Let's consider two spherical chickens

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4:45 a.m. (jan. 2015, Pisa)

Phone rings. Only one person can call me this early. "Can you make it to be here in the office in 15 minutes?" boss says, with no preambles. The only admitted answer in these cases is "Yes Sir!" and is implicitly assumed. "Its about three suspected murder cases, and cor blimey if they are not related! We only have to find how. By the way, we are talking 10 thousand bucks... Hurry up, son!" Click.

The strength of our Investigation Agency is, famously, the rapidity of our arrival to the crime scene, wherever it is located in space, time, or... elsewhere. Well, this morning we have really outperformed ourselves!

7:00 a.m. (523 b.c., Crotone, southern Italy)

Glauco, the young fisherman, is shocked. He can't stop looking at the pale face of his older brother Daippo, whose corpse lies on the sand in front of their humble thatched hut. That's where he found the poor man earlier this morning, back from nightly fishing. As two compassionate women take care of the body, I convince Glauco to answer some questions.

He has no doubts: not an accidental drowning, but homicide.

"Last year Daippo had been admitted to the *mathematikoi*, the restricted group of followers of Master Pythagoras. They are severely required not to disclose their findings, but he could not keep secrets with me. So, one evening

Figure 1: Two almost spherical, almost self-aware substructures (SAS).

he came back home with this device." He shows me a sort of rudimentary, one-string guitar. Must be the famous Pythagorean monochord!

"It seems (but we are not supposed to know!) that Phytagoras has been experimenting a lot with this object lately. Nice pleasant sequences of sounds can be obtained by placing a nut under the rope, but only at precise locations, in order to reduce its length by simple fractions - in particular 1/2, 1/3, 1/4, 1/5."

The four locations are indeed marked on the instrument, and now Glauco is plucking the rope, each time positioning the nut at one of them, in sequence. To my surprise, the ascending four-note melody turns out to be *exactly* the opening of Richard Strauss' epic theme 'Also sprach Zarathustra', made famous by the movie '2001, A Space Odyssey'.

"Nice, isnt it?" he continues. "Daippo was fascinated too by the Pythagorean jingle (that's how they call it at the School). But he thought he could do better, and built that thing."

He points to a table where five parallel ropes of equal thickness run a couple of inches above the wooden surface, fixed at a bridge at one side, and pulled at the other, through pulleys, by sand-filled bags of different weights. Basically, a gigantic, fretless bass guitar!

"Following the intuition of his Master, Glauco started by filling the bags with amounts of sand in the same simple ratios: 1 cup in bag 1, 2 in bag 2, up to 5 in bag 5. But the ascending tune obtained by plucking the ropes in sequence turned out to be ugly. Glauco then noticed that the nice interval perceived between the nut-free sound and the sound with the nut at position 1/2 in Pythagoras instrument is the same that we perceive between string 1 and string 4 of his instrument. Essentially, ratio 2 in the first case corresponds to ratio 4 in the second!

This has suggested my brother to *square up* the amounts of sand in the successive bags, filling them with, respectively, 1, 4, 9, 16, 25 sand cups. That's how they're filled now. Oh, sorry, this 'squaring' operation is a technicality that turns out to be useful also in a theorem that Pythagoras... Well, never mind. The end of the story is that, when plucking the ropes of bags 4, 9, 16, 25, in that order, you get back, guess what?

"The Pythagorean Jingle?" I suggest.

"Bingo! Glauco was so excited about it that he started playing the tune over and over. At some point I was so fed up that I decided to go fishing. During the night some member of the brotherhood must have overheard the tune, immediately reporting back to Pythagoras. The end of the story is under your eyes now". He turns to the white cloth wrapped around his beloved brother and explodes in a desperate crying. It is clear that his testimony is over.

I walk to the beach, pick up a thin stick and write on the foreshore Mersenne's formula for the frequency f of a string, as I recollect it from high-school:

$$f = \frac{1}{2L}\sqrt{\frac{sandCups}{\mu}} \tag{1}$$

where L is the rope length, *sandCups* is the amount of sand in the bag, and μ is the string mass per unit length. The formula tells clearly that one obtains the same frequency interval by multiplying the string length L by $1/n$, or the sand quantity by n^2. Good, I think I have some useful clues here.

