Entropy, entanglement and utility

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Basic message:

- There is a far-reaching analogy between problems in classical thermodynamics, entanglement in quantum theory and utility in decision theory.
- Roughly, a common element in these problems is that of representing an (incomplete) ordering structure quantitatively.
- While I would not claim that one of these problems can be "reduced" to any other, the analogy shows that theorems proved in one of these fields (decision theory) can contribute substantively to thermodynamics and quantum theory. other fields.

Classical Thermodynamics and its Second Law

Clausius (1850), Kelvin (1851), Planck (1897), Carathéodory (1909):

- concerned with macroscopic systems (e.g.: gas or mixture of vapor, liquid and solids) in thermal equilibrium states.
- Each such system has a state space Γ consisting of equilibrium states X. (E.g.: the (p, V) diagram for a fluid.)
- There exists a real-valued function S on Γ. $S(X)$ is called the *entropy* of the system in state X.
- Second Law of thermodynamics:

  > *The entropy of a system can never decrease as a result of an adiabatic process.*

Problems with classical formulations of the Second Law

- These treatments of the Second Law are not mathematically sophicated. They make abundant use of silent assumptions.
- Does the Second Law hold for all TD systems? (Or only for "simple" systems?
- Is there a unique entropy function?
- Exactly what state transformations for a given system can be realized by adiabatic processes?

Axiomatic approach of Lieb and Yngvason (1999)

Notation

$X \precsim Y$: $\exists$ an adiabatic process that transforms state X into Y

$X + Y$: the composition of two systems in states $X$and Y.

tX : a "scaled version" of state X

Axioms:

1. Reflexivity: $X \precsim X$.
2. Transitivity: If $X \precsim Y$ and $Y \precsim Z$ then $X \precsim Z$.
3. Composition: If $X \precsim Y$ and $X' \precsim Y'$then $X + X' \precsim Y + Y'$
4. Scaling: $X \precsim Y \iff tX \precsim tY$ for all $t \in \mathbb{R}^+$.
5. Decomposition& combination:$X \sim tX + (1-t)X$ for $0 \leq t \leq 1$
6. Stability: If $X + \epsilon Z \precsim Y + \epsilon Z'$ for $\epsilon \longrightarrow 0$, then $X \precsim Y$.
7. Comparability Hypothesis:
   For all X, Y in the same state space: $X \precsim Y$ or $Y \precsim X$.

Second Law à la L&Y:

Theorem

If Axioms 1–6 and the CH hold, then there is a function S such that:

$$X \precsim Y \text{ in the same state space } \iff S(X) \leq S(Y),$$

and which is additive and extensive, i.e.:

$$S(t_1 X_1 + \cdots + t_n X_n) = \sum_{i=1}^{n} t_i S(X_i)$$

Moreover this function is unique upto affine transformations.

Remarks

- No differentiability of S or assumptions on the topology of Γ needed. This is much more general than classical treatments.
- Main weakness is the Comparibility Hypothesis. This is only claimed to hold for "simple" systems.
- What would happen to the theorem if we drop CH?

Entanglement

Consider quantum system composed of two subsystems (far from eachother) in a pure state

- If the state of this system takes the form

$$|\Psi\rangle = |\psi\rangle|\phi\rangle \tag{1}$$

  one can also attribute a pure state to its components. The state is called separable.

- But if it takes the form

$$|\Psi\rangle = \sum_i c_i|\psi_i\rangle|\phi_i\rangle \tag{2}$$

  it is no longer possible to assign pure quantum states to the components. The state is called entangled.

Theorem (Schmidt's biorthogonal decomposition theorem:)

For every bipartite state $|\Psi\rangle$ there are orthonormal bases on $\mathcal{H}_1$ and $\mathcal{H}_2$ such that

$$|\Psi\rangle = \sqrt{p_1}|\alpha_1\rangle|\beta_1\rangle + \cdots \sqrt{p_n}|\alpha_n\rangle|\beta_n\rangle \qquad (3)$$

with $p_i \geq 0$ and $\sum p_i = 1$. The values $p_\Psi = (p_1, \ldots, p_n)$ are called Schmidt coefficients of state $|\Psi\rangle$.

