FQXi Contest 2020 "Undecidability, Uncomputability, and Unpredictability"

Calculate "*as if* "… but be careful

suggests Eckard Blumschein.

Shut up and calculate according to axiomatically formalized mathematics *as if* there were no arguments against Hilbert's hope for general decidability, computability and predictability. However be careful and don't speculate as if such calculation was realistic or at least logically correct. Hilbert's trust in the law of excluded middle was formally challenged by Gödel's incompleteness theorem and Turing's more practical halting problem. Unseen examples of possible mistakes with relevance to physics will be shown.

§ 1 <u>Why just ***as if***?</u>

Feynman's advice was meant helpfully: Well, you cannot at all understand quantum mechanics. Therefore shut up and calculate! [As if there was nothing to worry about.] Such pragmatism proved utterly successful as a trial and error method in the whole history of science. Friends of the "as if" in Germany supported Hans Vaihinger [1] who offered an almost forgotten theory of the "as if". Vaihinger's fictionalist theory was not the only and not a fertile alternative to the widespread trust in a rigid formalism which was vehemently propagated by David Hilbert [2] although it was soon formally shown untenable by Kurt Gödel [3], also by Alonzo Church [4], by Alan Turing [5], and by others.

Hilbert's successor Hermann Weyl was possibly correct when he rejected large parts of Cantor's set theory and warned: "We are less certain than ever about the ultimate foundations of mathematics" [6]. However, in which case do mathematics and physics need putatively ultimate foundations at all? Weyl did not exclude that some correct basics were just not yet found: He confessed for instance in [7]: "The problem of the proton and the electron is discussed in connection with the symmetry properties of the quantum laws with respect to the interchange of right and left, past and future, and positive and negative electricity. At present no acceptable solution is in sight", cf. § 8.

Abraham Robinson, who stood in the tradition of G. Cantor, D. Hilbert, and A. Fraenkel, also reasonably suggested acting as if infinite sets did exist, although he still strived for rigor when he meant [8]: "Infinite totalities do not exist in any sense of the world (i.e. either really or ideally). More precisely, any mention or purported mention of infinite totalities is literally meaningless".

Laymen may understand the existence of reality as simply the logical negation of merely abstract ideas. They should agree on considering a street as real and <u>just as if</u> it was a line, in contrast to the Platonic opposite view. This essay intends to reveal in detail why models may look as if they were realities even if logical discrepancies are undeniable, while on the other hand decisive imperfections of some mathematical metaphors are stubbornly ignored.

§ 2 Calculate as if dots were points

People count the five fingers of their hand as if the thumb was not pretty different from the other ones. Therefore Leopold Kronecker was perhaps not quite correct when he ignored abstraction and meant: "The natural numbers were made by God. Anything else is manmade." When Euclid summarized ancient mathematics [9], he thoroughly dealt with the choice of unity as the basis of commensurability. Perhaps, he reacted to the conclusion that Pythagoras was strictly speaking wrong when claiming "all is number". Geometry led to ratios like a/b with $a^2=2b^2$ in which at best either a or b can be arbitrarily declared commensurable with 1. The other one is then irrational. Because rational numbers can approximate any irrational number as good as desired, incommensurability did for many centuries not hinder using rational approximations of irrational numbers as if they were rational. There is no risk of confusion.

Irrational numbers like π are - if not seen as tangible measure of length - attributed to likewise unique ideal points (Peirce spoke of mere potentialities) in contrast to dots alias the even more tangible pebbles of abacus. When one comprehends the rational numbers as an endlessly extended set-like discrete structure of dots, then an uncountable manifold of irrational numbers fills the remaining gaps. The real numbers constitute what Weyl called the sauce of continuum.

