Perturbative Proof of the Covariant Entropy Bound

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Area and Information

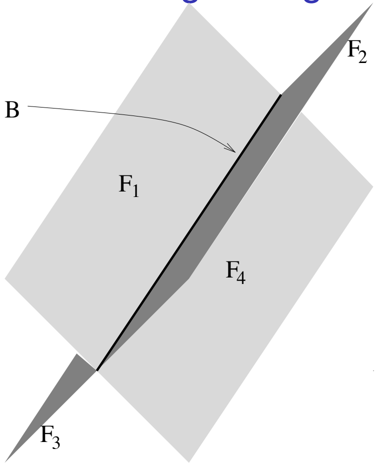
- There is evidence for a universal relation between geometry and information:
- The information on a light-sheet is bounded by the difference between its initial and final area.
- In this talk I will present a proof of this relation in a nontrivial limiting regime.

Covariant Entropy Bound

What is the Entropy?

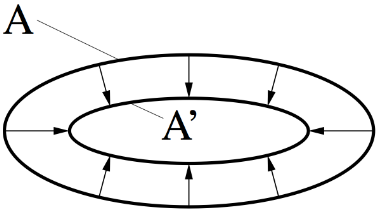
Perturbative Proofs of the GSL and GCEB

Surface-orthogonal light-rays



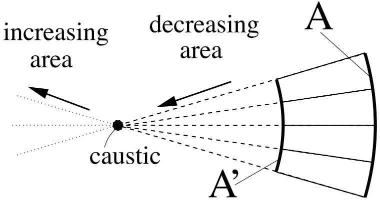
- Any 2D spatial surface *B* bounds four (2+1D) null hypersurfaces
- Each is generated by a congruence of null geodesics ("light-rays") $\perp B$

Light-sheets



An orthogonal null hypersurface is called light-sheet if the generating light-rays are nonexpanding away from the initial surface.

The Nonexpansion Condition

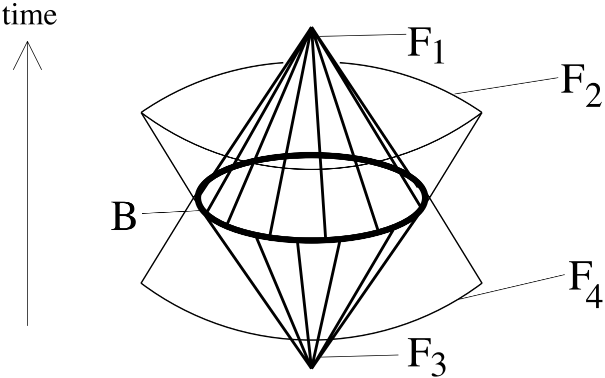


- $\theta \equiv \widehat{\nabla_a k^a}$, where k^a is the affine tangent vector field
- In terms of an infinitesimal area element $\mathcal{A}$ spanned by nearby light-rays,

$$\theta = \frac{d\mathcal{A}/d\lambda}{\mathcal{A}}$$

Demand
$\theta \leq 0 \quad \leftrightarrow \quad$ nonexpansion
everywhere on the
light-sheet.

Light-sheets



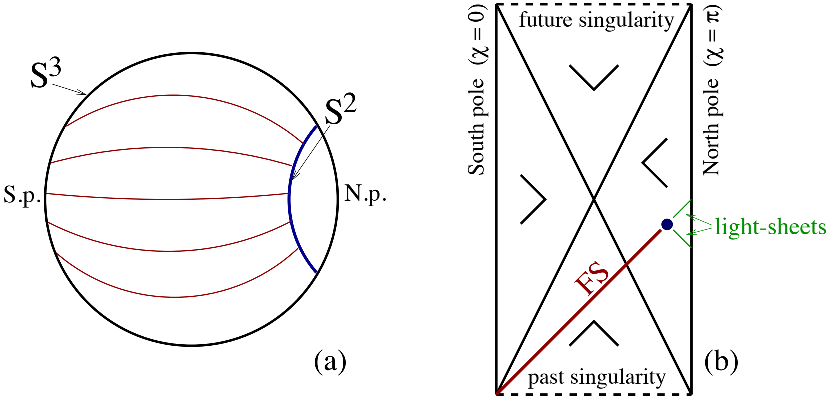
Out of the 4 orthogonal directions, usually at least 2 will initially be nonexpanding.

