How to faithfully map a mathematical structure to an experimental situation

FQXi Essay *Trick or Truth: the Mysterious Connection Between Physics and Mathematics*

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Abstract

Simple mathematical structures such as numbers or elementary geometry are directly tied to physical observations. Wigner pondered the existence of similar one-to-one correspondences between more advanced mathematical concepts, such as algebras, and the actual world. The compilation of such a list of "maps" is in itself a formidable research project *en route* to finding limits of the interplay between mathematics and physics. In this essay we will study the weighing problem, an example given in the 1930s to illustrate the idea of "complex experiments", and construct step by step the underlying group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and its representations. The concepts involved are advanced enough to highlight a non-trivial link between mathematics and physics without losing the idea midway through the formalism.

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It is impossible to study this amazing theory without experiencing at some times the strange feeling that the mathematical formulas somehow have a proper life, that they are smarter than we, smarter than their author himself, so that we obtain from them more than was originally put into them. — Heinrich Hertz [2]

It might be surprising to hear, but as an experimentalist I have to think about mathematical structures when I devise an experiment. This is because the measurement settings usually do not only form a simple set but there is a relation between them. For instance, if we have measurement settings

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"on" and "off" and measure the light intensity for both settings, yielding $(I_{\text{on}}, I_{\text{off}})$, then these two settings form the group $\mathbb{Z}_2$.[1] Representation theory then tells us we can calculate $I_{\text{mean}} = \frac{1}{2}(I_{\text{on}} + I_{\text{off}})$ and $I_{\text{diff}} = \frac{1}{2}(I_{\text{on}} - I_{\text{off}})$, which is known as the discrete Fourier transform. How obvious is it that we can calculate the mean and difference? Where do they come from? What if we define other quantities like $I_{\text{weird}} = -\frac{1}{3}(I_{\text{on}} + I_{\text{off}})$? What would they tell us? Is it even logical to calculate this quantity given the simple experimental setup? Without a proper understanding of how mathematics and physics are connected, those questions are hard to answer. The principal idea is that one cannot set up an experiment and independently define the mathematical structure describing it. These two things are inherently interwined, which we must keep in mind if we are looking for unbiased measured and calculated quantities.

With this in mind let us now look at the weighing problem, which was introduced by Yates [3] and Hotelling [4] to illustrate the idea of "factor analysis". Developed for the systematic study of the interaction of factors such as sunlight and weed killer on crop yields, it was soon realized that experiments whose outcome depends on multiple factors is a far more general scenario. As interesting as this is, we will not go into details here, restricting ourselves to the setup and result of the weighing problem. Assume you have seven light objects and you have to determine their weight with a simple single pan scale that has an intrinsic bias, which can be regarded as an additional weight. One approach is to weigh each object individually. A statistical uncertainty σ is associated with each of the eights weighings, including one for the bias, which can be reduced to $\sigma/\sqrt{N}$ for each mass by N repeated measurements.

The weighing problem can now be formulated: is there a better experimental design? The answer is *yes*. The key: simultaneous weighings done with a scale that has two pans. It is easy to show that, in this case, eight weighings are sufficient for a statistical error of $\sigma/\sqrt{4}$, which is substantially less than the 32 weighings needed with the simple scale. This is only the starting point for the exciting development of the field of optimal experimental design, but we will now concentrate on the mathematical features of this little problem.

[1] There are many ways of seeing this. Note that the measurement setting "off" records the background only whereas "on" measures both signal and background, i.e. the background is contained in both settings, hence the relation.