9:00 a.m. (late june 2008, Univ. of Vermont at Burlington)

A four-bed room in the undergraduate dorm. Four young girls in tears. A boy and two ropes (again!) on the floor. The boy is from Peru', his name is Quipo [1]. As an energetic nurse is attempting cardiopulmonary resuscitation, I interrogate the girls. They are here for the 'π Summer School', a two-week mathematics enrichment programme for teen-agers. Yesterday they had a very exciting lesson on Knot Theory by the distinguished mathematician John Horton Conway, who invited them in front of the young audience to volunteer for an experiment with two ropes. The girls had to stand at the corners of an ideal square, each holding firmly one rope end, and to move in a sort of dance that admits only two kinds of collective moves:

Swap The girls at the upper-right and lower-right corners swap their positions, the former holding the rope up over the latter while moving.

Rotate The whole quartet rotates counterclockwise by 90 degrees.

A fifth volunteer is keeping track of what happens by writing on the blackboard a sequence of numbers. Initially the two ropes are parallel and untangled, one at the upper and one at the lower square edge, and the first number is $x = 0$. At each move a new number x' is added to the sequence: $x' = x + 1$ for *swap*, $x' = -1/x$ for *rotate*. One of the girls hands me a crinkly sheet:

$tangleList = \{swap, swap, rotate, swap, swap, swap, rotate, swap, swap, swap\};$

"This is the 10-move sequence we started with", she says.

"And the result was a real spaghetti tangle", adds a second girl.

"Then Prof. Conway invited us to disentangle it, says the third.

"And everybody in the audience started shouting!", adds the fourth.

"But eventually we made it, and this is the 13-move sequence that completely disentangled the spaghetti", says the first girl, handing me another sheet.

$$untangleList = \{rotate, swap, rotate, swap, swap, rotate, swap, swap, swap, \\ rotate, swap, swap, swap\};$$

"Ok. That was yesterday morning. How about last night?" I ask.

"Well, Quipo had missed the morning lesson, so after dinner we invited him in our room for repeating the experiment." (Blushing.) "We even wrote a simple program to make double-sure that the numeric sequence would bring us back to 0." She shows me her tablet (Appendix A.)

"So what went wrong?" I ask, after checking that the second list actually terminates with '0'.

"Nothing went wrong. The mathematics was correct!" she proudly declares.

"I see that! But what happened to Quipo?". I'm losing patience.

"Well, for him to better understand the experiment, we invited Quipo to stand. . . at the center of the square. Then we moved around him, reproducing the two sequences with extreme care. At the end we started pulling, and pulling, and pulling, and pulling... If $x = 0$ the ropes must untangle, isn't it? It's a theorem! It's true! It must be true! Damn!". In saying this she starts crying, and in a moment all four are crying in unison.

[1]For the origin of this name, see http://en.wikipedia.org/wiki/Quipu

Fortunately, they are soon interrupted by the arrival of the gurney. The nurse announces that Quipo might survive, and quickly leaves the room with him, followed by the girls.

I wonder how they select the students for these programmes! No doubt the math correctly describes the behaviour of the physical system of *two* ropes. But, even ignoring Knot Theory, it is clear that, topologically speaking, Quipo was the *third* rope. Different system, different rules! Anyway, I start to glimpse a pattern here.

11:00 a.m. (circa 2075, New York state)

As I said, we land *anywhere*, fiction included.

I 've just entered the huge billiard room. Professor James Priss, two Nobel prizes for physics, is still wearing the dark protective glasses that were handed out to all participants. Edward Bloom, the multi-millionaire who built his fortune by engineering Priss's theoretical findings, is lying on the floor; a hole the size of a billiard ball has crossed his chest, completely removing his heart. All the media circus, gathered here to cover the match between the two life-long friends and rivals, is slowly leaving the room. The event, organised by Bloom for illustrating the exploitability of Priss' Two-Field Theory, has terminated in the most unexpected, tragic way.

