Mathematics as unifying force for science

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In this essay I will discuss the relation between mathematics (in short: math) and physics. Starting with a historical review, the close relation between math and physics is rooted in forecast of experiments in physics and engineering. Then math is simple a tool to tackle these problems. But math and physics changed by a cultural change of our thinking. Therefore, a more global view to problems was created leading to the consideration of general, abstract structures in math and physics as well. In particular, it was the need to understand invisible things like atoms or fields in physics. But math and physics met at this higher level again. In this essay I will also discuss the question why math was created. I see the roots in the requirement for abstraction necessary for a species with limited brain. But math is also limited as discussed by Gödel and Turing. The development of new math is a creative process which is bounded to our brain. So, I disagree with Plato: there is no independent world of ideas. Finally I will discuss the unifying power of math for all science in the future.

Dedicated to the memory of Alan M. Turing

I. PHYSICS AND MATHEMATICS

In 1938, Alan Turing wrote in his paper "Systems of Logic Based on Ordinals," section 11: The purpose of ordinal logic (published in Proceedings of the London Mathematical Society, series 2, vol. 45 (1939)): "*Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity. The activity of the intuition consists in making spontaneous judgements which are not the result of conscious trains of reasoning... The exercise of ingenuity in mathematics consists in aiding the intuition through suitable arrangements of propositions, and perhaps geometrical figures or drawings.*" Mathematics (in short: math) is not only driven by logic and formal systems of axioms but rather by intuition and creativity. Even the interaction between physics, as a kind of applied mathematics to the description of nature, and mathematics was very close to physics over many years but divorced later. There is no exact starting point for this fruitful connection. Physics was seen as part of philosophy for a long time, starting with the Greek period. But in the 16th century with the work of Galilei and Newton, it changed. Now mathematics became a main part of physics. More importantly, Newton developed a new mathematical method, the calculus, to describe the dynamics in mechanics. Later, nearly all mathematicians were also physicists at the same time. Lagrange, Hamilton and others developed analytical mechanics, Maxwell used methods of Gauss, Stokes and Helmholtz (known as vector analysis) to obtain the equations of electrodynamics, Minkowski unified space and time to get a description of special relativity etc. The situation changed in the Thirties of the last century. Mathematician started to look into the foundation of mathematics. Abstract algebra was developed (Noether and his school), algebraic topology took its shape, category theory (inspired by algebraic topology) was founded, algebraic geometry was newly founded and connected to abstract algebra etc. The group 'Bourbaki' was formed, the collective pseudonym under which a group of (mainly French) 20th-century mathematicians, published a series of books with the aim of reformulating mathematics on an extremely abstract and formal but self-contained basis. Physics and mathematics were divorced: physics had trouble to formulate quantum field theory and mathematics got a new basis. This situation changed in the sixties where elementary particle physicists tried to get a classification for strongly interacting particles and started to use group theory (mainly Lie groups) to realize the program (Gell-Mann, Zweig etc.). It was the starting point for gauge theory including the connection to vector or fiber bundles in the seventies. At the same time, one discovered the first topological solution (e.g. interpreting the monopole as non-trivial principal fiber bundle) or used topology to obtain the singularity theorems in General relativity (Hawking and Penrose). At the same time, the mathematicians conjectured and proved many unexpected relations between different topics like the Atiyah-Singer index theorem (analysis, topology, geometry), Taniyama-Shimura-Weil conjecture (number theory, elliptic curves/functions), Langlands conjecture (number theory, group representations), Thurstons geometrization conjecture (geometry, topology of 3-manifolds, group theory) etc. But it was also the beginning for a new fruitful connection between physics and mathematics again. The crystalization point for this new relation was the development of string theory, in particular by the work of Witten. There is a controversy about the physical content of this theory but from the mathematics point of view this theory is really interesting. Many different topics were connected like Teichmüller theory, Calabi-Yau spaces, differential cohomology, index theory in loop spaces etc. Totally new topics were born like quantum cohomology or the geometric Langlands program as well some conjectures like the Monstrous moonshine conjecture were solved reflecting this fruitful connection. Therefore it was not a surprise that at least one of the Clay Millennium prize problems is inspired by physics (foundation of Yang-Mills theory and proof of the mass gap).

But after this historical excursion, many question remain: Is it artificial or natural that mathematics is the foundation of physics? Is mathematics also useful or effective in other sciences (e.g. humanities, biology)? What are the borders of a mathematical description? One word about my abilities: my mathematical knowledge is partly limited to topology. Therefore I will mainly argue with this topic but I'm sure that the argumentation can be also extended to other topics (like number theory) with the same conclusions.