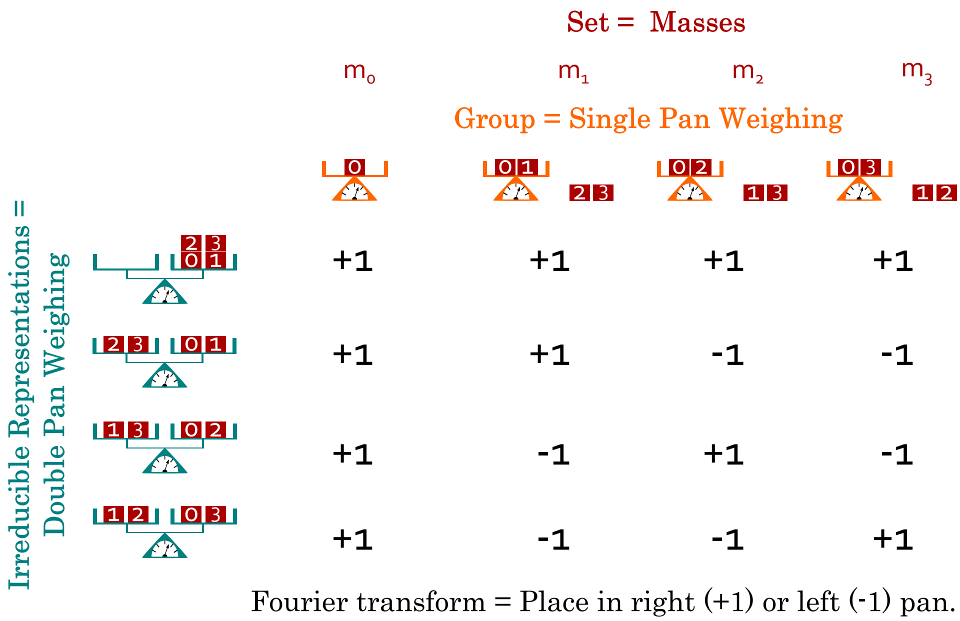

Figure 1: Physical realization of the Fourier transform over $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ by weighing objects. The masses M form a set and the single pan experiment defines, via the bias m_0, a relation on M which renders it to the group G (see text). The minimum uncertainty weighings require a double pan scale, which allows one to measure differences. Each weighing with the double pan corresponds to an irreducible representation of G. The act of placing the objects in the respective pans in each weighing constructs the character table of G.

Trick or truth?

We simplify the original formulation a little bit and consider the weighing problem for three light objects m_1, m_2, m_3 and a bias m_0, forming the set $M = \{m_0, m_1, m_2, m_3\}$, as shown in Figure 1. What we see is how the single pan weighing defines the masses experimentally and how this is structurally related to the more elaborate double pan weighing. Surprisingly, the Hadamard (transform) matrix H_4 shows up, which is nothing less than the character table of — or Fourier transform over — the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let me emphasize that this is not a transform that we do on a piece of paper or a computer, but by placing masses in pans.[2] Why do we obtain the group

$$G = (\{\tilde{m}_0, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3\}, \cdot) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{1}$$

when we devise the single pan weighing? (Note that we use $\tilde{m}_i$ in G to denote m_i *as defined by the experiment*.) Where does this structure, which is

[2]This is what interferometers do: instead of measuring the spectrum directly with a spectrometer, an interferogram is recorded and the spectrum is obtained by application of the inverse Fourier transform. This can reduce the measurement errors, leading to the so-called Fellgett advantage.

imposed on the set M, come from? Why is the Fourier transform over a group appearing in plain sight?

First observe that the single pan weighing allows us to make sense of expressions like $m_1 + m_2$ because we can put both m_1 and m_2 on the scale. However, differences like $m_2 - m_1$ cannot be measured with this scale. The natural combination of elements of M, and thus the most direct way of weighing each mass, is determined by the unique element m_0:

$$\begin{aligned} m_0 &\rightarrow m_0 \\ m_1 &\rightarrow m_1 + m_0 \\ m_2 &\rightarrow m_2 + m_0 \\ m_3 &\rightarrow m_3 + m_0 \end{aligned} \tag{2}$$

This gives us the group elements (see Figure 1):

$$\begin{aligned} \tilde{m}_0 &:= \text{Weigh } m_0. \\ \tilde{m}_1 &:= \text{Weigh } m_0, m_1 \text{ together and leave out } m_2, m_3. \\ \tilde{m}_2 &:= \text{Weigh } m_0, m_2 \text{ together and leave out } m_1, m_3. \\ \tilde{m}_3 &:= \text{Weigh } m_0, m_3 \text{ together and leave out } m_1, m_2. \end{aligned} \tag{3}$$

which partitions the masses into two sets: weighed or not being weighed. The identity element is obviously $\tilde{m}_0$.[3] Also, each element is its own inverse because m_1, m_2 and m_3 are related to m_0 individually but not jointly. This should become clearer soon when we answer the more pressing question: what is the group multiplication? Here things start to get less obvious. The following excerpt of Fisher's group theoretical account of factor analysis shows that the physical picture is still in the mists [5]:

A group may be formed, of which the elements are all the selections that can be made of none or more out of n letters. The order of the group is 2^n. The product of any two elements is formed by combining the letters they contain, deleting any they may have in common. The group is, therefore, Abelian.

Let me translate this to our situation. Each group element is specified by the masses that are not put on the pan:

$$\tilde{m}_0 = () \quad \tilde{m}_1 = (m_2, m_3) \quad \tilde{m}_2 = (m_1, m_3) \quad \tilde{m}_3 = (m_1, m_2) \tag{4}$$

[3] Again, this is the bias of the scale. In other areas such as medicine this would correspond to the control treatment or group.

Then one can combine two elements $\tilde{m}_i$ and $\tilde{m}_j$ as follows:

$$\begin{aligned}\tilde{m}_i \cdot \tilde{m}_j &= (\text{masses left out by } \tilde{m}_i) \cdot (\text{masses left out by } \tilde{m}_j) \\ &:= \begin{cases} \text{put mass left out by both weighings on pan} & \text{if} \quad i \neq j \\ \text{put nothing on pan} & \text{if} \quad i = j \end{cases}\end{aligned} \tag{5}$$