Dedekind's claim having created with his cut a new, an irrational number [10] does not just sound a bit presumptuous. In contrast to the useful definition by G. Cantor, Dedekind's cut only declared already common practice an axiom. Nonetheless, Dedekind's cut may be regarded as a radical step from original metaphor of a very small round mark to an ideal point according to Euclid's good old definition: something that has no parts. As still did Ebbinghaus [11] who asked what makes the rational numbers different from the real ones, Dedekind ignored the bridge between quantity and quality, finite and infinite, a subtlety that can be demonstrated by infinite division. Look at the infinite sum 1/1 + 1/2 + 1/4 + 1/8 etc. = 2. Incidentally, in ancient time the 1 was not yet seen a number. Presumably, numerical mathematics doesn't need such bridge, in contrast to the more aristocratic point-based geometry of the ancient Greeks. Today we may calculate as if dots were points, and rational numbers were still remaining bone-like when submerged in the fictitious actually uncountable continuum of real numbers.

Are irrational numbers nonetheless numerically distinguishable? This question led to a so called fundamental crisis of mathematics. Brouwer disagreed with Hilbert who vehemently defended his general use of the logical distinction between true and false: "Taking away this law of the excluded middle from the mathematician would be as if one did not allow to an astronomer a telescope or to a boxer using his fists". Brouwer's impressionism and Weyl's constructivism tried to replace Hilbert's rigid formalism. So far they didn't prove superior, just more complicated. On the first glimpse, this controversy looks undecided and rather futile, similar to disputes on Schrödinger's cat. However § 7 will show that Dedekind's distinction between dot and point has nonetheless consequences being relevant for mathematics and physics although so far ignored.

§ 3 <u>**Calculate as if the relative infinity by Leibniz was actually infinite**</u>

Abel criticized what he felt the improper basics of mathematics. Maybe, until he died much too young, he had not yet digested the formal remedy offered by Leibniz: a kind of "as if", the relative infinity. This fiction replaced rigorous logics by feasible mathematics. It equates rather than just approximates differentiable functions with a sufficient, declared as "dense" amount of discrete elements still in terms of numbers. Having also dealt with freedom vs. necessity, Leibniz even early envisioned digital computation. George Berkeley did certainly not understand Leibniz when he ridiculed the infinitesimals: "They are neither finite quantities, nor quantities infinitely small not yet nothing. May we not call them the ghosts of departed quantities?"

In mathematics, finitists are certainly correct. Hilbert even defended Cantor's naïve transfinite numbers as "simply counting in excess of infinity" (einfaches Hinüberzählen) [12]. Let's instead appreciate that already Johann Bernoulli, de l'Hospital, and Leibniz made decisive contributions to what Cantor dreamed of: an infinitum creatum. Let's comment on the three levels of infinity which were distinguished by Leibniz:

God is the highest level, the perfect property of being endless. Aristotle had not yet accepted such ideal fiction. When Aristotle argued infinity is merely potential but "actu non datur" he assumed that being infinite means: there is absolutely no imaginable limit to refer to. Aristotle did see infinity as strictly logically as still also did Salviati (Galileo Galilei) who concluded that "the relations equal to, and smaller or larger than are invalid for infinite quantities". Indeed, the logical property of natural numbers to be infinite is literally the opposite of having a limit. Nowadays, engineers don't hesitate writing $\tan(\pi/2) = \infty$ although mathematics cautiously forbade division by zero. These engineers may already feel that <u>*every*</u> real number is just something ideal, an ideal point rather than a dot, something <u>*without*</u> existing physical correlate.

Leibniz gave then examples for a metaphysical lower level: the whole space and eternity. They seem to be speculative and therefore not different from the discussed above highest level.

Important in mathematics is the lowest level including anything that is <u>*larger than*</u> any nameable quantity. Such pragmatic notion of infinity is clearly a relative one, something that may be used as if it was actually the perfect end of a convergent process although it strictly speaking still contradicts to endlessness. It may be praised as the key to numerical mathematics. Leibniz himself called his infinities and infinitesimals as well as the imaginary numbers altogether just fictions and he added: They are fictions with a fundamentum in re, not arbitrarily chosen ones. His relative infinity combined the unquantifiable infinity with quantifiable convergence. De facto, it already provided the sufficient basis for differential and complex calculus leading to a huge treasure of successful applications in science and technology.

