QFT. It was pointed out by Pryce [9] that the observer independent center of mass in special relativity has non-vanishing Poisson brackets among its different components, in fact they depend on the total momentum and spin (total angular momentum with respect to the center of mass) of the extended object — this expression can be easily inferred from equation (3) below. This non-trivial expression stems from the fact that the relativistic center of mass weighs the position of the particles by their energies, rather than their masses, which, incidentally, allows massless particles to contribute to the location of the center of mass. As the particles' energies depend on their momenta, the center of mass can be thought of as a function of the position and momenta of the constitutive particles of the extended object, a fact that leads, generally, to non-vanishing Poisson brackets for its components. As is pointed out in reference [13], the resulting form of the Poisson brackets is remarkably similar to expressions found in rather different approaches.

It is also noteworthy that, in contrast with most works on non-commutative geometry, this non-commutativity arises naturally, and, moreover, it is compatible with special relativity. The point is that the non-commutativity here is tied to the object probing space-time and not to the underlying space-time itself. However, in view of the operational point of view we have adopted in this work, it seems that the first distinction would become meaningless in the context of FTS where geometry and its probes are so intimately connected. On the other hand a relational/operational point of view seems to create difficulties in connection with Lorentz invariance: In essence, we would not be able to consider an empty Minkowski space-time, simply because its geometry would not be defined in the absence of any matter probes. For a complementary discussion see Ref. [14].

Using the Poisson brackets involving the positions and momenta of point-like free particles in curved background, we can compute the Poisson brackets corresponding to our effective particle description of the extended objects representing the physical probes. The derivation is outlined in the technical section at the end of the paper. What we find is that the curvature modifies these expressions in a rather complex way. Let X^μ be the coordinates of the center of mass world-line and P^μ the components of the total momentum at X, then the

[2]In the generic QFT situation, this task faces several additional complications as compared to the N-particle situation we have been discussing. See for instance [11, 12].

corresponding Poisson brackets have the form

$$\{X^\mu, X^\nu\} = \frac{S^{\mu\nu}}{M^2} + \frac{S^{0\mu}P^\nu}{M^2 P^0} - \frac{S^{0\nu}P^\mu}{M^2 P^0} + \mathcal{A}(R, P, X, S), \quad (3)$$

$$\{X^\mu, P^\nu\} = g^{\mu\nu} - g^{0\nu}\frac{P^\mu}{P^0} + \mathcal{B}(R, P, X, S), \quad (4)$$

$$\{P^\mu, P^\nu\} = P^\rho \Gamma^\mu_{\rho\sigma}\left(g^{\nu\sigma} - g^{\nu 0}\frac{P^\sigma}{P^0}\right) - P^\rho \Gamma^\nu_{\rho\sigma}\left(g^{\mu\sigma} - g^{\mu 0}\frac{P^\sigma}{P^0}\right) + \mathcal{C}(R, P, X, S), \quad (5)$$

where $S^{\mu\nu}$ are the components of the spin of the extended object, $M = \sqrt{-g_{ab}P^a P^b}$ is its total mass, $g^{\mu\nu}$ and $\Gamma^\mu_{\rho\sigma}$ stand, respectively, for the components of the inverse metric and the associated Christoffel symbols at X, and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ represent complicated series of terms involving the space-time curvature, as well as X, P^μ and $S^{\mu\nu}$, which vanish in flat space-time. The point is that the curvature of the underlying space-time affects the Poisson structure of the classical theory in highly non-trivial ways. This should have important implications for the quantum characterization of the MDF in any emergent space-time, as the one that would represent the classical geometry obtained as an appropriate limit of the quantum space-time within FTS. The fact that fundamental Poisson brackets for the canonical variables corresponding to our extended particles in curved space-times depend on curvature in such a way, indicates that one can expect curvature effects in the quantum theory of these objects as well. Therefore, the minimal coupling assumption behind the general relativistic prescription will be hard to satisfy if we try to recover standard physics from a program such as the one considered in this work. However, it seems that such a program, or something very close to it, would be the framework that might allow us to claim that our fundamental theory reduces to standard physics in the appropriate limit.

4 Discussion

We have considered some obstacles that will have to be faced when trying to recover classical geometry from a fundamental theory of quantum gravity, in particular, one involving space-time discreteness. We have argued that, when recovering the classical space-time notions of GR, one would have to rely on the identification of states of the MDF that could play the role of free particles and, moreover, that the effective geometry should be read-off from the center of mass world-lines characterizing such objects. This has lead us to consider the characteristics of the center of mass of extended objects. We have done that in a simpler setting where a background space-time is assumed as given, a simplification that would not be available in any realistic situation. Nevertheless, we consider that similar problems, perhaps even more daunting, would have to be faced when recovering from FTS a classical characterization at the level of GR.