II. MATHEMATICS = UNDERSTANDING STRUCTURES TO FORECAST THE FUTURE?

As described in the previous section, physics need (in some sense) mathematics but why? Let us start with a general remark: historically, physics grew from a connection between experiment and theory. "Ask a question to nature (theory) which will be answered (experiment)" is a common expression for this relation. This fruitful relation started with Galilei who made the first quantitative experiments. Then the idea was not far away that an analysis of past observation can be used to forecast future observations. The invention of calculus by Newton and independently by Leibniz gave the chance to forecast the trajectory of objects or particles. Right according to the principle: if we know the past then we can forecast the future. Then the central topic was the measurement of observable amounts like velocity, distance or time intervals as initial values. These numbers were transformed by the calculus into a set of new numbers for later time steps. One of the greatest triumphs of this method was the discovery of the planet Neptune by analyzing the deviation of the planet Uranus. In 1846, the french mathematician Urbain Le Verrier calculated the position of the unknown planet from the deviations. Finally Johann Gottfried Galle in Berlin (after a letter from Le Verrier) found the planet at the calculated position. Everything seemed at the right order. Mathematics with calculus was the method to produce this order. But calculus was partly inspired by physics and the need to make a forecast for trajectories. But the further development of calculus acted back on mathematics too. It was the necessity to introduce the continuum as the completion of the rational numbers in the course to understand the calculus. Then one started to define natural numbers (Peano axioms) and introduced the rational numbers. This development culminated into the construction of abstract numbers like complex numbers, quaternions and Cayley numbers (or octonions). But then mathematics changed its face: from pure calculations and solutions of equations to the understanding of the underlying structure. But why does it happens? It started with an innocent looking problem: finding the roots of a polynomial. In the beginning of the 19th century, the general solution of a polynomial equation up to forth order was known. Then two young mathematicians, Evariste Galois and Niels Henrik Abel (both died young) found the surprising solution: there is no general formula (using the usual operations: $\pm, \cdot, /, \sqrt[n]{}, {}^n$) for the roots of the polynomial equation of grade five or higher. But what about special polynomial equations? There is a complete solution to this problem by using the abstract formulation of this problem via field extension and group theory. Both topics were founded for this problem. It was the starting point for a deep change of mathematics. Mathematics became a theory about structures: group, rings, fields, module etc. This change had far reaching consequences: it opened up new connections between previous disjoint areas like geometry and groups (Klein's Erlanger program). This approach culminated in the development of category theory where these structures are clustered into categories and theories are described by fixing a map between categories (called functor). At this stage, mathematics and physics were divided at the first view. Of course, this exchange between mathematics and physics was never closed but it was a weaker connection. But in the meantime, physics also changed the way of thinking. Mechanics as founded by Newton and others has the concept of a force acting directly to all bodies. But with the advent of electrodynamics, Faraday developed the concept of a field which can be only visualized by the action to test particles. The complete theory was worked out by Maxwell. The local action via a field was born culminating in a change of mechanics into special relativity (for large velocities near the velocity of light). But the concept of a field was not the only conceptional change. Inspired by steam engines, thermodynamics was developed including the three laws of thermodynamics (zeroth, first and second law of thermodynamics), i.e. existence of a thermal equilibrium and temperature (zeroth law), energy conservation (first law) and existence of dissipation (second law: 'Heat cannot spontaneously flow from a colder location to a hotter location.'). Every law is connected with a scalar observable: temperature, energy and entropy where entropy is the most mysterious one. Thermodynamics started as a pure theory to explain all processes with heat. But with the appearance of the second law of thermodynamics, the whole theory became much more abstract. The concept of the entropy was connected with the order of a system as developed in statistical physics. Today, entropy is a general concept used in information theory (Shannon entropy), quantum theory (von Neumann entropy) or in general relativity (Bekenstein-Hawking entropy of a black hole). Also, thermodynamics together with statistical physics is now a general method to describe many particle systems, a step of a further abstraction in physics. With the advent of quantum theory and the theory of relativity, the relation between math and physics became closer. Mathematicians like Herman Weyl or Johann von Neumann worked at the sharp end of theoretical physics. The new mathematical methods, like Riemannian geometry or Hilbert spaces, were useful for physics at the first glance but came to an end in quantum field theory. At the same time, there was an explosion in activity for developing topology, abstract algebra, number theory etc. with no direct application to physics (see the previous section). In the 1970s, topology and differential geometry became important to understand special solution of gauge field theories (monopoles, instantons etc.). With the appearance of string theory, the relation between physics and math is now very fruitful for both sides. Now mathematical topics like quantum topology were created by questions inspired by physics and math helps physics (in particular string theory: classification of D brane charges using K theory).

