Leaving Cantor's Paradise through Paul Cohen's Golden Door

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" Idealists argue that the hexagonal rooms are the necessary shape of absolute space, or at least of our perception of space. They argue that a triangular or pentagonal chamber is inconceivable. (Mystics claim that their ecstasies reveal to them a circular chamber containing an enormous circular book with a continuous spine that goes completely around the walls. But their testimony is suspect, their words obscure. That cyclical book is God.) Let it suffice for the moment that I repeat the classic dictum: The Library is a sphere whose exact center is any hexagon and whose circumference is unattainable."
The Library of Babel, Jorge Luis Borges [Bo].

"No one shall expel us from the Paradise that Cantor has created."
On the infinite, David Hilbert [H].

Introduction

Despite its success, Quantum Mechanics is still a mysterious theory. Paraphrasing Feynman, we can safely say that nobody really understands it. Why is it so difficult? In a few words, because it has not been possible to completely adapt the language we use to describe and understand Classical Theories, Set theory and Classical logic, to Quantum Mechanics. We use mathematics not just to describe or measure reality, but we also use them to make questions, to understand and explain the world around us. Therefore, the important thing is not that we can make predictions about planet's motion but that we can ascribe such motion to the curvature of spacetime[1].

So far, classical logic has been the only one we have used to understand the

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[1]Here I am paraphrasing Steven Weinberg but in an opposite sense, in his book Gravitation and Cosmology [W], he wrote: *"The important thing is to be able to make predictions about images on the astronomers photographic plates, frequencies of spectral lines, and so on, and it simply doesn't matter whether we ascribe these predictions to the physical effects of gravitational fields on the motion of planets and photons or to a curvature of space and time."*

world we live in. We have mixed our rulers, pointers and classical logic to construct real numbers and many more abstractions that have allowed us to be successful in our description and understanding of reality. Every time we have found a new inexplicable phenomena, the classical framework has given us the tools to create more abstractions to measure and understand such phenomena. But with quantum mechanics this success has not been completely achieved.
We have found for example, that observables in quantum mechanics do not satisfy some laws of classical logic and that when we try to reformulate the theory to fix this, the situation seems to persist. You may think this is because we have not been smart enough to find a satisfactory mathematical abstraction to describe the theory in complete agreement with classical logic. Shockingly the theory not just violate the laws of classical logic, but also it can be used to formulate new logical operations [D1] . Then, even if we can find a mathematical formulation of quantum theory that respects all the laws of classical logic, we will have to include new laws to rule this new logics operators. It seems then, that this time we cannot save classical logic.
What should we do to include non classical logics in our description of reality? First we have to identify what logic is more appropriate to describe quantum reality and then built our rulers and pointers, under this chosen logic, in the same way as real numbers are conceived using the laws of classical logic. This is the utilitarian part of the model and it has been tackled in many forms during the last decades, from the seminal work of Birkhooff and Von Neumann on quantum logic until the more recent approaches developed by Chris Isham and others in the context of topos theory. Second we have to start to meta-reasoning within non classical logic, being careful to avoid notions linked to classical logic that are not valid in the new logic. This second point has been ignored unconsciously until now, but as I said above we use mathematics no just to measure or predict reality but also to understand it.
How important is this point becomes evident when we analyse how the problem of quantum gravity is normally tackled. I will show in this essay, how until now our search for an unified theory has been infested with many concepts of classical logic that probably are the core reason of our evident failure in finding this unified theory. Then I will show how introducing non classical logics, new remarkable perspectives to solve the unification problem appear.

