Before giving my annual description of the Puzzle Corner ground rules, let me pass along Tom Terwilliger's enthusiastic recommendation of Rosenhouse and Taalman's *Taking Sudoku Seriously*.

Now for the rules. In each issue I present three regular problems, the first of which is normally related to bridge, chess, or some other game, and one "speed" problem. Readers are invited to submit solutions to the regular problems, and two columns (i.e., four months) later, one submitted solution is printed for each regular problem; I also list other readers who responded. For example, the current issue contains solutions to the regular problems posed in May/June.

The solutions to the problems in this issue will appear in the January/February column, which I will need to submit in mid-October. Please try to send your solutions early to ensure that they arrive before my deadline. Late solutions, as well as comments on published solutions, are acknowledged in subsequent issues in the "Other Responders" section. Major corrections or additions to published solutions are sometimes printed in the "Better Late Than Never" section, as are solutions to previously unsolved problems.

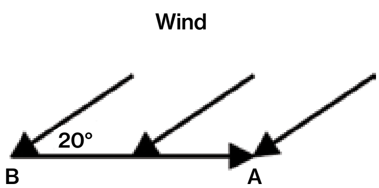
For speed problems the procedure is quite different. Often whimsical, these problems should not be taken too seriously. If the proposer submits a solution with the problem, that solution appears at the end of the same column in which the problem is published. For example, the solution to this issue's speed problem is given on the facing page. Only rarely are comments on speed problems published.

There is also an annual problem, published in the first issue of each year, and sometimes I go back into history to republish problems that have remained unsolved.

Problems

S/O 1. We begin with another wink problem from Rocco Giovanniello, who this time wants you to start with a 4 × 6 board, with the square (4,3) empty and the others containing a wink. As usual, you are to find a sequence of horizontal and vertical jumps so that only one wink remains.

S/O 2. In his "earlier years," Alan Faller used to sail around Monhegan Island in Maine each summer and wondered about the correct direction to tack.



In particular, as shown above, Alan wishes to travel from B to A against a head wind at 20°. At what direction should he head initially, assuming that he will change direction once and that the speed of the boat through the water is $v = V \sin(2\pi ß/360)$, where V is a constant (depending on the wind speed, shape and size of the sails, etc.), and $ß$, the angle between the boat's direction of progress and the wind's direction, can be chosen from 0 to ±90°?

S/O 3. Tim Malony wants you to show that all solutions of the complex equation

$$e^z = \frac{z-1}{z+1}$$

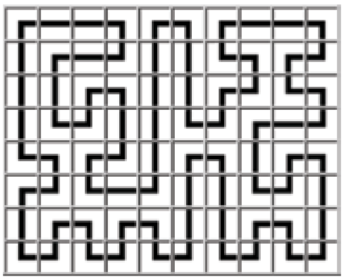
lie on the imaginary axis.

Speed Department

Avi Ornstein asks for an "interesting property" satisfied by 5, 6, 25, 76, 376, 625, 9,376, and 90,625 and no other number below 100,000.

Solutions

M/J 1. I received several beautifully drawn solutions to Frank Rubin's Coraline (for *cor*ners and *line*s) puzzle. Here's Ken Haruta's solution to what Frank calls "a corner puzzle for the Puzzle Corner."



M/J 2. I received a number of very fine solutions to Fred Tydeman's sock problem. No one found a closed-form solution; indeed, it is not clear that one exists.

Jerrold Grossman derived recurrences and had Maple do the computations. He also had Maple simulate the process, and the analytic answers agree with the simulation data. The recurrences and the simulations are on the Puzzle Corner website.

His answers for 2, 3, 4, and 5 socks are 5/3, 7/3 (although 35/15 seems like a better way to look at it—the denominators are all odd factorial numbers, i.e., products of the first $n - 1$ odd positive integers), 311/105 (about 2.96), and 3,377/945 (about 3.57). For 10 socks and 20 socks, Grossman obtained, respectively, 4,248,732,053/654,729,075 (about 6.49) and 226,261,084,752,832,183,400,743/18,813,587,457,228,104,165,625 (about 12.03).

Grossman contacted Milton Eisner (apparently the originator of the problem) and Geoffrey Pritchard (who wrote a paper on the problem with Wenbo Li), and the latter confirmed that asymptotically the distribution of the maximum number of socks on the bed is normal, with mean $n/2$ and variance $n/4$.

Richard Hess also attacked the problem computationally. His values agree with Grossman's above, and then he gives some "non-exact" answers. In particular, for 100 pairs of socks, that value is approximately 53.915.

The following solution from Donald Aucamp includes an example and a diagram to illustrate the technique.

Define state variables i and j based on a given realization, where at a given point i is the number of socks drawn from the hamper and j is the number of socks still on the bed. Define U_{ij} and D_{ij} as the transition probabilities of going up or down by one sock on the next draw if currently in state (i,j). Then:

$D_{ij} = j/(2n - i)$
$U_{ij} = 1 - D_{ij}$

D_{ij} is based on the fact that there are $2n - i$ socks still in the basket and j socks on the bed, so the number of socks on the bed will go from j to $j - 1$ if the next sock chosen matches one of them. Now let p_{ij} be the probability of a realization reaching state (i,j). Then:

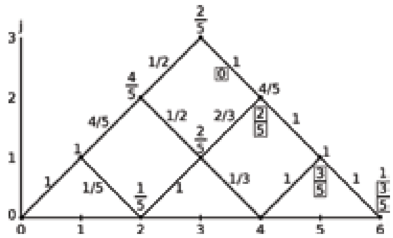
$$p_{ij} = p_{i\text{-}1,j\text{-}1}\, U_{i\text{-}1,j\text{-}1} + p_{i\text{-}1,j+1}\, D_{i\text{-}1,j+1}$$

Since $p_{00} = 1$ and all the Us and Ds are known functions of i and j, then all the ps can be solved sequentially by incrementing i. This procedure can be manipulated to find $E(x)$, as follows: Let $P(x)$ be the probability of x (the maximum number of socks seen on the bed), and let $F(x)$ be the cumulative, where $F(x) = F(x - 1) + P(x)$. Note that $P(0) = F(0) = 0$. If $D_{i,x+1}$ is reset to $D_{i,x+1} = 0$ for all i, then the above system will not allow re-entry back to j = x once a realization crosses above it. Thus, F(4) in the perturbed set-up is the probability of reaching the terminal node $(2n,0)$. That is, $F(4) = p_{2n,0}$. Since all the Fs can be found by this method, then all the Ps can likewise be found, and $E(x)$, the expected value, follows from: $E(x) = \Sigma \times P(x)$.

The final answers for various n are:

n	1	2	3	4	5	10	20
$E(x)$	1	5/3	7/3	311/105	3.5735	6.4892796	12.02647211

Suppose $n = 3$ pairs of socks, so there are six socks in the hamper.



The figure shows the 10 feasible nodes with the p_{ij}s above them. The other nodes have $p_{ij} = 0$. The transition probabilities D_{ij} and U_{ij} are given on the lines connecting these nodes. For example, if $i = 3$ and $j = 1$:

$D_{31} = j/(2n - i) = 1/(2\text{x}3 - 3) = 1/3$
$U_{31} = 1 - D_{31} = 2/3$
$p_{31} = p_{i\text{-}1,j+1} \times D_{i\text{-}1,j+1} + p_{i\text{-}1,j\text{-}1} \times U_{i\text{-}1,j\text{-}1} = (4/5) \times (1/2) + (1/5) \times 1 = 3/5$

Note that $p_{60} = 1$, so the probability of eventually reaching the end node is unity. Now suppose $x = 2$ and $F(2)$ is to be found. Set all $D_{i,x+1} = 0$ to eliminate moves back from above $j = x = 2$. In this case set $D_{33} = 0$. This altered transition probability is shown enclosed in the box on the leg connecting the nodes at (3,3) and (4,2). The p_{ij}s that are thereby changed are $p_{42} = 2/5$, $p_{51} = 3/5$, and $p_{60} = 3/5$. These new values are shown in the boxes. Thus, $F(2) = p_{60} = 3/5$. Similar calculations yield $F(1) = 1/15$ and $F(3) = 1$. So $P(1) = F(1) - F(0) = 1/15$, $P(2) = F(2) - F(1) = 8/15$, and $P(3) = F(3) - F(2) = 2/5$. Accordingly, $E(x) = \Sigma \times P(x) = 1 \times (1/15) + 2 \times (8/15) + 3 \times (2/5) = 7/3$.

M/J 3. Everyone agrees that the solution to Gerald Giesecke's solid cross-section problem is $\gcd(s,t)$, the greatest common divisor of s and t. The following two proofs, from Richard Schooler and Dan Katz, use (related) results from number theory and abstract algebra, respectively.

Schooler writes, "We number the surfaces from 0 to s, arbitrarily choosing one as zero and numbering the rest sequentially around the cross-section.

"The twist t maps surface p to $p + t$ mod s. So surfaces $(p + nt)$ mod s are identified for all integers n. The set of nt mod s is an ideal whose representative m, or smallest number such that all elements are multiples, is $\gcd(t,s)$. For any smaller number $0 < q < m$, the surfaces p and $p + q$ are disjoint by construction. So m is the number of disjoint surfaces.

"That $m = \gcd(t,s)$ follows from Bézout's identity. The set nt mod s is the set $(at + bs)$ mod s for all integers a and b. And all $at + bs$ are multiples of $\gcd(t,s)$. $\text{Gcd}(t,s)$ is also the complete intersection of factors of t and s, so any smaller candidate is missing one of those factors, and cannot be in the linear combinations of t and s.

"An interesting special case is prime s, in which case the gcd reduces to s for $t = 0$ mod s, otherwise 1."

Katz writes, "Number the surfaces 0 to $s - 1$. When the ends are connected, each surface k 'merges' with the surface $k + t$. So we need to identify each element k in the set $\{0, \ldots, s - 1\}$ with the element $k + t$, and see how many distinct elements remain.

"Algebraically, this is equivalent to counting the elements of the quotient group G/H, where G is the set of integers mod s, and H is the subgroup of G generated by the element t. It is well known in abstract algebra that H is the set of multiples of $\gcd(s,t)$, and the quotient group will be the set of integers mod $\gcd(s,t)$. This group has $\gcd(s,t)$ elements, which is the resulting number of surfaces."

Other Responders

Responses have also been received from J. Chandler, B. Currier, A. Goel, S. Gordon, J. Hardis, R. Hess, G. Perry, K. Rosato, E. Sheldon, W. Sun, T. Terwilliger, E. Underriner, and K. Zeger.

Proposer's Solution to Speed Problem

If you square the number, the original appears at the end. For example, $376^2 = 141{,}376$.

Send problems, solutions, and comments to Allan Gottlieb, New York University, 715 Broadway, Room 712, New York, NY 10003, or to gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.