OCTONIONS and FERMIONS

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I: Octonions
II: Dimensional Reduction
III: Leptons
IV: What Next?

DIVISION ALGEBRAS

Real Numbers:

$$\mathbb{R}$$

Complex Numbers:

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i$$

$$z = x + yi$$

Quaternions:

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j$$

$$q = (a + bi) + (c + di)j$$

Octonions:

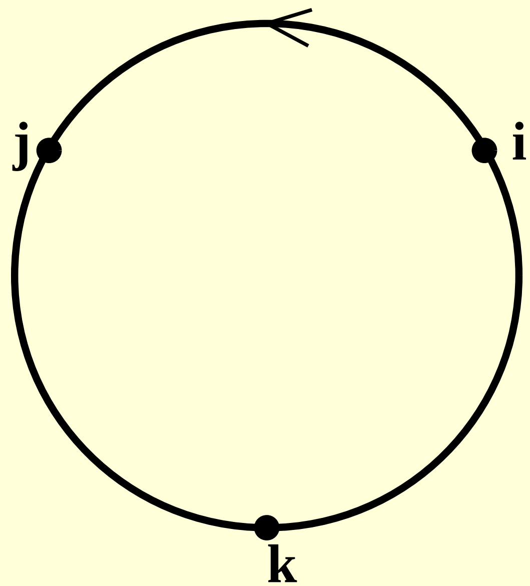
$$\mathbb{O} = \mathbb{H} + \mathbb{H}\ell$$

$$\boxed{i^2 = j^2 = \ell^2 = -1}$$

QUATERNIONS

$$k^2 = -1$$
$$ij = +k$$
$$ji = -k$$

not commutative

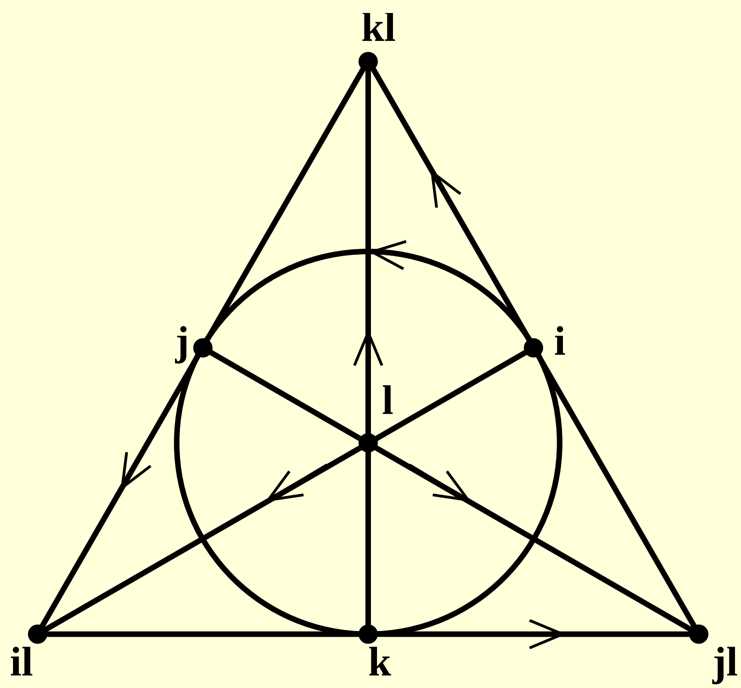


OCTONIONS

each line is quaternionic

$$(ij)\ell = +k\ell$$
$$i(j\ell) = -k\ell$$

not associative



VECTORS I

$$v = bi + cj + dk \longleftrightarrow \vec{\boldsymbol{v}} = b\,\hat{\boldsymbol{\imath}} + c\,\hat{\boldsymbol{\jmath}} + d\,\hat{\boldsymbol{k}}$$

$$vw \longleftrightarrow -\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$$

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Dot product exists in any dimension
but
Cross product exists only in 3 and 7 dimensions

VECTORS II

$$\boldsymbol{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad \longleftrightarrow \quad \boldsymbol{X} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

$$\boldsymbol{X}^\dagger = \overline{\boldsymbol{X}}^T = \boldsymbol{X}$$

$$-\det(\boldsymbol{X}) = -t^2 + x^2 + y^2 + z^2$$

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- {vectors in (3+1)-dimensional spacetime} $\longleftrightarrow$ {2×2 complex Hermitian matrices}
- determinant $\longleftrightarrow$ (Lorentzian) inner product
- $\boldsymbol{X} = tI + x\sigma_x + y\sigma_y + z\sigma_z$ (Pauli matrices)

WHICH DIMENSIONS?