For a separable (i.e. not-entangled) state, $p_\Psi = (1, 0, \ldots, 0)$
For a 'maximally' entangled state $p_\Psi = (1/n, \ldots, 1/n)$.

Measures of entanglement and LOCC operations

- There have been many proposals of entropy-like measures of entanglement, or to prove the uniqueness of such a measure. (e.g.: Vedral & Plenio (1997), Rudolph (2001), Donald e.a. (2002), Vedral & Kashefi (2002))
- There is no consensus on this issue; in particular for mixed states.
- However, consensus exists on the following claim: Whatever one means by the "amount of entanglement" of a state $|\Psi\rangle$, it is not possible to increase the amount of entanglement by LOCC operations.

Definition

LOCC operations: all state transformations that can be achieved by local operations (such as local unitary evolutions, local interactions with third systems, or local measurements), assisted by classical communication.

Theorem (Nielsen's Theorem)

State $|\Psi\rangle$ can be transformed (with certainty) into $|\Psi'\rangle$ by LOCC operations if and only if p_Ψ is majorized by $p_{\Psi'}$

Where

Definition (Majorization)

probability distribution $p = (p_1, \ldots, p_n)$ is majorized by $q = (q_1 \ldots, q_n)$ (Notation: $p \precsim q$) if p can be writtten as a convex mixture of q and permutations of q:

$$p_i = \sum_j \alpha_j (\Pi_j q)_i \text{ for permutation matrices } \Pi_j \text{ and } \alpha_j \geq 0, \sum \alpha_j = 1$$

- In words: p is more "uniform", "spread out", or "disordered" than q.

- A minimal requirement on a putative measure of entanglement, say $E(|\Psi\rangle)$ is thus that it should respect majorization in the sense that

$$p_\Psi \precsim p_{\Psi'} \implies E(|\Psi\rangle) \geq E(|\Psi'\rangle)$$

A partial analogy between LOCC and adiabatic operations

- Majorization is a preorder (i.e. Axioms 1 and 2 above hold)
- Interpret combinations as products:

$$X + Y \leftrightarrow p \otimes q := (p_1 q_1, p_1 q_2, \dots, p_n q_n)$$

  Then Axiom 3 holds.
- Interpret scaling as taking multiple copies:

$$tX \leftrightarrow p^{\otimes t} = p \otimes \cdots \otimes p, \quad t \in \mathbb{N}$$

  Then Axiom 5 holds in the integer form $p^{\otimes(n+k)} \sim p^{\otimes n} \otimes p^{\otimes k}$.
- Axiom 6 holds in the integer form

$$\text{If } p^{\otimes k} \otimes z \precsim q^{\otimes k} \otimes z' \text{ for } k \longrightarrow \infty \ \text{ then } p \precsim q.$$

- But Axiom 4 (Scaling) fails in this case. One may have

$$p \not\precsim q \quad \text{but} \quad p \otimes p \precsim q \otimes q$$

- Also, a lemma following from Axiom 1–6 is Cancellation:

$$\text{If } X + Z \precsim Y + Z \text{ then } X \precsim Y$$

  The corresponding statement for majorization fails.

- By Nielsen's theorem, this last failure means that there are quantum states $|\Psi\rangle$and $|\Psi'\rangle$ such that it is impossible to transform $|\Psi\rangle$into $|\Psi'\rangle$ by LOCC, but it is nevertheless possible to transform $|\Psi\rangle|\Omega\rangle$ into $|\Psi'\rangle|\Omega\rangle$ for a suitable state $|\Omega\rangle$! This is known as entanglement catalysis (Jonathan and Plenio, 1999).

Extending the analogy

- Entanglement catalysis suggests introducing a preorder that extends majorization, called trumping (Nielsen).

Definition (Trumping)

Distribution p is trumped by q ($p \precsim_T q$) iff $\exists$ a probability distribution $z = (z_1, \ldots z_m)$ such that $p \otimes z \precsim q \otimes z$.

Clearly: $p \precsim q \implies p \precsim_T q$.