But let us stop here with the historical facts. Why is the relation between math and physics so close? Above, I analyzed the root of this relation: the appearance of the experiment in modern physics and the need to forecast as well understand quantitatively the experimental results (using numbers as part of math). But later the kind of the relation changed. Both sciences, math and physics, used more and more abstraction, so the relation remained. Also today, math and physics are connected by many abstract ideas, e.g. the usage of gauge theory in the theory of smooth 4-manifolds. I interpret this change of the relation from the quantitative forecast of experiments to the structural understanding of nature as caused by a cultural change in our thinking. The close relation is rooted in the need for detailed calculations in the application of physics to engineering. But our thinking changed in the last 200 years away from the vissible objects (wheels, stones etc.) to the hidden objects (fields, atoms etc.). At the same time, our thinking was able to grasp more and more complex relations between objects. This change was revolutionary in physics but can be seen also in other sciences like biology (cells, virus, genome but also evolution etc.) or chemistry (atoms, structural chemistry etc.). I see the close relation between math and physics as a result of the beginning of modern physics (forecast of experiments). The today's relation is more the result to understand complex systems by using a thinking into structures and concepts, i.e. we use qualitative methods to understand complexity. But mathematics is the science of structures, but why? In the next section, I will show a deeper root of math in our thinking.

III. HOW IS MATH CREATED?

It is the concept of abstraction from which we are able to get any knowledge about the world. But this ability is rooted in our species. As a baby on a way to learn our language, we learned the words by examples. Then the following questions were central for the baby. What are the objects which are denoted as ball: small or big, round, painted red or blue. There are many possibilities but we learned to see the characteristics which made an object to a ball. This process is abstraction and for me the development of the language was one of the first abstraction processes of our mankind. Abstraction is necessary concept for our species: we have a limited memory in our brain and a limited number of sensors to sense the world. Therefore, we have to simplify many relations in the world to understand them. But abstraction is also the root of mathematics: numbers as an abstract count of objects was the beginning. In Greek philosophy, the structure of space is given by the relational description of objects. Then one needs only some simple axioms to deduce geometry. It was this deductive approach of geometry which served as a blueprint for other deduction processes in math. In this sense, one can understand the logic of Aristoteles as inspired by math. But math is in particular a relational theory. Let us consider Euclid's geometry. One needs some obvious basic objects like point, line or surface which is not defined. Then the axioms are given by the relation between the three objects (like: the intersection between two lines is a point). In principle all axiom systems are of this kind. The power of deduction was very inspiring for all mathematicians. So, it was very natural that David Hilbert asked for a complete deduction of the whole mathematics by axiom systems at the International Mathematician congress in Paris in 1900. At this time, it seemed possible. But Kurt Gödel found a contradiction in this approach. It was the first limit in our approach to understand the world. Alan Turing expressed it in his model of a machine which deduces from the axioms all mathematical result. It is not possible because this machine will never stops. Then Turing claimed: Mathematics is a creative process which cannot be automatized by a simple machine. I see this statement as the most important result of the 20th century mathematics.

If one looks into the mathematical breakthrough of the last years then this creative process is an unconventional combination of different areas. As an example, the Feynman diagrams of the Chern-Simons theory underlying Wittens approach to the Jones polynomial are used to create the chord diagrams in the theory of Vassiliev invariants. Even the Vassiliev invariants were inspired by singularity theory used to understand the bifurcation of solutions in dynamical systems. For me, it is rather a miracle that everything in math is connected to each other then this close relationship to physics.

So, math is the result of an abstraction process which is necessary for our species to understand the world around us. But abstraction is only the beginning of math, which also enforces the introduction of structures into math. The unconventional combination of ideas from different areas is the stronger driving force for the creation of new math. Here, I disagree with Plato: there is no independent world of ideas. It is our brain which makes the division of our world in more or less independent set of objects with some relations between them. But this division is not the same for all people. It can vary and induces a different view on a problem by different people leading to a creative process for one people which found the useful division.

