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The Problem Section

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The Problem Section

Michael Ecker, Ph.D., Problem Section Editor

Bryan Wilson, Solutions Editor

Albert Natian, Contributing Editor

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Our Purpose and Interest. The problem section seeks lively and interesting mathematics problems and their solutions. Although we favor proposals with relevance to community college faculty or students, we encourage everybody to participate with a variety of challenges and levels. The typical problem might entail one hour to one month to solve. Technology may supplement a problem solution but should never supplant it. A good problem will stretch your knowledge to go beyond your textbook, routine, and comfort zone. A very good problem may have some generalization, special case, or nice extension into another area. It might afford a novel solution beyond mere formulaic chug-and-plug. Quicker Problems and our NEW Students' Quicker Challenges are shorter, easier problems that do not quite make the cut because they can be handled more quickly. Their solutions appear in the same issue just before the solutions to earlier problems. If you have one to share, or believe you have a better solution to one printed, please send it. If we agree, we may publish it in a future issue.

Send new problem proposals to Dr. Michael Ecker, Problem Section Editor, at DrMWEcker@aol.com. Send all solutions to Prof. Bryan Wilson, Solutions Editor, at bryan.wilson@slcc.edu, and cc: Dr. Ecker. Prepare each document as a separate Word file saved in the older .doc format or newer .docx. Use Math Type or Equation Editor to lay out mathematics. For all submissions, please include your name, affiliation, city/ state, and full problem statement with each item.

Quicker Problems (Q), Students' Quicker Challenges (SQ)

(Solutions to *all* Q and SQ Problems appear immediately following the New Problems.)

SQS-0 Proposed by Michael W. Ecker, Problem Section Editor. A price rises by \$1.23, which turns out to be exactly 5% of the original price. What was the original price?

SQS-A Proposed by Michael W. Ecker. Imagine a straight line of idealized shops represented by points. There are two that you wish to visit, at distinct points A and B on said line. Imagine that you can park your idealized bike right on the line itself, so there is no walking from parking lot to the line or a shop on the line. You will walk directly from one of the two shops to the other, and then back to your bike. You wish to minimize your total amount of walking. Where should you park your bike to accomplish this? (Hint: Just draw a picture. Also see S-1, this issue.)

SQS-B Proposed by Allen Fuller, Contributing Editor. Describe the graph of the function $y = \frac{1}{2} \cos^2(x) + \sqrt{3} \sin(x) \cos(x) - \frac{1}{2} \sin^2(x)$. (Note: The intent is to manipulate, not to use technology.)

SQS-C Proposed by Michael W. Ecker. Solve for all positive real numbers x satisfying $x^{1/x} = \sqrt{x}$.

SQS-D Proposed by Michael W. Ecker. Can both a sphere and a cylinder of the same radius have equal volumes and equal surface areas?

QS-0 Proposed by Michael W. Ecker. For integers a, b, c, k with b odd and a nonzero, under what conditions can the equation $ax^2 + bx + c = k$ have a double root?

QS-1 Proposed by the late Peter Lindstrom. Prove that $\sum_{i=1}^n i \cdot i! = (n+1)! - 1$.

QS-2 Revived by Michael W. Ecker. Let F denote a field with at least two distinct elements. Prove that its zero 0 and its unity 1 are necessarily distinct.

QS-3 Proposed by Michael W. Ecker. For the Fibonacci numbers F_n , we know that $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \phi = \frac{-1 + \sqrt{5}}{2} \approx 1.618$. For m a fixed positive integer, what is the value of $\lim_{n \rightarrow \infty} \frac{F_{n+m}}{F_n}$?

QS-4 Proposed by Michael W. Ecker. Solve $f(x) + g(y) = \sin(x) + \cos(y)$ for all solution pairs f, g .

QS-5 Proposed by Michael W. Ecker. Give a single equation whose graph precisely consists of: (a) the two coordinate axes; (b) all points in the closed first quadrant.

QS-6 Proposed by Stephen L. Plett (Emeritus), Fullerton College, Fullerton, CA. Under what conditions will 3×3 matrix M with $m_{1,1} = 1$ have an LU decomposition of the form

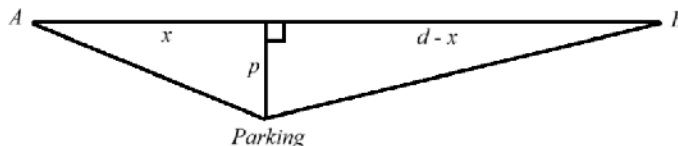
$$\begin{bmatrix} 1 & 0 & 0 \\ \# & 1 & 0 \\ \# & \# & 1 \end{bmatrix} \begin{bmatrix} 1 & \# & \# \\ 0 & \# & \# \\ 0 & 0 & \# \end{bmatrix} = M, \text{ with 0's, 1's fixed?}$$

New Problems

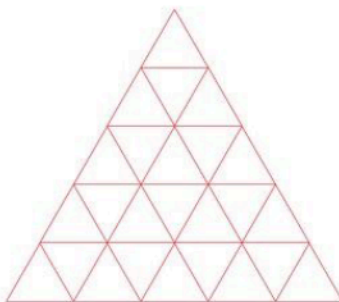
Solutions to problems in this issue are due by 30 January 2027.