- Idea:Replace majorization by trumping. Then all the Axioms 1–6 hold!

Where the analogy breaks down

- But there is no way the Comparability Hypothesis can hold in this case (except for qubits); if $n > 2$probability distributions are simply not completely ordered by trumping.
- However, we would like to get rid of this hypothesis anyway.

Recenly, Turgut (2007) and Klimesh (2007) obtained the following:

Theorem (Turgut-Klimesh)

Suppose $p$and q only have non-zero elements. Then $p \prec_T q$ holds if and only if for all $r \in \mathbb{R}$

$$M_r(p) < M_r(q) \quad \text{if } r > -1 \text{ and}$$
$$M_r(p) > M_r(q) \quad \text{if } r < -1$$

Here:

$$M_r(p) := (\sum_{i=1}^{n} p_i^{(r+1)})^{1/r} \quad r \in \mathbb{R} \tag{4}$$

Remark

- Main point: trumping holds iff there is inequality in a family of entropy-like expressions. Moreover, one has additivity and extensivity since

$$\log M_r(p_1^{\otimes t_1} \otimes \cdots p_n^{\otimes t_n}) = \sum_i t_i \log M_r(p_i) \quad \forall r \in \mathbb{R}$$

Exploiting the analogy: I. From QM to TD

- The Turgut-Klimesh theorem suggests the problem: if we drop the Comparibility Hypothesis, can one still derive the Lieb-Yngvason second law for a family of entropies?
- Yes! Surprisingly, the relevant theorem is proved in utility theory.

Utility

- Imagine a "rational" agent who has preferences amongst prospects (i.e., the possible consequences of his acts) $X \in \Gamma$. E.g.: $\Gamma = \{shivering, sweaty, comfortable\}$.
- These preferences are represented by some ordering relation on Γ: $X \precsim Y$.
- The agent can extend preferences to bets that yield prospect X_i with probability p_i, $i = (1, \ldots n)$.
- Von Neumann and Morgenstern prove (under some assumptions) the existence of a (unique upto affine transformations) utility function S such that

$$X \precsim Y \iff S(X) \leq S(Y)$$

- Their main assumption is that preferences are completely ordered.
- But that assumption often seems too strong, e.g. for group decisions.

Theorem (Shapley & Baucells (1998))

Let Γ *be a (compact) mixture space, i.e.:*
if $X, Y \in \Gamma$ *then also* $tX + (1-t)Y \in \Gamma$ *for* $0 < t < 1$.

- *Let* $\precsim$ *be a preorder on* Γ *that satisfies*

$$X \precsim Y \Longleftrightarrow tX + (1-t)Z \precsim tY + (1-t)Z \quad (\forall t \in [0,1]$$

$$\{t \in [0,1] : \ tX + (1-t)Z \precsim tY + (1-t)Z'\} \text{ is closed}$$

then $\exists$ *a convex closed family* $\mathcal{S}$ *of additive and extensive real-valued functions* S *on* Γ *such that:*

$$X \precsim Y \Longleftrightarrow S(X) \leq S(Y) \quad \forall S \in \mathcal{S}$$

Exploiting the analogy: II. From TD to QM

- Nielsen's Theorem applies only to pure quantum states.
- For mixed states ρ, we may still use the axiomatic approach and interpret $\rho \precsim \sigma$ as 'ELOCC-accesibility'. (ELOCC for LOCC assisted by entanglement catalysis).
- Then the result of Shapley & Baucells will also apply to mixed states.
- Of course, the question remains to identify this class of functions $\mathcal{S}$ in this case.

Conclusions

- There is a striking analogy between axiomatic thermodynamics and entanglement theory, which is revealed by focusing accessibility under a class of state transformations rather than on entropic measures.
- Whenever such an accessibility relation is not a complete ordering it is foolish to expect that a single entropy measure is able to represent it.
- L&Y write "The Comparability Hypothesis is crucial for the existence of entropy". No! It is only crucial for the uniqueness of entropy.
- Analogy is a two-way street. Results in TD can inspire new approaches in QM and vice versa.