Collapsing to a Classical Logic Limit

The affirmation that Newtonian gravity is the classical limit of general relativity is a proposition about our reality that we can perfectly understand. Indeed we know that when gravity is weak, sources move slowly, and material stresses are negligible the equations of Einstein tend as a truly limit to Newton's equations. On the other hand the problem of unification is normally posed in very similar terms:

We should find a new theory which reduce to quantum mechanics in the *limit of weak gravity and which reduces to general relativity in the limit of large actions and that is able to explain situations where both strong gravity and quantum effects play a fundamental role, like in the formation of black holes.*

But there is a big difference with the Einstein/Newton case, quantum reality is not ruled by the same logic of general relativity. What most of our classical approaches to quantum gravity are missing is that this new theory should include also the notion of how a world ruled by a non classical logic can converge to a world ruled by classical logic. This sounds too crazy (or too obvious) for being taken seriously. But mathematicians have already developed some machinery where we can *collapse* structures ruled by non classical logic to structures governed by a classical one. The surprising fact is that these same tools are being already used to find new foundations of quantum mechanics and physics in general, and they seem to be very promising. As far as I know, how this tools allow us to relate theories ruled by different logics is something that has not being noted or has not been considered seriously. I will show how this point is the key to understand what it means to think in a non classical logic sense.

Now I just want to say that the word "collapse" above is not a coincidence. We will see that the way this collapse to classic logic works, shows how to construct a beautiful mathematical machinery to relate the many worlds or consistent histories interpretation of quantum mechanics with the classical Copenhagen interpretation. We will see how a many world structure ruled by non classical logic can be collapsed to a one unique world ruled by classical logic.

The First Hilbert problem, the golden Door

The first of the 23 problems that Hilbert posed in his famous conference the 1900 at Paris, was the problem of the Continuum Hypothesis (CH for short). The problem was to demonstrate that every subset of the real numbers is either enumerable or has the cardinality of the continuum. In more simple terms the problem says that if we have a collection of real numbers we can either, assign to each of them one different natural number (i.e. we can enumerate them), or, if this is not possible, we can find a bijective function between our collection of numbers and the real numbers (the continuum).
The problem was solved 63 years later in a shocking way that probably Hilbert and Cantor, who originally posed the problem and worked hard on it, would have ever imagined. In 1940 Goedel proved that the CH cannot be disproved from the classical axioms of set theory, which we use to define the real numbers. What does this mean? Let see a more familiar example to explain this result. We know there are geometries where the five Euclidean postulates are valid, for example the plane, then we cannot disprove the fifth postulate from the other postulates because this would imply that the postulate is not valid on the

plane, but we know it is valid [2]. In the same way we can find universes where the axioms of set theory hold and when the respective Continuum (the "new real numbers" in that universe) satisfies the CH. This means we cannot disprove the result from the axioms because we have a model where the axioms and the result are valid. After 1940, twenty three years later Paul Cohen showed that we cannot prove the result either. He showed it is possible to find also one Universe where the axioms of set theory hold but where the CH is not valid for the "real numbers" in that universe. How does all this relates to physics? We will see this in more detail in the next sections, for now keep in mind the notion of "new real numbers", what does this concept mean? a new measurement tool?
The method used by Paul Cohen to prove the independence of the CH (i.e. the fact that we cannot prove it or disprove it) is probably one of the greatest achievements in mathematics. In his original work Cohen [Co] dealt with models of classical set theory, it means within classical logic. Some years later R. Solovay and D. S. Scott extended the theory to include models based in logics where the true values are not just "true" or "false", like in classical logic, but values that can variate in an arbitrary Boolean algebra. Independently Lawvere, in his attempt to axiomatise the category of sets, found the notion of topos. Later, working with Terney, they found that this, along with the notion of sheaf allows to explain the method used by Solovay and Scott in terms of topoi. Here is where the marvellous circle close, in the words of Saunders Mac Lane and Ieke Moerdijk [MM],

"A startling aspect of topos theory is that unifies two seemingly wholly distinc mathematical subjects: on the one hand, topology and algebraic geometry, and on the other hand logic and set theory"[3]

I think it is this startling connection where we can find the key to understand quantum mechanics and its link with general relativity. But not everything is as good and easy as it looks. Topos theory and in general category theory, are very good, however, not very friendly to work with. Nowadays categories are widely used in many areas of mathematics, nevertheless they are still too abstract to think that we will be able to understand, or even express, our physical theories in their language. To put it in terms of Vladimir Arnold, at the moment, in Category Theory algebra's devil is winning the battle against geometry's angel. Some people like John Baez and Chris Isham are working hard to find categorical foundations of fundamental physics, however, if we want to find a new language to understand reality, it has to be one close to the language we already have. We should be able to use this language to easily talk about "new real numbers", "new manifolds", "new Hilbert spaces" etc. Even if topoi are universes similar to our universe of classical sets, Category Theory tools blur dramatically the

[2] We can show the result is valid in the plane because it is possible to use coordinate or Cartesian geometry to prove it, not because we can use the other four postulates to prove it.