The scene is exactly as I pictured while reading 'The Billiard Ball' by Isaac Asimov. The two poles of a huge anti-gravitational device are positioned above and below the billiard table, perfectly aligned with a 12-inches hole at the center. Only a few minutes ago everybody could see the glowing cylinder of the zero-gravity field materialize between the two poles, across the hole. Then Bloom invited Priss to take the cue and direct the ball into the cylinder, to be the first in history to experience the effects of the zero-gravity field on a massive body. Priss accepted with apparent reluctance, but he took a significant amount of time, as if pondering behind his dark glasses (which he touched and adjusted several times), before deciding about the optimal trajectory. In light of the life-long rivalry between the two characters, and of Priss' exceptional skills both in physics and in billiard, the suspect of a perfect homicide is quite legitimate.

"Professor Priss, do you think that, once programmed with all the equations of your Two-Field Theory, and all other known theories, including, say, the physics of elastic collisions, a *gigantic supercomputer* could, very hypothetically speaking, ehm…figure out the relation between the incoming and outgoing trajectories and velocities of a billiard ball crossing a zero-gravity cylinder?"

"I suppose your *gigantic supercomputer*", he smiles somewhat sardonically, "is also aware of the exact initial conditions - distribution of planets, stars and so on, around the cylinder, right?" he remarks.

"Oh, sure!"

He takes a deep breath. "My friend, you don't need to beat around the bush! You see, whatever I tell you now will not appear in Asimov's novel, which is the only place where I exist. My worldline is completely determined."

He takes off his dark glasses and hands them to me. "A gigantic computer? That's a century old technology. Ever heard about Google Glass v.18? You see, it all fits in the side temples. Look."

As I wear the glasses, an augmented reality scene appears to me, with stylised billiard balls and virtual dotted trajectories dancing before my eyes.

"What not even Asimov knew is that a mole in Bloom's team was keeping me informed about his plans. So I had plenty of time for programming this little toy. At the right moment, it was easy to swap glasses and have them assist me in the perfect shot."

1:00 p.m. (jan. 2015, between New York and Pisa)

I'm driving back to the Agency headquarters, in good company with the 'Police' and a triple sandwich. The illumination comes as Sting starts one of my favourites: 'Murder by Numbers'.

That's obvious! The common denominator of the three crimes is *the unreasonable correspondence between mathematics and reality.* We reliably use simple numeric ratios for describing nice musical intervals, algebraic operations for describing and predicting the behaviour of knots, and compact equations for telling the behavior of vibrating strings, massive bodies and fields. I think I have a culprit, and I can't wait calling the boss.

"So, your point is that some people here have died because math supposedly describes the physical world fully, reliably and persistently?"

"Exactly! Imagine a world *not* ruled by mathematics: the Pythagorean jingle could't have been conceived, let alone reproduced, and the sequence of knots, or the trajectory of the billiard ball, could not have been directed to their lethal final states. Can you imagine such a messy world?"

"Of course I can!", says the boss "That's exactly *our* world! I can mention numberless cases of people who were killed precisely because the world is *not* ruled by math. Randomness is the greatest killer! Do you remember that case down in Vegas? Was it not the outcome of a totally random event?".

"You mean the Russian roulette case? Well, the position where the revolver cylinder stopped was *not* random. Of course it was unpredictable to the poor guy but, had he been given all the initial values, he could have computed the outcome."

"Look, if math were so effective in describing what *really* happens in this world, I would have hired a mathematician, not you! Math may well be *unreasonably* effective, but only for describing a few *unreasonably* idealized cases! Remember the brilliant brains we consulted for the henhouse case? They started with *two perfectly spherical chicken*! (Fig. 1.) Adding a third immediately got them into trouble! In the Russian roulette case, initial conditions involve the movement of the hand which transmits the rotation, the brain that controlled that hand, the neurochemical processes in the brain, the physics behind them, and so on ad infinitum. How can you expect to mathematically tame this infinite mess?"

