# Puzzle corner

Send problems, solutions, and comments to Allan Gottlieb at New York University, 60 Fifth Ave., Room 316, New York, NY, 10011, or gottlieb@nyu.edu. For other solutions and back issues, visit the Puzzle Corner website at cs.nyu.edu/~gottlieb/tr.

I have two milestones to report: I just turned 75, and three days later I broke my arm for the first time.

Unfortunately, I am very much right-handed, and (of course) that is the arm I broke.

Please forgive any shortcuts I must use in this column.

## Problems

**N/D 1.** Jorgen Harmse sent us the following, which is based on an actual deal.

| | | |
|---|---|---|
| | ♠ 52<br>♥ A843<br>♦ A9654<br>♣ 86 | |
| ♠ ??<br>♥ ??<br>♦ ??<br>♣ Jt532 | | ♠ ?3<br>♥ ??<br>♦ ??<br>♣ 7 |
| | ♠ AKQ<br>♥ KQt6<br>♦ K<br>♣ AKQ94 | |

The contract should have been 7 hearts (unless you were warned about the clubs), but you're in 6 no-trump. West leads the jack of clubs, and you immediately win two club tricks. What are your chances of making the contract?

**N/D 2.** David Mayhew has been given the task of evaluating the sequence $1^3 + 2^3 + 3^3 + \ldots + n^3$ for large $n$ and without computing hardware. A straightforward term-by-term calculation seems virtually impossible to accomplish in a lifetime. Prove that the formula $c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4$ will evaluate the sequence for all $n$ and find the coefficients that make it happen.

**N/D 3.** Richard J. Morgen offers a math problem that stems from a quiz show on TV in which three contestants get to spin a wheel. Each contestant can choose to spin one or two times. The wheel has 20 equal-size segments, marked 5, 10, 15, 20, on up to 100. Each contestant can decide, in turn, whether to stop after one spin or spin again. Anyone who goes over 100 is out. A tie (which could be two-way or three-way) results in a one-spin spinoff.

What is the best strategy? Specifically, what number should you stop on if you spin first? What if you spin second? (Obviously, if you are second, you take a second spin if your first spin does not match or exceed that achieved by the first spinner. But you need more strategy than that.) The third spinner only has to decide whether to stop if he or she ties someone.

## Speed department

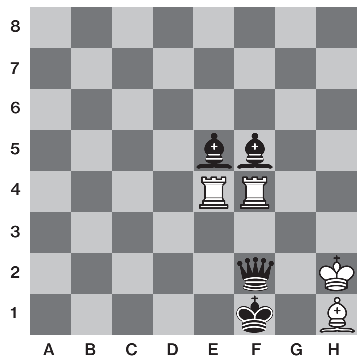
Ermanno Signorelli wonders what the following sets of numbers have in common: {13, 84, 85}, {15, 112, 113}, {28, 195, 197}, and {35, 612, 613}.

## Solutions

**J/A1.** Duffy O'Craven offers what I might call our first "tit for tat" chess problem. A few years ago, O'Craven submitted our only "helplessmate" problem. This time you are to find a legal chess position where the player-to-move is in check and this player's only legal move is to deliver checkmate. The mating move must be neither a capture nor a discovered check.

The following solution is from Michael Branicky.

Black has just moved to Qf2+. The only legal move is Bg2#.




My approach requires at least six pieces:

**1.** Both kings (B1 and W1)

**2.** A Black piece initiating check (B2)

**3.** A White piece that can be placed between B2 and W1 to both block the check and force the mate (W2)

**4.** A White piece that pins B2 so that it cannot capture W2 after that block (W3)

**5.** A Black piece that pins W3 so that it can't capture W2 to end the initial check (B3)

**6.** Other pieces as needed to force the mate on B1 and to force the move of W2 as the only legal move for White: that is, to cut out all moves of both kings.

**J/A2.** David Dewan's cryptarithmetic problem, which John Ebert notes had previously appeared as 2017 M/J2, requires you to substitute a digit 0 to 9 for each letter in the equation

$$HA^{PPY} = NEW + YEAR$$

Distinct letters get distinct digits, and if a letter appears multiple times, the same digit is substituted each time.

As promised, I gave preference to solutions that were not computer searches of all the possibilities. Greg Muldowney offered the following.