[3] I hope the reader is feeling how strong this affirmation is. Even if the approach I am trying to justify here is not the right one, I hope you agree after this quote, that soon or later topoi will play a fundamental role on physics.

logic that governs them, the way the semantic of these logics work, and how this all can be connected to our classical notions. Nevertheless ...

Understanding Topoi, grasping reality

What do we mean when we say that something is true? We can say for example that the proposition “there is just one even prime number” is true, because we can prove it and we agree that the proof is valid, and we will surely agree on this in a hundred years (or not?). On the other hand, if we consider the proposition “every even number greater than 2 is the sum of two prime numbers” , the Goldbach conjecture, we cannot say that the proposition is true or false. We know that 4=2+2, 6=3+3, 8=5+3, but we have not yet a proof to know if the result is valid for all even numbers greater than two. Maybe in future we will find a proof or an even number, greater than two, that cannot be written as the sum of two primes. In any case, at the moment we can only say that there is just a set of even numbers which can be written as the sum of two primes, this set will grow in future as we find new prime numbers, in some sense this growing behaviour is like an evolving condition of the degree of truth.
In classical reality the situation is analogous, we say then, that something is true if we agree that it is true in the present and it will be still true in future. For some propositions we cannot say they are true or false but that they are “so far true” or “so far false”. But in quantum reality the situation changes. We do not ask if one proposition is true or not, we ask “if we take a measurement is the proposition true or false?” But what if we do not take a measurement? The point is that, when we take a measurement we are collapsing our non classical logic universe into a classical one; where the two values, $truth$ and $false$, make sense. In our non classical logic universe we cannot ask if something is true or false, but if you want to keep your classical intuition, a more appropriate approach to this would be the idea of potentially true or potentially false.

Therefore, classical reality shows that “true” in the classical world is related with some kind of extension or neighbourhood (in the examples above from present to future) and Quantum reality shows that “true” is a potential feature. We can take this potential feature as multivalued logic worlds (many worlds), that are related in a way that affect how the classical world is perceived. This seems to be very confusing, but there are some beautiful mathematical models where a very similar situation can be expressed in a spectacularly neat way. These models were introduced originally by X. Caicedo on his outstanding paper: *Logic of Sheaves of Structures* [C]. These models are based on the idea of topos and variable sets, but they simplify a lot the weirdness of the categorical approach, and can be used to give a new elegant proof of the independence of the Continuum Hypothesis [B1], [B2]. I will show now, how these models are used in order to prove the independence of the Continuum Hypothesis; and how, in the same way, we can use them in physics.