"Not ad infinitum. Maybe there is a bottom" I suggest.

"Then you must fully specify the initial conditions down there - real variables that require infinite decimal expansions: unfeasible. Add the butterfly effect, and things become *totally* unpredictable. Randomness strikes back!"

"Maybe the bottom is discrete and finite instead, and can be described *completely*, like the input data of a computer program" I timidly propose.

"Ok buddy, I admit that we have often cracked hard cases by making bold hypotheses, but this is the craziest conjecture I've ever heard!" Click.

That made me thirsty. As I grab the bottle with one hand, driving with the other, a voice from the back makes me jump. "The right question is not *whether* mathematics describes the physical world - which is obviously true - but *why*!"

"Prof. Priss, what are you doing here?"

Priss quietly smiles at me. "I'm just taking advantage of my status. But let's not digress. The topic is intriguing, and I hear you hire scientists for consultancy. Two Nobel prizes are not peanuts, and Physics must have progressed a bit from 2015 to 2075, don't you think?"

"Of course! You mean that by 2075 everything has been figured out?"

"Unfortunately not. But first, how about a pizza?"

2:00 p.m. (jan. 2015, pizza place near Pisa)

After a 'napoletana' and two glasses of local wine, Priss is ready for conversation. The following is a faithful transcript from my voice recorder.

"So let's start where your phone call was interrupted. You were completely right: physical reality *does* sit on a discrete, *deterministic* bottom layer. This is clear now - I mean, in 2075! It was 't Hooft who gave the strongest contribution to the grand return of classicality, and the downgrading of quantum mechanics to the status of a mere tool. In his 2014 paper [6] he started showing that Einstein was right: *at its most basic level, there is no randomness in nature, no fundamentally statistical aspect to the laws of evolution.*"

Now, more than *sitting* on that mathematical layer, physical reality *is* that layer, and that layer is all there is: all the rest is redundant baggage - convenient names and concepts that we humans have invented for structuring our descriptions of that reality, for matching the limited bandwidth of our senses and lab equipment. Understanding physical reality means simply *discovering* the mathematical properties of that glimmering piece of math at the bottom"

"Wait! Are you saying that those bizarre ideas attributing an equal status of existence to *all* mathematical structures - the platonic Level IV multiverse - are still being considered in 2075?" [7, 8]

"There is nothing bizarre in viewing the fabric of the universe as a nice piece of math. What else? Take any slice of the spacetime cake and try to provide a baggage-free description of it. When *human baggage allowance* drops to zero (see, I'm using Tegmark's terminology) what is left can only be a mathematical structure. *What else*?"

"Put it differently", he continues, "as you increasingly magnify the fabric of reality, familiar properties tend to vanish. The smell of this pizza? The taste of this wine? The mass of that particle? All gone. What else can be left, other than a purely abstract structure? Now, what is the most abstract, featureless thing that you know? One without smell, color, extension, spin?"

"The point! "

"Good. You start to follow! So now take a few points and some relations among them. What mathematical object do you get?"

"A graph?"

"Fair. What properties does it have?"

"None! No smell, no taste. Nothing!" I am really starting to follow.

"Wrong! It has symmetries, and they can offer you a lot."

"Oh, ok. Anyway, a graph is only one out of many possible abstract structures. How about the others? Are there infinite parallel mathematical universes?"

"Few physicists still worry about this question, in 2075."

"Why?"

"Because of the Priss-Gödel-Priss Theorem (2031):

> *All mathematical structures entailing conscious entities: (i) are defined by total, computable functions, and (ii) are isomorphic.*

The first part proves that a physical universe hosting consciousness only requires *total computable functions*, those that can be computed by a Turing machine that halts for each input. The Theorem proves and refines a conjecture by Tegmark [7, 8], and is consistent with a fundamental discreteness assumption, since the computable functions from N to N (N are the naturals) can be put in one-to-one correspondence with N itself. As you know, the set of all functions from N to N is much larger than that; those that are left out are not computable; it is remarkable that we don't need all that stuff for defining conscious entities. On the other hand, the ontological status of mathematical structures involving non-computable functions, or even restricted to computable functions, but not entailing consciousness, is, in my opinion, irrelevant". Priss takes a sip of coffee, and continues.