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$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \longmapsto$

$$\boldsymbol{X} = \begin{pmatrix} p & \overline{a} \\ a & m \end{pmatrix} \qquad (p, m \in \mathbb{R}; a \in \mathbb{K})$$

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$$\dim \mathbb{K} + 2 = 3, 4, 6, 10 \text{ spacetime dimensions}$$

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supersymmetry

$$SO(2,1) \approx SL(2,\mathbb{R})$$
$$SO(3,1) \approx SL(2,\mathbb{C})$$
$$SO(5,1) \approx SL(2,\mathbb{H})$$
$$SO(9,1) \approx SL(2,\mathbb{O})$$

WEYL EQUATION

Penrose spinor:

$$\psi = \begin{pmatrix} c \\ \overline{b} \end{pmatrix}$$

Weyl equation: (3 of 4 string equations!)

$$-\widetilde{P}\psi = 0$$

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General solution: $(\xi \in \mathbb{O})$

$$\psi = \theta\xi$$

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P, ψ quaternionic

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Weyl equation: ($\mathbb{H}$ Penrose spinors)

$$\Psi = \begin{pmatrix} \theta \\ \eta \end{pmatrix} \longleftrightarrow \psi = \eta + \boldsymbol{\sigma}_k \theta$$

$$(p^\mu \, \widetilde{\boldsymbol{\sigma}}_\mu - m \widetilde{\boldsymbol{\sigma}}_k) \, \psi = 0$$

DIMENSIONAL REDUCTION

$$SO(3,1) \approx SL(2,\mathbb{C}) \subset SL(2,\mathbb{O}) \approx SO(9,1)$$

Projection: $(\mathbb{O} \to \mathbb{C})$

$$\pi(p) = \frac{1}{2}(p + \ell p \bar{\ell})$$

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Mass Term: $P = \pi(P) + m\,\sigma_k$ $\qquad \sigma_k = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$

$$P = \begin{pmatrix} p^t + p^z & p^x - \ell p^y - km \\ p^x + \ell p^y + km & p^t - p^z \end{pmatrix}$$

SPIN

Finite rotation:

$$R_z = \begin{pmatrix} e^{\ell\frac{\theta}{2}} & 0 \\ 0 & e^{-\ell\frac{\theta}{2}} \end{pmatrix}$$

Infinitesimal rotation:

$$L_z = \left.\frac{dR_z}{d\theta}\right|_{\theta=0} = \frac{1}{2}\begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}$$

Right self-adjoint operator:

$$\hat{L}_z\psi := (L_z\psi)\bar{\ell}$$

Right eigenvalue problem:

$$\hat{L}_z\psi = \psi\lambda$$

ANGULAR MOMENTUM REVISITED

$$L_x = \frac{1}{2}\begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix} \qquad L_y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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$$\psi = e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \Longrightarrow$$

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Simultaneous eigenvector!

(only 1 *real* eigenvalue)

LEPTONS

ψ $\qquad$ $P = \psi\psi^\dagger$

$$e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix} \qquad e_\uparrow e_\uparrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$e_\downarrow = \begin{pmatrix} -k \\ 1 \end{pmatrix} \qquad e_\downarrow e_\downarrow^\dagger = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