IV. MATHEMATICAL STRUCTURES FOR A QUALITATIVE UNDERSTANDING OF SCIENCES

As described above, the relation between math and physics is not accidental. But the discussion above also implies that math is a general concept for whole science. But at the first view, only physics has this strong relation, why? I see the reason in the different complexity in science. Physics describes the dynamics of simple objects. Even at the beginning of physics, there were simple models which served as universal models to understand the underlying processes. The harmonic oscillator is one model which is used in mechanics (pendulum), thermodynamics (heat bath), electrodynamics (oscillator of Hertz) and quantum physics (harmonic oscillator, free field quantization in quantum field theory). But most interesting models (like the Navier-Stokes equation) cannot be solved. Physicists were very creative to overcome these problems in some models and developed new approximation methods for better calculations. All these more or less simple models describe the world quite successfully. Even this success is another reason why math and physics are so close to each other. But in the meantime, math went a step further and developed qualitative methods to describe complex system, called topology, i.e. pictures instead of equations. But more importantly, the solution don't depend on the idealized model because every deformation of the model is also described qualitatively in the same way. These ideas were firstly applied in dynamical systems as founded by Henri Poincare. There, one studies the critical points of a time-dependent system, for instance systems described by ordinary differential equations (like gradient systems $\dot{x} = f(x)$). From the deformation point of view, the neighborhood of every fix point can be divided in a finite number of cases. Then, one can understand the solution of the dynamical system by connecting the fixpoints. Of course the concrete form of the solution depends on the equation. But from the qualitative point of view, every system with the same fix points have a very similar behavior. Here, topology helps to describe classes of systems with a similar behavior. For systems described by partial differential equations, one can use foliation theory to obtain classes of systems with a similar behavior. But even more is true: one can define invariants of the foliation, i.e. invariant expression w.r.t. deformation. A large part of topology deals with the general theory of deformations, also called homotopy. Even in the last years, there was a great success to extend this concept to many structures in mathematics. So, one can deform not only topological spaces but also algebraic structures (or whole categories). In physics, it is also every important to know the ground state of a many-particle system (or a field theory). This ground state depends on the underlying space (where the system is defined). Here, the famous Atiyah-Singer index theorem is able to give restrictions on the ground state. In quantum field theory, there was a strong interaction with topology since the 1980's (anomalies, topological quantum field theory, string theory etc.). I also think that the dualities in quantum field theory are also deeply rooted in topological properties of the spacetime.

But what about other sciences? It seems that only physics and math have a well-established relationship. Other sciences like biology or sociology based on very complex systems. There are two problems: the system is too complex to admit a realistic description (for instance biology: the human brain) or the rules of the system are in principle not known (in sociology: experiments are not repeatable). Of course, in the first case one can try to make computer simulations but this simulations are useless if one don't understand qualitatively the system. Then, one is able to make progress if one understands the systems in a qualitative way. There are first signs like the usage of knot theory to understand the protein or RNA folding. Therefore I think that the increasing complexity of the models can be handles qualitatively by using topological methods in the future. But what about the second problem: finding the rules? Mathematics is the theory of structures, i.e. math can be used to formulate a problem in structured way. Usually this approach started with a concrete system or model and give a mathematical formulation which can be generalized (from special to general). For instance, some aspects of sociology can be understand using network theory. Or, economics used game theory. In the simplest fashion, one has the 2-person-games of Nash (having the Nash equilibrium). Now one can put these 2-person-games on an evolving network to get a realistic behavior of a market. Then the special system is not important. But it is likely that there are invariants in the system which determines also the global behavior. In some sense, these invariants reflect the rules of the system. But it is not an easy way to extract the rules following this idea. As a prerequisite, one can analyze the structure of systems from the logical point of view. A notable trial is George Spencer Brown "Law of forms" which is a structural approach (including an own language for description) to general laws. This approach is quite necessary but I'm sure that it is only a beginning. Category and model theory are much more appropriate to describe the complex laws of social-economic systems. In my opinion, it is worthwhile to try it.

Finally, I see modern mathematics as the basic stone of all science. Even the structural aspects is most important for sciences except physics.

V. CONCLUSION

I have presented a certain number of ideas:

1. Math and physics are historically strongly related starting from the time of Newton up to now. This relationship changed over the years like both areas changed.
2. The origin of this close relation was the need in physics (and engineering) to understand and describe experiments. Then the formulation of any laws in physics must be done by math to get in contact with the experimental data.
3. Math changed around 200 years ago and in particular after the Second World War. Now it is the science of structures like groups, modules, categories etc. instead of elegant calculations. Interestingly, physics also changed in the same period, so that this change can be understood as a cultural change in our thinking. Now we try to grasp the global questions where we need to understand the large connections in a system.
4. Math is a purely creative process. It was born in the need of abstraction. Without abstraction, our species with a limited brain is unable to reflect the world. At the first glance, math can be identified with the process of abstraction (in the spirit of Greek philosophy).
5. Math is the unifying force for all sciences. The relation to physics is mainly caused by the simple (in contrast to other sciences) calculable problems in physics. But in future, even the structural aspect of math will be the driving force for the mathematical development of other sciences like sociology, biology etc.

Before concluding, I will mention the discussions with Carl H. Brans, Jerzy Król, Helge Rosé and in particular with my wife Andrea about the use of math in other sciences.