Covariant Entropy Bound

In an arbitrary spacetime, choose an arbitrary two-dimensional surface B of area A. Pick any light-sheet of B. Then $S \leq A/4G\hbar$, where S is the entropy on the light-sheet.

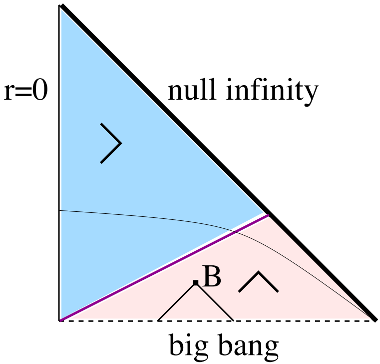
RB 1999

(1) Closed universe



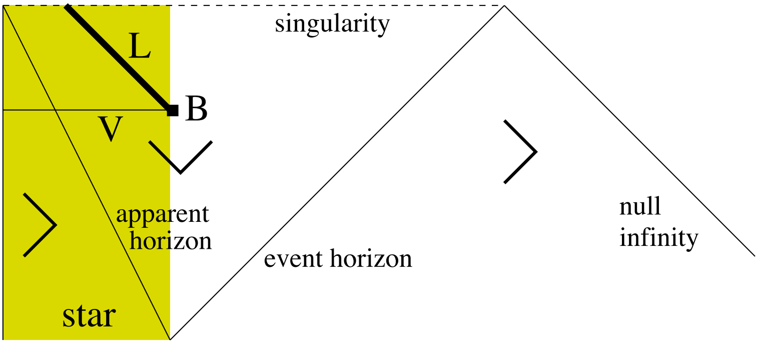
- Consider a small 2-sphere on a closed spatial manifold.
- The light-sheets are directed towards the “small” interior, avoiding an obvious contradiction.

(2) Flat FRW universe



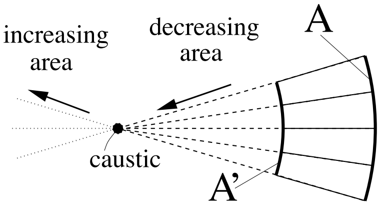
- Sufficiently large spheres at fixed time t are anti-trapped
- Only past-directed light-sheets are allowed
- The entropy on these light-sheets grows only like R^2

(3) Collapsing star



- At late times the surface of the star is trapped
- Only future-directed light-sheets exist
- They do not contain all of the star

Generalized Covariant Entropy Bound

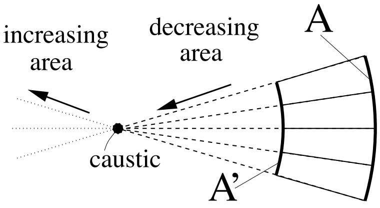


If the light-sheet is terminated at finite cross-sectional area A', then the covariant bound can be strengthened:

$$S \leq \frac{A - A'}{4G\hbar}$$

Flanagan, Marolf & Wald, 1999

Generalized Covariant Entropy Bound



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This bound is can be nearly saturated in the weak-gravity regime.

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- . . . the matter system fit on a light-sheet of a surface of area A.

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- Under what conditions can we consider a matter system to "fit" on L?
- What is S if some relevant field modes are not fully localized to L?
- This is generic: In QFT, there is no sharp separation of subsystems.
- Even the vacuum, restricted to L, has (divergent) entropy.

How is the entropy defined?

- In general, no satisfactory definition of the entropy on a light-sheet has been given.
- However, the GCEB is nontrivial even in the perturbative regime, where matter has small backreaction on the geometry.
- In this regime, a sharp definition of S is possible.

How is the entropy defined?