$$\nu_z = \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad \nu_z \nu_z^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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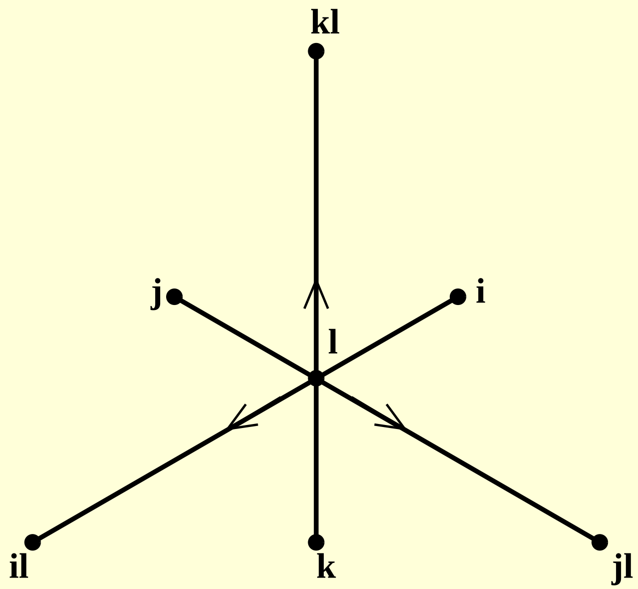

Answer: $3!$

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WHAT NEXT?

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What about $\emptyset_z$?

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Neutrinos have just one helicity!

What about $\varnothing_z$?

Want:

WHAT NEXT?

Have:

3 generations of leptons!

Neutrinos have just one helicity!

What about $\not{O}_z$?

Want:

- interactions
- quarks/color ($SU(3)$!)
- charge

EXCEPTIONAL GROUPS

$\boldsymbol{F_4}$: "$SU(3,\mathbb{O})$"

$$(\mathcal{M}\mathcal{X}\mathcal{M}^\dagger)\circ(\mathcal{M}\mathcal{Y}\mathcal{M}^\dagger) = \mathcal{M}(\mathcal{X}\circ\mathcal{Y})\mathcal{M}^\dagger$$

$\boldsymbol{E_6}$: "$SL(3,\mathbb{O})$"

$$\det\mathcal{X} = \frac{1}{3}\,\mathrm{tr}\,((\mathcal{X}*\mathcal{X})\circ\mathcal{X})$$

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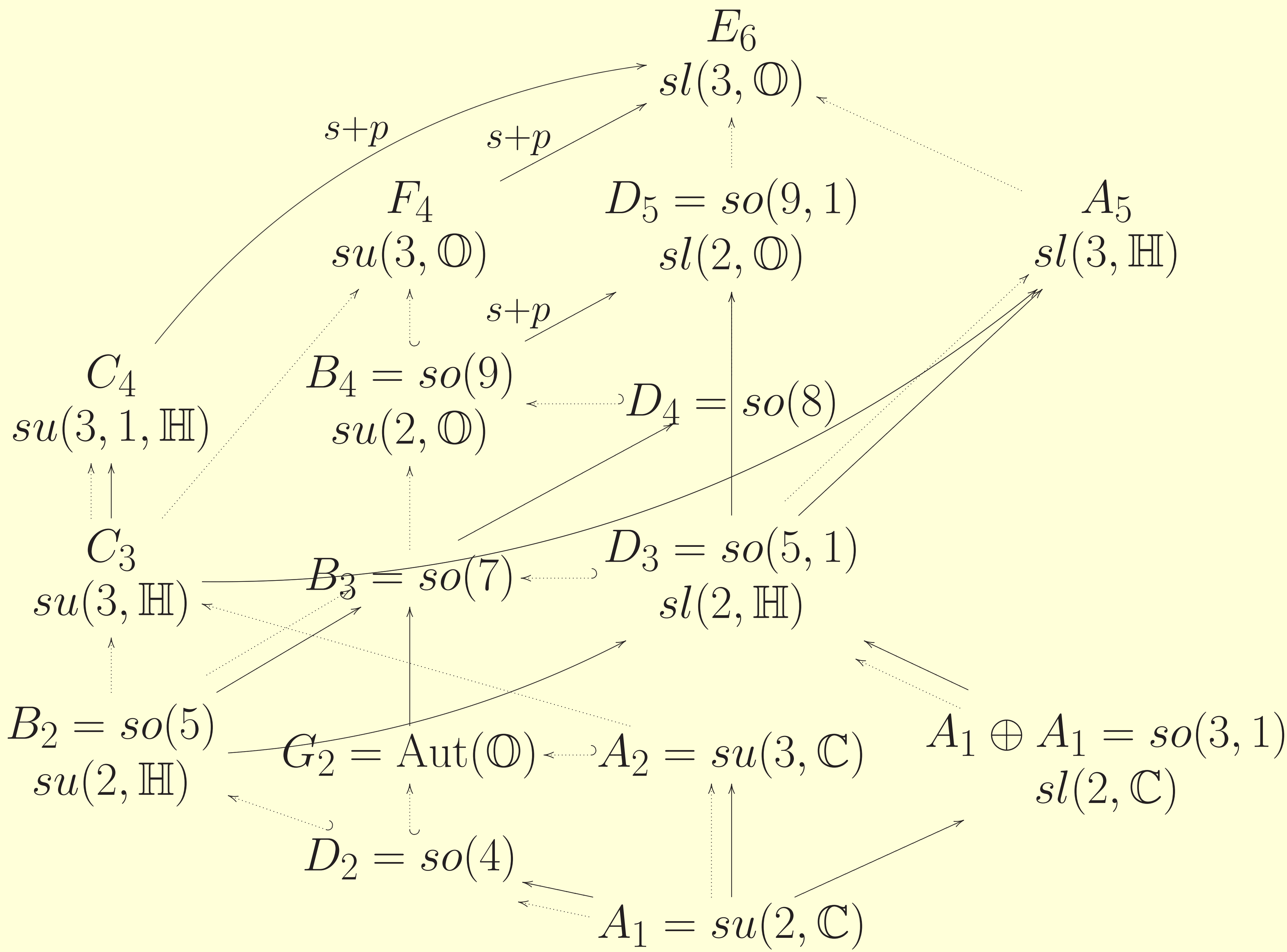
Exceptional quantum mechanics:

(Jordan, von Neumann, Wigner)

$$(\mathcal{X}\circ\mathcal{Y})\circ\mathcal{X}^2 = \mathcal{X}\circ\left(\mathcal{Y}\circ\mathcal{X}^2\right)$$

A MAP OF E_6



E_6
$sl(3, \mathbb{O})$
$s{+}p$
$s{+}p$
F_4
$su(3, \mathbb{O})$
$D_5 = so(9,1)$
$sl(2, \mathbb{O})$
A_5
$sl(3, \mathbb{H})$
$s{+}p$
C_4
$su(3,1,\mathbb{H})$
$B_4 = so(9)$
$su(2, \mathbb{O})$
$D_4 = so(8)$
C_3
$su(3, \mathbb{H})$
$B_3 = so(7)$
$D_3 = so(5,1)$
$sl(2, \mathbb{H})$
$B_2 = so(5)$
$su(2, \mathbb{H})$
$G_2 = \mathrm{Aut}(\mathbb{O})$
$A_2 = su(3, \mathbb{C})$
$A_1 \oplus A_1 = so(3,1)$
$sl(2, \mathbb{C})$
$D_2 = so(4)$
$A_1 = su(2, \mathbb{C})$

CAYLEY SPINORS

$$\mathcal{X} = \begin{pmatrix} \boldsymbol{X} & \theta \\ \theta^\dagger & n \end{pmatrix}$$

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{M} & 0 \\ 0 & 1 \end{pmatrix}$$

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Cayley/Moufang plane:

$$\begin{aligned} \mathbb{O}\boldsymbol{P}^2 &= \{\mathcal{X}^2 = \mathcal{X}, \operatorname{tr}\mathcal{X} = 1\} \\ &= \{\mathcal{X} * \mathcal{X} = 0, \operatorname{tr}\mathcal{X} = 1\} \\ &= \{\mathcal{X} = \psi\psi^\dagger, \psi^\dagger\psi = 1\} \end{aligned}$$

$(\psi \in \mathbb{H}^3)$

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Furthermore: $\mathcal{X}^\dagger = \mathcal{X} \Longrightarrow \mathcal{X} = \sum_{n=1}^{3} \psi_n \psi_n^\dagger$

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leptons, mesons, baryons?

Life is complex.

Life is complex.

It has real and imaginary parts.

Life is octonionic...

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THE END