We pointed out the fact that the center of mass world-lines do not follow the geodesics of the background space-time. This suggests that, in extracting the effective geometry of space-time, one should identify *a priori* the center of mass world-line of a free extended object as a geodesic of an “effective geometry”. Thus, when considering that that the physical space-time geometry should be the one accessible in experiments, one would need to rely on such extended objects as the only ones available in nature. In particular, we noted that there would be a difficulty in finding all the components of the effective connection because several extended objects are needed to probe the given region in all space-time directions. In addition, it is not clear if the results would be independent of the nature of the extended objects, particularly in the case where the underlying space-time is discrete and has therefore high curvatures localized on the support of the quantum geometry state. We have also described a method for obtaining an effective curvature tensor by considering the relative acceleration between the center of mass world-lines of two nearby extended objects, and found that the results depend strongly on the detailed characteristics of such objects.

These difficulties in reading a space-time geometry with extended objects suggest that the task would be even harder when dealing with quantum probes. This raises the possibility that, in a regime where the quantum properties of the particles cannot be neglected, the geometrical language itself might cease to have meaning and a fundamental role, as it might not be properly defined. Of course, this issue will need to be overcome when trying to recover the GR limit from a theory of quantum gravity.

We have also shown that the covariant definition of the center of mass is not associative in the sense that it cannot be used to represent the parts into which one might divide an extended object. That is, in order to calculate the center of mass of the complete extended object, all the individual particles have to be identified as such, as neglecting their internal compositeness would lead to inconsistencies. This can have practical consequences when using the center of mass to characterize an extended object in regions with high curvature, but it is also interesting at the conceptual level because it suggests that the center of mass is only well defined if fundamental particles exist. This non-associativity of the center of mass construction would have also important consequences at the quantum level. In particular, it would cast serious doubts on the whole renormalization program for a quantum field theory that can be considered as effective and emergent from the underlying FTS. To clarify this point we recall that the notion that one can integrate over certain degrees of freedom (usually taken to be those corresponding to the ”short wavelengths”) and then consider as new fundamental variables the effective ones obtained from such integration, is at the basis of the renormalization group approach. The lack of associativity of the process of obtaining the center of mass world-lines, is clearly at odds with the notion of integrated variables and the mobility of the integration boundaries.

We also studied the issue of how to generalize the calculation of the Poisson brackets of the components of the center of mass to curved space-times and considered how this might indicate a problem when recovering the quantum characterization of the MDF. Interestingly, this problem seems to persist in situations where one would believe that the underlying quantum nature of space-time might be neglected. The non-trivial appearance of the space-time curvature in the commutators of the free particles implies that there could be curvature effects on the quantum theory of “point particles” extracted from the underlying FTS. This indicates in turn that, at the quantum level, one can expect obstacles to recovering the minimal coupling assumption behind the GR prescription. Effects characterizing the granular structure of space-time, of the form suggested in [15, 16], might then show up. Also, as the fundamental MDF cannot be taken as infinitely localized, it is unclear how to state the Equivalence Principle at the operational level.

There is still a lot of work to be done along the lines described in this manuscript. In particular, it is intriguing to see if it is possible to generalize the construction of the center of mass to situations where the MDF are described in the realm of quantum field theory in curved space-times. It seems that this would be a required first step in order to define a covariant position operator which could be used to determine some canonical world-lines that could be used to define an effective geometry. Naturally, such program would encounter similar difficulties to those we have described in this work. In fact, it can be expected that those difficulties would appear aggravated simply because, in that context, one needs to regularize and renormalize the energy-momentum tensor. In view of the non associativity we have discussed above, those steps indicate we should question the meaning of an effective geometry which would be tied to the motion of real physical matter probes. On the other hand, it is rather doubtful whether something that is not intimately related with the behavior of such matter probes should be rightfully described as the physical geometry of space-time.

After facing all these problems, in what seems to be a natural approach for extracting classical GR from FTS, we could be tempted to conclude that the underlying theory should be such that nothing like what we presented here ever arises. In that case, it seems, the underlying FTS should be of a rather different nature from the usual approaches and, in particular, it should not involve any fundamental space-time discreetness. At this point we do not advocate any of these postures but it seems that the only alternative would require to confront the various problematic aspects we have discussed in this work.