"Well, the story behind the proof is also quite interesting. Initially I started with simple toy physical universes - no life, no consciousness, say just flat Minkowski space - trying to associate them with the right mathematical structures. Easy, you might guess. No way! As I was kicking a non-computable function out of the door, another was entering through the window! Then I went the opposite extreme, and considered a system of two moderately conscious entities having an interaction in empty space - what got to be known as Priss' *two-chickens problem* (Fig. 1) - modelling them along the lines of Tononi's seminal work on consciousness and integrated information [1]. Surprisingly, all non-computable functions disappeared for good, and all the other pieces of the inert physical world popped out and found their place in the picture. *In a way, it is the very presence of consciousness that makes the physical world so simple, and mathematically describable.*"

(Mental note: this is definitely going to be the grand opening of my report to the boss.)

"The second part - the isomorphism result - essentially means that there is *only one* possible structure which can host conscious beings like you and me... well, maybe just you. Of course, the Theorem does not put any limitation to the variety of shapes that consciousness may assume in our unique universe, which are not restricted to soliton forms. A generalisation of the two-chicken system, for example, led me to discover sophisticated forms of consciousness in vacua from the Standard Model - something that popular wisdom has believed since antiquity. Needless to say, symmetry groups play a crucial role in their classification."

"Crucial, of course... Hmm, Prof. Priss, I'm afraid it's time to leave now. This is so interesting!"

3:00 p.m. (jan. 2015, Pisa headquarters)

Parking near the Agency. "Here we are, Prof. Priss! I'd be delighted to introduce you to my boss. Oh, don't forget your bags."

"Thanks, but I always travel *baggage-free*. And, if you don't mind, I'd like to pay a visit to Galileo's pendulum first. I see the cathedral behind the corner.

I'll be back soon."

"You silly!", says my boss, as I complete my report, "you set free the only suspect who confessed his murder! How can you expect him to come back?"

"He will. He has nothing to lose: his worldline is frozen. Like yours and mine, for that matter. We are part of a unique, finitely describable, discrete mathematical structure that includes everything. Free will and the flow of time are pure illusions in the eyes of us frogs - they disappear in the bird's view".

"Birds, frogs, chickens: are you sure Priss is a physicist? Anyway, while you were being indoctrinated by that criminal, I pondered on our phone conversation, and came to some conclusions. Let me tell you.

Your crazy idea was to place a discrete, finite structure, like the input of a computer program, at the bottom of the universe. What's the obvious crazy thing to do next? Look for the right program, run it, and get this universe out! After some googling, I actually found that people like Zuse [11], Fredkin [3], Wolfram [9] have been playing around the idea for decades. For example, programs as simple as elementary cellular automata (CA) can mimic some key features of Nature: not only pseudo-randomness, or deterministic chaos (my favourite!), but also self-similarity and particle interaction."

"I know. For that you need, respectively, automata 30, 18, 110 [9]. But then you get three distinct universes, not one."

"False! Look at Fig. 2. This is three-color CA n. 738926016882 [5]. It proves that you can get those *three* fundamental aspects of physical reality out of *one* deterministic algorithm, even starting with the simplest initial condition! The algorithm is indeed just a function $f : S^3 \rightarrow S$, where $S = \{0, 1, 2\}$ (Appendix B) so, in a way, Tegmark is right in saying that Nature is math [7], but, I add, *only at the bottom*, and only in an *algorithmic* sense. The algorithm, which need not be a CA [2], starts baggage-free - it initially ignores the fundamental constants of the Standard Model, for example - but *creates* baggage during the journey, by emergence, as the computation unfolds. Its function can be