There is a decisive advantage of digital over analog technology: Digital signals may cope with the noise-caused loss of decidability. By getting refreshed they may perform errorless computation, transmission, and storage of signals. If noise exceeds by chance a tolerable limit, such operations can be repeated.

However be careful. Don't interpret all calculated results as if they did apply in physics too. Instead of accepting that the calculated infinite field strength around a line-shaped conductor approaches infinity, one should use a realistic model with the radius of conductor larger than zero. Abstracta like the absolute infinity and true singularities don't have correlates in reality.

§ 4 Calculate as if there was no causality

Who trusts in reality as negation of mystic ideas should never deny causality. However, since the laws of nature were abstracted from reality, they are no longer temporally or locally bound to concrete points on the actual scales of time or space. There is no such restriction anymore at the level of abstracted quantities. The laws are here invariant under operations like flipping, shifting, expanding or shrinking these scales. Models are just a bit too flexible. Here one may even compute as if the past was predictable. While Shannon's common sense view tells us that this is unreal, there are theoreticians who do not accept that they are just operating with mere fictions. Some of them are even trying to speculate causality away.

So far it is reasonable practice to tolerate so called non-causalities for instance in the current theory of signal processing for the sake of elegant calculability. Of course, students are challenged to think more carefully. They are obliged being able to formally calculate as if there was no causality but they are not obliged to believe that a mouse can leave a box before it was in it. And they may feel sympathetic with Einstein who as a believer denied not just in a letter of condolence the distinction between past and future [13]. Einstein did effectively so already in his dispute with Karl Popper who called him a Parmenides. Hilbert also denied the now. By the way, Einstein wrote "past, *present*, and future" as if present was a state in the middle between the two immediately adjacent to each other ideal states.

Despite of many benefits from calculating as if there was no causality, ultimate awareness of causality is definitely an important achievement of mankind and still important for its survival. It doesn't matter how much we may cherish metaphors like closed loop, ideal circle, the endless to both sides sinusoidal function, and in physics the harmonic oscillator. Using them, we have always to arbitrarily choose an added point of reference, a beginning, an end, and a direction.

While the metaphor of life as a spiral progress may be a good reminder to us as the mankind of our true ethical responsibility for future, ideal spirals are only academically appealing.

Although in general, nature did not yet and can certainly never obey manmade arbitrarily constructed mathematics, we all have no chance but to be pragmatic and humbly accept the impossibility to command Laplace's demon. We should therefore be ready to simplify on the basis of our experience in terms of discrete patterns and fuzzy generalization. Anything flows.

§ 5 Notice: Expansion of mathematics at will cannot expand nature

The practice to freely define axioms more or less at will in addition to the traditional Euclidean ones started perhaps with Dedekind's cut and G. Cantor's transfinite numbers. It opened the door for a considerable expansion of mathematical theories. Dedekind put the different kinds of numbers under a hierarchic system of nested umbrellas. Integer numbers are a special case of the rational ones which on their part together with the irrational ones constitute the so called real numbers. The latter together with imaginary numbers are complex numbers. This system was further extended up to octonions and beyond.

The name real numbers is perhaps not a good choice, not just because a complex number is a real number and may hence consist of its real and imaginary parts. "Real" was originally meant as more tangible in contrast to the distrusted less intuitive imaginary number. Nonetheless, mathematics was made a seemingly perfect construct and Dedekind's friend G. Cantor was possibly correct (from the view of so called pure mathematics) when he claimed [14]:

"*The essence of mathematics is its freedom*".

Earlier, at the time of Fourier, there was a widespread opposite opinion [15]:

"… *mathematical analysis is as extensive as nature itself, it defines all perceptible relations, measures times, spaces,* ..."

Meanwhile there are not many physicists who still believe as did Fourier in a mathematically constituted reality even if this doesn't mean that they entirely share Georg Cantor's antithesis. Unfortunately Dedekind did not devote the due attention to the preceding expansion, the at least equally important expansion of the natural numbers into the more artificial realm of integer numbers. This step by John Wallis (?) has serious consequences for physics up to now.