- Using definitions of S due to Casini [2008] and Wall [2011], I will present a perturbative proof of the GCEB.

  RB, Casini, Fisher, Maldacena 2011

- Despite the limited regime, this result is remarkable as it does not require any explicit assumptions of a relation between entropy and energy, nor of energy conditions on matter.

Casini Entropy

Consider two field theory states in Minkowski space: the vacuum, $|0\rangle$, and some excited state $\rho_{\rm global}$. In the absence of gravity, $G_N = 0$, the geometry is identical in all matter states, and one can restrict both states to a subregion V:

$$\begin{aligned} \rho &\equiv \mathrm{tr}_{-V}\, \rho_{\rm global} \\ \rho_0 &\equiv \mathrm{tr}_{-V}\, |0\rangle\langle 0| \end{aligned}$$

Due to vacuum entanglement entropy, the van Neumann entropy of each density operator diverges like A/ϵ^2, where A is the boundary area of V, and ϵ is a cutoff. However, the difference is finite:

$$S^C \equiv S(\rho) - S(\rho_0)\ .$$

Marolf, Minic & Ross 2003, Casini 2008

Properties of the Casini Entropy

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  $\rightarrow$ No Species Problem
- The observer-dependence is physically appropriate: an observer with access only to V is unable to discriminate an arbitrary number of species due to thermal effects.

Covariant Entropy Bound

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Perturbative Proofs of the GSL and GCEB

Relative Entropy

Given any two states, the (asymmetric!) *relative entropy*

$$S(\rho|\rho_0) = -\mathrm{tr}\,\rho \log \rho_0 - S(\rho)$$

satisfies positivity and monotonicity. That is, under further restrictions of ρ and ρ_0 to a subalgebra (e.g., a subset of V), the relative entropy is nonincreasing.

Lindblad 1975

Modular Hamiltonian

Definition: Let σ be the vacuum state, restricted to some region V. Then the *modular Hamiltonian*, K, is defined up to a constant by

$$\sigma \equiv \frac{e^{-K}}{\text{tr}\, e^{-K}} \ .$$

A Central Result

Positivity of the relative entropy implies

$$S^C \leq K\ .$$

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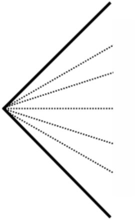
This is useful because K can be related to

- the area increase of a causal horizon Wall 2011
- the perturbative area difference of an "optimized" light-sheet RB, Casini, Fisher, Maldacena 2011

leading to proofs of the Generalized Second Law, and of the GCEB in the semiclassical limit.

GSL for Rindler Space



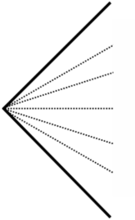
The modular Hamiltonian on the Rindler horizon ($x^+ = x + t > 0$) is given by

$$K = \frac{2\pi}{\hbar} \int d^2x^\perp \int_0^\infty dx^+ \, x^+ \, T_{++} \, ,$$

where $T_{++} = T_{ab}k^a k^b$ and k_a is the affine tangent vector to the horizon.

Bisognano, Wichmann 1975

GSL for Rindler Space



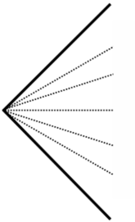
By integrating the Raychaudhuri equation

$$-\frac{d\theta}{dx^+} = \frac{1}{2}\theta^2 + \sigma_{ab}\sigma^{ab} + 8\pi G_N T_{ab}k^a k^b$$

twice, at leading order in G_N one finds that the Rindler horizon grows in area from $x^+ = 0$ to $x^+ = \infty$ by

$$\Delta A = 8\pi G_N \int d^2x^\perp \int_0^\infty dx^+ \; x^+ \; T_{++} \; .$$

GSL for Rindler Space



Hence one finds

$$S^C \leq K = \frac{2\pi}{\hbar}\frac{\Delta A}{8\pi G} = \frac{\Delta A}{4G\hbar}$$

and thus, the Generalized Second Law of Thermodynamics for the Rindler horizon.