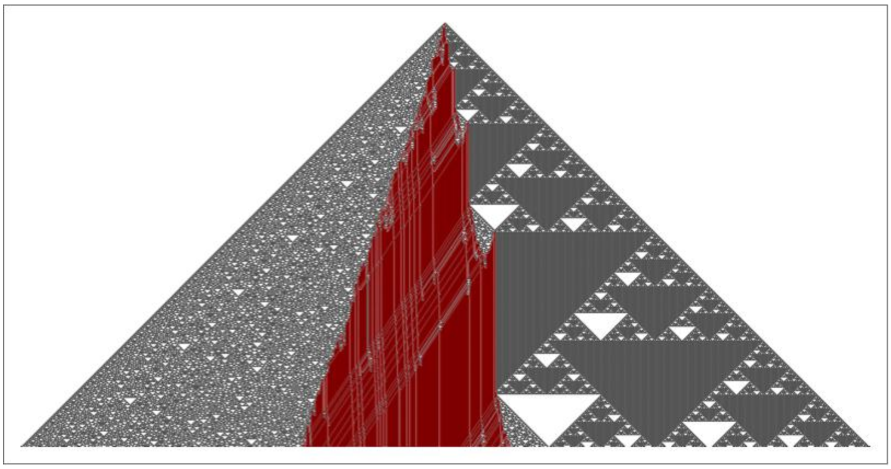

Figure 2: Randomness, 'particles' and selfsimilarity in the same CA. Mathematica call: `ArrayPlot[CellularAutomaton[{738926016882, 3, 1}, {{1}, 0}, 880]]`.

written on Priss' T-shirt (Appendix B), but the computation, with its amazing architecture of emergent phenomena, can't. And I bet it is a non-terminating computation".

"No output?" I ask, thinking of Priss' *total* computable functions.

"No output. Who cares whether the final output is really 42? What matters is what emerges during the trip. By the way, don't forget that, whenever a model of computation is Turing-universal, some of its instances *must* be non-terminating, for some input. [2] Divergence is a virtue, not a sin!"

"Then, if the universe is a finite mathematical structure only at the bottom, how do you explain the unreasonable effectiveness of math in describing *also* the phenomena that emerge during the unfolding?".

"I told you. Math is only *partially* effective in this. Consider again Fig. 2. Math can trivially describe the r.h.s part of the computation, and, with some effort, also the physics of the emergent 'particles' in the middle. But you cannot find a theory for the messy l.h.s component, because that part of the computation is *irreducible*. See, physical theories are nothing but data compressors: they can squeeze huge amounts of experimental data into compact formulas. Not in this case. I mean, the formula is there - function f at the bottom - but there is no way to predict the configuration at step k other than running k steps of the computation. No short-cut.

What we can describe efficiently by mathematics is only the portion of universal algorithmic evolution that is compressible, due to its regularity. Fortunately, it is provably much more likely that something regular and compressible emerge in a computational universe - e.g. in the behaviour of a universal Turing machine running a random program - than from a monkey typing at random on a typewriter [4, 10].

The Universe is not a static mathematical structure - a huge, pre-defined set of elements and relations that we progressively discover. It is the unfolding of a computation, [3] and a relentless source of novelty; its future gifts are mathematically unknowable until they actually come into existence. I have no objection in attributing a respectable status of existence to beautiful pieces of math such as the Mandelbrot set. They did exist at the Plank era, or, if you prefer, they inhabit a platonic, timeless space. I can't say the same about atoms and molecules, frogs and birds, you and me.

In conclusion, no doubt physicists and mathematicians will continue building (or discovering) the world of mathematics, but they should put much more effort in exploring the computational universe too, before we get convinced that chicken are indeed spherical! And for doing this, we need a *gigantic computer*."

"Or a tiny one, as Priss' glasses!" In saying this, I realise that his little jewel is still in my pocket.

[2] Diagonal argument: if each instance F_i ($i \in N$) of model F converged on each input $j \in N$, then computable function $G(j) := F_j(j) + 1$ would not be in F, thus F would not be universal.

[3] This conjecture is not to be confused with *the Matrix* - the idea that we live in a simulation.