In particular, restrictions in reality set a lot of decisive conditions for decidability, computability, and the degree of predictability. To begin with the simplest, one should already not conclude from the existence of negative numbers that any physical object must a priori have negative or even imaginary values, even if such values might be computable. As is well known, nearly all original physical quantities can be reduced to primarily non-negative values that refer to a natural zero. Then one may arbitrarily refer instead to a clever agreed zero, e.g. temperature to the melting point of ice or sound pressure level to the pressure of air.

§ 6 A fundamental mistake between mathematics and physics has been revealed

Fourier was wrong: Mathematical analysis is more extensive as nature itself. Not even Laplace and Lagrange realized this although they hesitated for a while to accept that Fourier identically replaced a *given function* f(x) by combined infinite series of sinuses and cosines

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right).$$

Why should one question this equation? Today it is even available in Word as a most frequently used one, and nobody doubts its correctness. One should however be aware of the option to express the sinus functions by the cosine functions or vice versa. They are related to each other by $\sin^2(x) + \cos^2(x) = 1$. Fourier [15] demonstrated: If f(x) is an even function then there are only cosine functions; If it is an odd one then there are only sine functions. Equivalent to above sum is the complex representation where there is instead of b_n and a_n only one coefficient c_n, and an additional variable phase ϕ emerged. From where? In physics, the value of ϕ is automatically chosen at will with agreeing on a reference point x=0. In case of frequency analysis of measured data, ϕ and t = 0 are irrelevant because nature cannot depend on human choice. The tacit introduction of a redundant phase is already hidden in the words "given function" in the sense of "arbitrarily chosen function".

For instance the human ear as a frequency analyzer has no agreed reference t = 0. It must be understood as "referring" to the actual zero of elapsed time: the current end of the incoming signal. Future data are not yet available in advance. There is no plausible reason for extending the integral f(t) over the full range from minus infinity to plus infinity if there are no future data. When mathematical terminology only speaks of missing support for t > 0 in R, this does intentionally deny the possibility to calculate within only positive real numbers R+ instead in the mandatory realm R of positive as well as negative real numbers.

The elegant way to perform a complex Fourier analysis of the "half-sided" real-valued signal is Heaviside's splitting it into two fictitious components, an even and an odd one which cancel each other for t > 0. Perhaps there is no physiologically realistic correlate for such tricky maneuver. Nonetheless, Heaviside's trick looks mathematically correct on condition it is consequently performed up to correct interpretations. First of all, complex transforms are artificial, in contrast to Hilbert "transform" which is simply an equivalence relation. Transform from real domain into the fictitious complex ones is a jump. It means addition of an imaginary component to the real domain or the other way round omission of an imaginary one from the complex one. Accordingly, one has to jump back after performing calculations in complex domain, and do not forget recombining the split signal within the original real domain. Be sure, although an original signal in the domain of only positive time is hence represented in an apparently redundant manner in complex domain, each of the four components (positive and negative real and imaginary parts) is necessary as to correctly perform inverse transformation and reduction back again into a real-valued one-sided result without redundancy.

While on one hand application of complex quantities in engineering proved convenient and in almost all cases not at all problematic, history of modern physics illustrates on the other hand that it is at least risky to lazy interpret all results in terms of physical quantities already in complex domain.

For instance, some physicists rejected the negative frequencies because these are indeed nonsensical in the original time domain. Negative frequencies are however indispensible in complex domain as to correctly encode the original real-valued and unilateral signal. A real-valued Hamiltonian which was accordingly introduced in complex domain implies a somewhat strange analytic (complex) signal after inverse transformation into time domain. Calculating as if reality was something in a complex domain is superior for several plausible reasons. It was just sometimes inappropriately generalized.

For instance, there is still no evidence for Alan Oppenheim's claim having improved Tukey's cepstrum [16] by making it complex.