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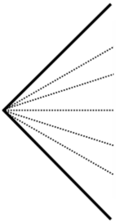

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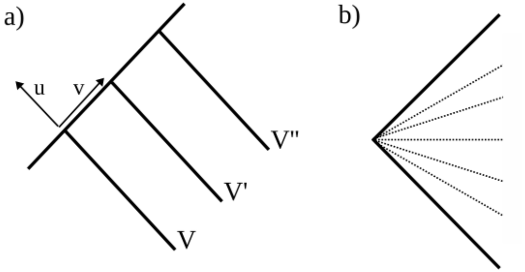
This and all subsequent results obtain at leading order in G_N (weak backreaction).

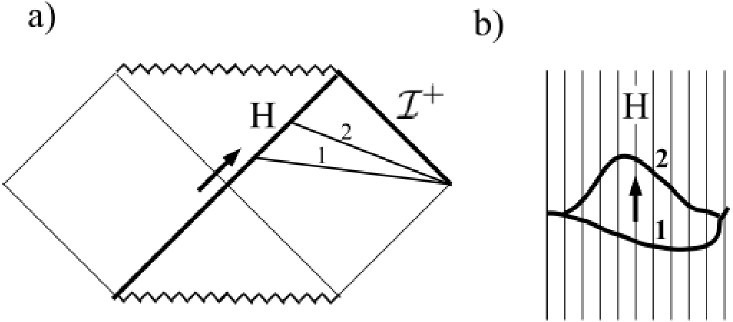
GSL for Rindler Space



- This result applies only to the process where all of the entropy in the wedge passes through the Rindler horizon.
- By exploiting monotonicity, it can be generalized to the GSL for a sequence of horizon slices (nested Rindler wedges).

Wall 2010

GSL for Causal Horizons



- By quantizing directly on the light-front, one can further generalize this to arbitrary horizon slices, of arbitrary causal horizons (black hole, de Sitter) Wall 2011

Finite regions

In finite volumes, the modular Hamiltonian K is generally nonlocal. However, again one finds that null hypersurfaces have special properties: K simplifies dramatically. On a finite light-sheet

$$K = \frac{2\pi}{\hbar} \int d^2x^\perp \int_0^1 dx^+ \; x^+(1 - x^+) \; T_{++} \; .$$

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But it is easy to find quantum states for which $T_{++} < 0$. In fact, explicit examples can be found for which $S^C > \Delta A/4G\hbar$, if the initial expansion vanishes ($\theta_1 = 0$).

Initial expansion and negative energy

- If the null energy condition holds, initially vanishing expansion is the “toughest” choice for testing the GCEB.
- However, if the NEC is violated, then $\theta_1 = 0$ does not guarantee that $\theta \geq 0$ holds everywhere in $0 \leq x^+ \leq 1$.
- A valid light-sheet may have to contract initially.

Proof of the GCEB

Combining the light-sheet condition with the generalized area difference

$$\Delta A = A_0\theta_0 + 8\pi G_N \int_0^1 dx^+ x^+ T_{++} \; ,$$

it is possible to show that

$$K \leq \Delta A / 4 G_N \hbar \; .$$

The Generalized Covariant Entropy Bound follows.

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- Observer-dependence of entropy is crucial.
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- Demanding nonexpansion on entire light-sheet is crucial.
  (As opposed to, e.g., demanding only initial nonexpansion plus some averaged version of the NEC.)
- No energy conditions needed. Existence of vacuum (which goes into the definition of the modular Hamiltonian) apparently captures all the necessary restrictions.

Questions

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- What is the analogue of the reference state ρ_0 for the non-perturbative case?

Take-home Slogan

Entropy bounds appear to require a detailed relation between entropy and (light-ray focusing) energy.

Without first understanding this connection from a more fundamental theory, it seemed difficult to prove any such bounds.

We have made some progress by working with the (operationally relevant) Casini entropy, which is observer-dependent.

Information tells spacetime how to curve; spacetime tells information how to disappear.