The integer values of the set {A,E,H,N,P,R,W,Y} satisfy:

$$(10H + A)^{110P+Y} = 1{,}000Y + 100E + 10A + R + 100N + 10E + W$$

The right side is in the thousands, hence $P$ must be 0—an exponent in the hundreds is too large unless $H$ is 0 and $A$ is 1, in which case the left side is 1. Further, $Y = 1$ as an exponent is too small, hence $Y \geq 2$. Grouping terms containing $Y$, $H$, and $A$ on the left gives:

$$(HA)^{Y} - (1{,}000Y + 10A) = 110E + 100N + (R + W)$$

The extremes of the right side are 317 and 1,803, corresponding to $\{E,N,R,W\} = \{9,8,7,6\}$ and $\{1,2,3,4\}$ respectively. These extremes set bounds on $HA$ for a chosen value of $Y$:

$$1{,}000Y + 317 + 10A \leq (HA)^{Y} \leq 1{,}000Y + 1{,}803 + 10A$$

Initially the $10A$ is neglected. For $Y = 2$, $2{,}317 \leq (HA)^2 < 3{,}803$, hence $HA$ is 49 to 61—omitting 50, 52, 55, and 60, which repeat a digit. For $Y = 3$, $3{,}317 \leq (HA)^3 \leq 4{,}803$, hence $HA$ is 15 or 16. Updating the bounds with $10A$ moves the minimum at $Y = 2$ to 51; all else is unchanged. For $Y \geq 4$, $HA < 10$, which would imply $H = 0$, duplicating $P$. Thus there are only 10 $(Y, HA)$ cases to evaluate, eight for $Y = 2$ and two for $Y = 3$. Each is pursued by seeking to match $[(HA)^{Y} - (1{,}000Y + 10A)]$ with $(110E + 100N + R + W)$ using four of the six remaining digits. The lower-bound cases (2, 51) and (3, 15) are infeasible because 0, 1, and either 2 or 3 are assigned, forcing $N$ to be at least 4. Of the other eight cases, only one solves the problem:

$$16^{003} = 829 + 3{,}267$$

both sides being 4,096. A trivial variant is obtained by swapping $R$ and $W$, i.e., the 7 and 9.

**J/A3.** Burgess Rhodes notes that all lines intersecting at a fixed point in $R^3$ fill $R^3$. No two of these lines are parallel. Also, all lines parallel to a fixed line in $R^3$ fill $R^3$. No two of these lines intersect. The question is whether $R^3$ can be filled with lines no two of which are parallel *and* no two of which intersect.

Dale Worley submitted the following solution:

It's geometrically obvious that $R^3$ can be partitioned into a set of nested one-sheet hyperboloids of revolution, plus the $z$ axis. There are probably a lot of different ways to construct such sets of hyperboloids. And it's well known that one-sheet hyperboloids can be "ruled," or partitioned into a set of lines (in two ways!).

In this case, the important thing is that all the lines on a particular hyperboloid of revolution have the same slope relative to the $xy$ plane, and so we need to arrange that the lines on each hyperboloid have a different slope from the lines on every other hyperboloid. Using the Wikipedia page on hyperboloid as a reference, we set the equation of the (symmetric) hyperboloid as

$$x^2/a^2 + y^2/a^2 - z^2/c^2 = 1$$

and parameterize the lines on it with the vector equation

$$(a \cos \alpha, b \sin \alpha, 0) + t\,(-a \sin \alpha, b \cos \alpha, c)$$

where $\alpha$ parameterizes where on the hyperboloid the line is and $t$ parameterizes the points along the line.

To see the slope of a line relative to the $xy$ plane, we choose a line of constant $y$, which forces alpha = $\pi/2$. The line is parameterized

$$(0, b, 0) + t\,(-a, 0, c)$$

which shows that the slope of the line in the $xz$ plane is $-a/c$.

So we need to assemble nested hyperboloids that each have distinct values of $a/c$. This can be done by choosing the family with $c = 1$, which means that every point (without $x = y = 0$) is on the hyperboloid in the family with

$$a = \sqrt{(x^2 + y^2)/(1 + z^2)}$$

Every point is on exactly one of these hyperboloids, or the $z$ axis, so they partition $R^3$. All the lines on a hyperboloid have a different slope relative to the $xy$ plane than the lines on any other hyperboloid, so two lines on different hyperboloids are not parallel. All of the lines on one hyperboloid are skew to each other.

# Better late than never

**M/A1.** David Patrick and Michael Branicky independently discovered the same flaw in our solution. Patrick writes: "I believe that the solution to M/A1 in the recent July/August issue of MIT News has a subtle but significant error. In particular, in the given explanation of why the White king cannot have moved last, the claim 'There is no previous Black move that could have put the White king into check' is incorrect. A Black bishop could have been on b8 and moved to c7, discovering a check from the Black queen, and then the White king could have captured that bishop on c7." Patrick's full response is on the Puzzle Corner website.

# Other responders

Y. Aliyev, Anonymous, S. Berkenblit, G. Chan, J. Chandler, J. Ebert, A. Jakobčič, T. Johnson, J. Kingsnorth, P. Kramer, J. Larsen, T. Maloney, T. Mita, B. Rhodes, and Y. Zussman

# Solution to speed problem

Remove one digit from each number and you get Pythagorean triplets. {3, 4, 5}, {5, 12, 13}, {8, 15, 17} and {5, 12, 13}.