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Technical notes

Covariant center of mass in curved space-times

We consider an extended object, made up of a finite number N of free point particles, and define, following Dixon, its center of mass. Let x be an arbitrary space-time point and U an arbitrary 4-velocity at x. We construct the simultaneity surface Γ, which depends on x and U, generated by the geodesics through x and orthogonal to U. Denote by $y^{(i)}$ the point where the world-line of the i-th particle intersects Γ. By assumption, there is only one geodesic connecting $y^{(i)}$ and x, so that we may assign "position vectors" $\Xi^{(i)}_{x,U}$ to each particle, such that

$$\exp_x \left[\Xi^{(i)}_{x,U}\right] = y^{(i)} \,, \tag{6}$$

where $\exp_x$ is the exponential map.

Call $\tilde{p}^{(i)}$ the four-momentum of the i-th particle, parallel transported from $y^{(i)}$ to x along the geodesic connecting them. The total momentum $P_{x,U}$ of the extended object, with respect to x and U, is defined as

$$P_{x,U} = \sum_{i=1}^{N} \tilde{p}^{(i)} \,. \tag{7}$$

In general, $P_{x,U}$ will not be parallel to U. Define then a special U_0, which can be shown, under suitable hypotheses, to exist and be unique [6], by requiring that P_{x,U_0} be parallel to U_0 — this can be thought of as the four velocity of an observer at x, comoving with the extended object. *The* total momentum of the extended body at x is then defined as $P_x = P_{x,U_0}$. Define also *the* position and energy of the i-th particle at x as $\Xi^{(i)}_x = \Xi^{(i)}_{x,U_0}$ and $E^{(i)}_x = -g_x(\tilde{p}^{(i)}, U_0)$, where g_x is the metric at x. For a given reference point x then, the position vector of the center of mass is defined as

$$\Xi_x = \frac{\sum_{i=1}^{N} \Xi^{(i)}_x E^{(i)}_x}{\sum_{j=1}^{N} E^{(j)}_x} \,. \tag{8}$$

Finally, the center of mass world-line is defined as those special points x_0 for which Ξ_{x_0} vanishes — under the hypotheses of Ref. [6], the collection of all x_0 is a unique, differentiable time-like curve.

Poisson brackets for a free point-like particle in a curved space-time

We calculate the Poisson brackets among the position and momentum components of a free test particle in a generic space-time. The starting point is the Lagrangian associated with a space-time foliation for a single particle of mass m and the canonical conjugated momentum, which are given by

$$L=-m\sqrt{-g_{\mu\nu}(q)\dot{q}^{\mu}\dot{q}^{\nu}}\,,\qquad \pi_i=\frac{\partial L}{\partial\dot{q}^i}=\frac{mg_{i\rho}\dot{q}^{\rho}}{\sqrt{-g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}}}\,, \tag{9}$$

where q^μ is the position of the particle, the dot represents a derivative with respect to coordinate time $t=q^0$, and $i,j,k=1,2,3$. From this last expression it is possible to write p^μ in terms of the canonical variables. In order to do so we define $\gamma=1/\sqrt{-g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu}$ and h^{ij} such that $g_{ij}h^{jk}=\delta_i^k$. It can be seen then that

$$\dot{q}^i=h^{ik}\left(\frac{\pi_k}{m\gamma}-g_{0k}\right)\,. \tag{10}$$

For point-like particles we can assume

$$p^\mu=\alpha\dot{q}^\mu,\qquad -m^2=g_{\mu\nu}p^\mu p^\nu, \tag{11}$$

with α positive. Using equation (10) we get

$$p^0=m\gamma,\quad p^i=m\gamma\dot{q}^i=h^{ik}\left(\pi_k-m\gamma g_{0k}\right). \tag{12}$$

The Poisson brackets in question may now be calculated. Since $q^0=t$ has vanishing Poisson brackets with the canonical variables, we can write

$$\{q^\mu,q^\nu\}=0\,. \tag{13}$$

With the expressions (12) and the canonical Poisson brackets we can also get the brackets involving p^μ,

$$\{q^\mu,p^\nu\} = g^{\mu\nu}-g^{0\nu}\frac{p^\mu}{p^0}, \tag{14}$$

$$\{p^\mu,p^\nu\} = p^\rho\Gamma^\mu_{\rho\sigma}\left(g^{\nu\sigma}-g^{\nu0}\frac{p^\sigma}{p^0}\right)-2p^\rho\Gamma^\nu_{\rho\sigma}\left(g^{\mu\sigma}-g^{\nu0}\frac{p^\sigma}{p^0}\right), \tag{15}$$

where $p^0=\sqrt{m^2+p^ip_i}$, and the metric and Christoffel symbols are evaluated at q. It is easy to check that these expressions reduce to the usual ones in flat space-time and Minkowski coordinates.These relationships, together with a suitable covering by "freely falling frames" of the regions of interest in the curved space-time, are taken as the staring point for the results described in Sec 3.