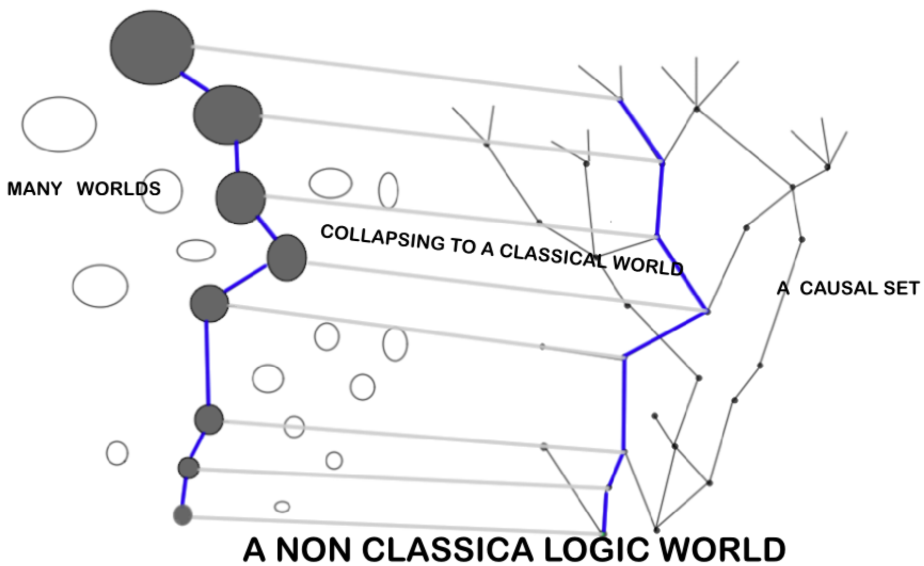

Figure 1: A quantum reality collapsing to a classical reality

From now on you should read looking at figure (1). We start choosing a partial order $\mathbb{P}$. Now, for each element of the order $p \in \mathbb{P}$ we can assign an universe $V(p)$. The structure of this universe depends strongly on how p is related with other elements in the order, and on the structure of their respective universes. To proof the independence of the Continuum Hypothesis each $V(p)$ is a model of set theory. These structures are ruled by a logic where something is true in $V(p)$ if it is also true in the universes of elements in an extension containing p (you can think of this extension as present to future like in the example above) and all the structure (i. e. the points plus their respective universes) are a typical model of an intuitionist logic. Then we construct the analogous of the natural numbers and the real numbers in the universe $V(p)$ for each p [4]. After this we can construct another set bigger than the "new natural numbers", in the sense that cannot be enumerate in this context; and smaller than the "new real numbers", in the sense that is contained in the "new real numbers" but it does not exist a bijective function between the two sets. Finally since the true is defined on extensions, we cannot assure that something is $false$ or $true$, but we can find some extensions where this is possible. Using these extensions, if we have chosen the order in the right way, we can collapse the intuitionist model to a classical model[5], where the Continuum Hypothesis does not hold.

[4]In the proof it is not use the analogous of the real numbers but a set that we know has the same cardinality.

[5]In mathematical terms we collapsed the multivalued logic universe to a classical one, taking a generic ultrafilter on the set of open subsets of the order topology on $\mathbb{P}$

How one made an analogous construction for quantum mechanics or to model the spacetime has been studied by C. Isham, J. Butterfield, F. Markopoulou and others (See for example [I1] for a nice introduction). Nevertheless how these tools can be used to collapse worlds ruled by non classical logics to worlds ruled by classical logic, it has not been studied so far. However, if you consider the analogy seriously the consequences can be revolutionary.

The second step towards A true Theory of Everything.

When physicists talk about a theory of everything they are normally talking about a theory that will explain the known fundamental forces [6] and matter in a self-consistent mathematical model. Then, this is more a partial theory of everything because does not intend to explain emergent phenomena like life, conciousness or computation. However, an appropriate understanding of the fundamental forces and matter, should explain how a smooth and classical higher level reality arise from a quantum fluctuating lower one. In this context the Planck scale limit has become a widely accepted notion to explain where our classical conception of spacetime should subside to a more fundamental conception of quantum spacetime, whatever this quantum spacetime represents. What I am trying to explain here is that this is probably a huge mistake caused by keep thinking using notions of classical logic. The notion of limit is fundamental to make classical logic work in the description of the continuum. We can define our real numbers as limits of Cauchy sequences because of this notion. On the other hand, this definition coincides with the Dedekind cuts definition within classical logic, but in other logics there is not an equivalent Cauchy limit definition. Therefore the idea that reality is not classic anymore in some kind of classical limit (a real number), to become a quantum reality is an incorrect way of posing this problem. The same mistake occurs in the usual approach to the measurement problem. There is not such a thing like a limit where the various states collapsed to a definite one, again we cannot use the classical notion of limit in a non classical logic description of reality.