Thus, each element is its own inverse and the group is Abelian. But is this description really satisfactory? This is where careful thinking starts. To illustrate the subtleties involved, look at the group identity in Figure 1, which is depicted as the experimental situation where only the single pan and its bias m_0 is relevant. Notice that we did not place m_1, m_2 and m_3 next to it. Why? Because that would be an inaccurate description of what the identity is all about. The single pan with its bias exists independent of whatever we are going to weigh with it. It would be the same if we decide to weigh seven objects. This reflects the unique status of the identity element. If, instead, we had defined

$$\hat{m}_0 := \text{Weigh } m_0 \text{ and leave out } m_1, m_2, m_3, \tag{6}$$

then we also need to redefine $\tilde{m}_0 = (m_1, m_2, m_3)$ in (4), which would lead to difficulties with (5). For instance $\tilde{m}_0 \cdot \tilde{m}_1$ would be "put both m_2 and m_3 on the pan", which is an undefined experimental situation. It is of course physically possible, but would require a different structure — another transformation — to disentangle the information about m_1 and m_2 in this joint weighing. But then, what is the relation between $\tilde{m}_i$ and $\tilde{m}_j$ ($i, j \neq 0$), given that m_i and m_j are not weighed together? It seems like the only relation is given by the simultaneous weighing of each mass m_i ($i = 1, 2, 3$) with m_0. However, we systematically leave out two masses in each weighing, which, for example, defines the relation between $\tilde{m}_1$ and $\tilde{m}_2$ as "both left out m_3 in their weighing". This sounds very abstract since by reading off the scale we only obtain information about the relation between m_i ($i \neq 0$) and m_0. But careful here. The *actual weighing* adds a little bit more structure to the existing setup by defining a function $R : G \to \mathbb{R}$. That is, for each weighing the scale produces a real number. The group is only concerned with what we have set up: there is a special mass m_0, for each measurement the masses are partitioned into being weighed and not being weighed, and one mass is weighed at a time (together with m_0). This definition is very general and in principle there is structural information contained in both the relation of the elements in "being weighed" and "not being weighed". It is the function R that specifies what information we use and what we discard.[4] Therefore, Figure 1 is a bit misleading because there the "meter" is already included in

[4]For sake of clarity, I did not define "information" and use it only to convey the idea.

the scales, hinting that we measure only relations that "are on the pan" and thereby already including the function R. This only exemplifies the struggle for a consistent and yet accessible physical description.

We have established that it is the experimental idea and measurement settings that define the group. It might therefore be surprising that the corresponding irreducible representations and Fourier transform are "so real". Irreducible representations of a group are special types of group homomorphisms, i.e. maps that preserve the group structure. Here, we physically construct the four of them $(\rho_1, \rho_2, \rho_3, \rho_4)$ by replacing the single pan with a double pan scale. This basically maps "leave out m_i and m_j" to "put m_i and m_j in the second (reference) pan", which can be seen in Figure 1. This allows us now to measure "real differences", unlike the inverses of G. Moreover, each of the four weighings $i = 1, 2, 3, 4$ **is** the Fourier transform of $\tilde{m}_0, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3$ at irreducible representation ρ_i because, as Figure 1 shows, by placing the masses in their respective pan, we form the transformation matrix "with our hands".

What have we learned? Given that the Fourier transform is clearly visible and knowing that it works to describe this experiment, it seemed like a trivial task to physically construct the mathematical structure around it. Though we ended up with a "faithful map" between the axioms of a group and elements of the experimental setup, its construction proved to be hard, which is not what I expected at the outset. A clear understanding of the physical-mathematical relationship, then, enabled us to associate the double pan weighing with irreducible representations by simply seeing that the only change, compared to the single pan weighing, is that everything that is not being weighed is now put in the second pan. Given that structurally nothing else changes we immediately know that they are irreducible representations. If we were just given the double pan weighing and we were asked to identify the mathematical structure describing the experiment, we would have had an equally hard job (if not harder).

What conclusions can be drawn with respect to the reality of mathematics? Certainly, there are "real" elements because the relationship between physical objects embody the mathematical structure, most prominently with the identity element.[5] Other parts, like inverses, seem more fuzzy but eventually every element of the mathematical structure has meaning. Let me point

[5] For example, Tegmark [6] points out that vector spaces have too much structure to support the idea of a homogeneous space. For all practical purposes, however, we have a reference point, e.g. center of gravity, which is why we need an identity element $\vec{0}$.

out that the transformation describing the experiment must be the character table of G. For instance, one cannot simply replace "-1" by "0" and say this is an equally valid transformation. It is certainly a possible transformation, but not for our experiment: the double pan scale really shows the difference between what is put in the right pan (+1) and what is put in the left pan (-1). Replacing -1 by 0 would mean to not use the left pan at all, effectively reproducing the situation with a single pan. In applications, for example the theory of optimal experimental design and Hadamard transform optics, the Hadamard transform can be used as a tool without ever mentioning representation theory, in which we would call it a character table. It just works, and in practical use the full representation-theoretical description must feel like a mathematical sledgehammer — which, not surprisingly, blures the relationship between mathematics and physics at times.

So what have we achieved? We constructed a proper map between elements of physics and mathematics. We did not just loosely connectoi elements (the obvious part), but have preserved their relations as well (which was tedious). But what have groups and representations to do with weighing objects in the first place? This is a question we can only answer after analyzing a large number of similar problems. Our job is to prove that a certain physical situation is a mathematical structure is disguise. We need to understand how nature chooses its mathematical dresses, and only then are we able to connect the dots and tackle the bigger questions.

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