The tacit introduction of an arbitrarily chosen redundant phase due to the above revealed fundamental mistake could easily be avoided by ***not*** calculating as if it was recommendable to ignore causality, in other words by calculating in R+ instead of R. Of course, this implies the consequence to abstain at least in principle from any use of negative ordinates. Common sense understands: Future data are unreal. Accordingly, one must mutilate an elegant symmetrically extended function of time with a caudal part that crossed the logical border t = 0 by restricting it to just positive ordinates. As illustrated in Fig. 1 of [17], this may be performed by reflection at t=0. Mathematicians are correct if they declare the physicists responsible for such corrections. Physicists are always responsible for their models. For instance, they have also to be careful with models that calculate down to r = 0 as if a conductor was a line and a charge was a point.

§ 7 Related arguments

As already indicated in §2, the easily imaginable metaphor of a dot alias pebble is often used as if it was identical with the ideal point. However this leads into problems which are so far evaded in mathematics by means of generalizing Rolle's mean value theorem [18] as if it was valid in case of discontinuities, too. Actually, it holds only for differentiable functions.

Is there in case of the function y = sign(x) at x = 0 a convincing value for y? No. The mandatory attribution is the middle value y = 0. Corresponding attributions can be found in integral tables. They are also unfounded and do not have any relevance in physics. They are confusing and should be deleted. The point that represents a real number does not cover any width. Ironically: Tertium non datur: There is no extended zero between positive and negative and no extended state between past and future.

When Terhardt dealt with Laplace transform [19] he tried without convincing avail to somehow reconcile the restriction of time to non-negative values with an integration symmetrically centered at t = 0. Mathematicians told him to start the integration quite a bit left to t = 0. This corresponds to Hausdorff topology but it is logically impossible Inside R+. The impossibility is quite obvious: Hausdorff postulates pebble-like surroundings bilateral around every number which means to both sides of zero. The same calamity does even more strikingly affect a metaphor shown in many textbooks: the bell-shaped time-symmetrical approximation of a Dirac impulse. Maybe, the engineers mentioned in § 2 don't need it?

§ 8 Shut up for good?

Consequent use of R+ instead of R did plausibly remove Weyl's worries about apparent symmetries. However, not just Niels Bohr, Richard Feynman, and Roger Penrose claimed that the use of complex numbers [or equivalent Hermitean matrices] is truly essential in quantum theory and not merely a convenience. Wolfgang Pauli even pointed out in [20] that only quantum mechanics differs from all classical physics in which the use of complex numbers is just an utterly favorable but in principle avoidable option.

Isn't quantum mechanics the first theory that was exclusively built on complex Hilbert space? Hilbert stood in the tradition of Fourier as Fourier stood in the tradition of Laplace and Lagrange who didn't see any problem with interpretation of complex quantities in physic although they hesitated to accept Fourier's theory because of lacking evidence. The evidence that supports Fourier's theory was later provided by Dirichlet. It is based on Rolle's mean-value theorem, see § 7.

Likewise one may not expect John von Neumann, who himself coined the expression Hilbert space, having questioned complex quantum theories just because he uttered towards Birkhoff that he did not believe in Hilbert space any more [21]. Unfortunately, one cannot even prove the theory of quantum mechanics logically wrong, because it was not logically derived but just heuristically fabricated by Schrödinger as well as by Heisenberg.

Majority tends to follow personalities including Fourier who dictated what is decidable, computable, and predictable. While Cauchy's lessons were felt unacceptable and he lost all but one of his students, Weierstrass had a lot of fellows. Weyl called Hilbert a pied piper to whom we all have to follow.

My suggestion "be careful" is not meant as a warning, rather as an encouragement: Be ready for an open-minded attitude. Then we may hope for getting rid of more mistakes. The "as if" lets certainly enough room for critical (rational while not necessarily common) sense reasoning between Vaihinger's agnosticism and Hilbert's intention to formally pinpoint logics as a rigid mathematical construct by claiming decidability and all that. When we calculate as if something is correct, this may but need not imply it is actually incorrect.

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