How do we approach these questions in a non classical logic framework? Probably the reader suspects from the Continuum Hypothesis example and the figure (1) what solution I have in mind: *The internal structure of the non-classical logic world explains what the Classical world see, in the same way as the intuitionist structure of the example above, determines what the classical logic collapse sees (i.e. that the Continuum Hypothesis does not hold).* This is a more fundamental notion of limit where multivalued logic universes (many worlds) collapse to a classical two-valued (*true* and *false*) world.
From figure (1) we see how these ideas can be applied in physics, the fundamen-

[6]The known fundamental forces are Electromagnetism, strong and weak nuclear force and gravitation.

tal problem is to determine what should be the analogous of the set $\mathbb{P}$. Here it is important to note that the set $\mathbb{P}$ has not to be a partial order; it can be an arbitrary topological space, then we have many possibilities. Chris Isham has important results in this context [I2]. He has proposed for example, to take the set $\mathbb{P}$ as a space of Boolean algebras which represent history propositions in the sense of the consistent histories interpretation of Griffiths, Omns, Gell-Mann and Hartle. Other idea is to take the set $\mathbb{P}$ as a causal set, i. e. a set that describes the causal order structure of the spacetime. This approach has been developed by F. Markopoulou [M], it is particularly interesting the fact that he takes as the universes $V(p)$ the respective extended notion of causal past of each point; this seems to be a very promising idea. Normally this causal set is take as a discrete structure following the ideas of Sorkin; however, there is nothing important about discreteness. The duality between discrete and continuum is just one more of the misunderstandings caused by a classical logic reasoning. In this way the causal set (i.e the analogous order $\mathbb{P}$ considered above) can be just our normal spacetime, a Lorenziant 4-dimensional manifold with the causal order. I think this is the most important example that has not been yet developed.

Reading those papers you will find how rich these tools are, but richness can become an enemy; we need a definitive result. How can we get one appropriate order or space to describe completely quantum reality? The key step is to consider what I am trying to expose here. We need to choose this base space in such a way that the non classical logic universe that arises from it, can be collapsed to a classical world that looks like the world around us. This is so far the most testable theory we can imagine. *For the first time we will be able to understand that if quantum reality were not chaotic and non deterministic, probably our classical world would not be so smooth and well behaved.*

This new way to relate two different realities (logics) is too impressive to not be taken seriously. It is a totally new perspective to understand better some emergent phenomena. For example, we can make an immediate analogy with the DNA sequencing, that is no more than a very special type of order from where a huge complexity arises. Probably our method is far from being useful in this context, but show us a new perspective to understand such realities: How upper level complex universes can arise from lower level simple universes (partial orders, topological spaces, causal orders, DNA sequences) which intrinsic properties determine the upper level structure. A closer analogy to physics where I think we can test this approach, if correct, is emergence in computation. If we can get a model of quantum reality in this context, for example taking as the base space some kind of causal set, we will have a complete description of the hardware. After this, the software is just the way we collapse to classical reality (i.e our ultrafilters), then we should be able to explain completely what we see now as emergent phenomena.

I consider this like a second step to a truly Theory of Everything. The first step is (or was) Quantum Theory of computation, maybe the only new insight aspect about reality that theoretical physics has given us in the last three decades. Why quantum computation is the first step is explained in David Deutsch's extraordinary book, *The Fabric of Reality* [D1]. To conclude, I would like to leave you with a short piece from this book:

> *What makes a theory more fundamental, and less derivative, is not its closeness to the supposed predictive base of physics, but its closeness to our deepest explanatory theories.*
> *Quantum theory is, as I have said, one such theory. But the other three main strands of explanation through which we seek to understand the fabric of reality are all high level from the point of view of quantum physics. They are the theory of evolution (primarily the evolution of living organisms), epistemology (the theory of knowledge) and the theory of computation (about computers and what they can and cannot, in principle, compute). As I shall show, such deep and diverse connections have been discovered between the basic principles of these four apparently independent subjects that it has become impossible to reach our best understanding of any one of them without also understanding the other three. The four of them taken together form a coherent explanatory structure that is so far-reaching, and has come to encompass so much of our understanding of the world, that in my view it may already properly be called the first real Theory of Everything.*

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