References

[1] David Balduzzi and Giulio Tononi. Integrated information in discrete dynamical systems: Motivation and theoretical framework. *PLOS Computational Biology*, 4(6):e1000091, 2008.

[2] Tommaso Bolognesi. Algorithmic causal sets for a computational spacetime. In *A Computable Universe*. H. Zenil (ed.), World Scientific, 2013.

[3] E. Fredkin. Five big questions with pretty simple answers. *IBM J. Res. Dev.*, 48(1):31–45, 2004.

[4] Leonid A Levin. Laws of information conservation (nongrowth) and aspects of the foundation of probability theory. *Problemy Peredachi Informatsii*, 10(3):30–35, 1974.

[5] Remko Siemerink. *Wolfram Science Summer School 2009 - Favourite three-color cellular automaton*. Pisa, June 2009. http://www.wolframscience.com/summerschool/2009/alumni/siemerink.html.

[6] Gerard 't Hooft. The cellular automaton interpretation of quantum mechanics, June 2014. http://arxiv.org/abs/1405.1548 [quant-ph].

[7] Max Tegmark. The mathematical universe, 2007. http://arxiv.org/abs/0704.0646 [gr-qc].

[8] Max Tegmark. *Our Mathematical Universe*. Alfred A. Knopf - New York, 2014.

[9] Stephen Wolfram. *A New Kind of Science*. Wolfram Media, Inc., 2002.

[10] Hector Zenil. The world is either algorithmic or mostly random. *FQXi Essay Contest 2011 - Is Reality Digital or Analog?*, 2011. http://fqxi.org/community/forum/topic/867.

[11] Konrad Zuse. Calculating space. Technical report, Proj, MAC, MIT, Cambridge, Mass., 1970. Technical Translation AZT-70-164-GEMIT. Original title: "Rechnender Raum".

Appendix A - Matematica code for Conway's tangle game

```
(* ------ CODE ------ *)

swap[x_] := x + 1
rotate[x_] := -1/x
play[init_, moveList_] := FoldList[#2[#1] &, init, moveList]

(* ------ DATA ------ *)

tangleList = {swap, swap, rotate, swap, swap, swap, rotate, swap, swap, swap};

untangleList =
{rotate, swap, rotate, swap, swap, rotate, swap, swap, swap, rotate, swap, swap, swap;

(* ------ RUN ------ *)

play[0, tangleList]
{0, 1, 2, -(1/2), 1/2, 3/2, 5/2, -(2/5), 3/5, 8/5, 13/5}

play[13/5, untangleList]
{13/5, -5/13, 8/13, -13/8, -5/8, 3/8, -8/3, -5/3, -2/3, 1/3, -3, -2, -1, 0}
```

Appendix B - The function for three-color CA n. 738926016882

The table below defines function f by providing the value of cell c_i at step t, given the values of cells c_{i-1}, c_i, c_{i+1} at time $t-1$.

```
{{0, 0, 0} -> 0},
{{0, 0, 1} -> 2},
{{0, 0, 2} -> 2},
{{0, 1, 0} -> 0},
{{0, 1, 1} -> 0},
{{0, 1, 2} -> 1},
{{0, 2, 0} -> 1},
{{0, 2, 1} -> 2},
{{0, 2, 2} -> 2},
{{1, 0, 0} -> 2},
{{1, 0, 1} -> 2},
{{1, 0, 2} -> 1},
{{1, 1, 0} -> 0},
{{1, 1, 1} -> 0},
{{1, 1, 2} -> 0},
{{1, 2, 0} -> 2},
{{1, 2, 1} -> 2},
{{1, 2, 2} -> 0},
{{2, 0, 0} -> 2},
{{2, 0, 1} -> 2},
{{2, 0, 2} -> 1},
{{2, 1, 0} -> 1},
{{2, 1, 1} -> 2},
{{2, 1, 2} -> 1},
{{2, 2, 0} -> 2},
{{2, 2, 1} -> 0},
{{2, 2, 2} -> 0}.
```