S-1 Proposed by Michael W. Ecker. This is a simple extension of SQS-A, this issue, but we now introduce the more realistic element of a parking spot distance p from the line, associated with the distance x from the point A (between A and B , specifically, two points still d apart). Where should

you park to minimize the total distance walking from parking spot to A plus from A to B plus from B back to parking spot?



S-2 **Revised Classic by Michael W. Ecker.** The figure shows what we might call *nested equilateral triangles*, the smallest of which have side 1 inch, and the largest of which has side n inches (but with $n = 5$ shown). We are interested only in those equilateral triangles whose vertices coincide with the vertices and intersection points within the figure. With the full figure having largest side n inches, find a simplified formula, as a function of n , for the total number of equilateral triangles within, each having integral-length sides of 1 to n inches.



$n = 5$ case of general question

S-3 **Proposed by Michael W. Ecker.** Given fixed point $P = (x_0, y_0)$ satisfying $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, find the closest point $Q = (x, y)$ satisfying $\frac{x^2}{a^2} + \frac{y^2}{b^2} = c^2$, with $c \neq 0, 1$. First consider $b = a$, then $a > b > 0$. (If you cannot give a complete description, tell whatever you can in the general case.)

S-4 **Proposed by Michael W. Ecker.** Let's say that a natural number n is *almost composite* iff n can be expressed as the sum of two composite numbers (not prime, not 1). Find all almost-composite numbers.

S-5 **Proposed by Albert Natian (with Michael W. Ecker).** Prove that
$$\int_0^{\infty} e^{1+x-e^x} dx = \cos^2(6\pi/5) + \sin^2(-\pi/5).$$

S-6 **Proposed by Michael W. Ecker.** Suppose that a particular player gets at least the average number of points available in each round of a multi-round game of n players, where the object is to get the least total score at the end of all the rounds. The game continues until at least one player

reaches a target score and just one player has the lowest score. (If there is a tie, they all play another round.)

It should be obvious that the particular player can't be the winner. Prove it! (Note: The number of points per round is usually constant but may vary in certain games. Do consider this.)

S-7 Proposed by Michael W. Ecker. Define a hand of 13 playing cards out of the usual 52 as *almost unbeatable* if it has all the highest cards in each suit that the hand actually has. (The hand could be void in at least one suit.) Define it as *absolutely unbeatable* if it has that property *and also* has at least one of each suit. What is the probability that an almost unbeatable hand is absolutely unbeatable?

SPECIAL PROBLEM 10 Proposed by Michael W. Ecker. a) *A polynomial with integral coefficients evaluated at an integer will always yield an integer. Is the converse true? That is, if a polynomial always assigns to each integer an integer, must that polynomial have only integral coefficients?* b) *Similarly, what if the polynomial evaluated at an integer always yields a perfect square? (Or a perfect cube? Higher power?) Must such a polynomial have integral coefficients?*

Send all solutions directly to Dr. Ecker at DrMWEcker@aol.com. There is no deadline.

BONUS: Is $e^\pi - \pi$ surprising? (Answer appears following solutions to SQ and Q problems.)

Solutions to Quicker Problems (Q), Students' Quicker Challenges (SQ)

(Solutions here are by the Proposer and/or the Problem Section Editor, unless noted otherwise.)

SQS-0 Since \$1.23 is $\frac{1}{20}$ (5%) of original price p , then $p = 20 \cdot \$1.23 = \24.60 .

SQS-A Let d = distance of A to B . Park your bike anywhere you wish between points A and B , inclusive. Then the total amount of walking you will do is $2d$, as long as you park somewhere on the line segment AB . (If you foolishly parking outside this line segment, you will wind up walking more than $2d$.) To see this, imagine you park x units away from A (but toward B). You have to walk x units to A , then d units to B , and then $d - x$ units back to your bike, for a total of $x + d + (d - x) = 2d$.

SQS-B $\frac{1}{2} \cos^2(x) + \sqrt{3} \sin(x) \cos(x) - \frac{1}{2} \sin^2(x) = \frac{\sqrt{3}}{2} \sin(2x) + \frac{1}{2} \cos(2x)$, which may be rewritten as $\cos\left(\frac{\pi}{6}\right) \sin(2x) + \sin\left(\frac{\pi}{6}\right) \cos(2x) = \sin\left(2x + \frac{\pi}{6}\right)$: sine graph, amplitude 1, period 2π , phase shift $-\pi/12$. Problems Editor's Comments: This is the kind of form change for $f(x) = a \cos(x) + b \sin(x)$ that shows its range and extreme values satisfy the double-inequality $-\sqrt{a^2 + b^2} \leq a \cos(x) + b \sin(x) \leq \sqrt{a^2 + b^2}$. Represent a winding number or angle θ as though an

acute angle in a right triangle with legs a and b , resulting in hypotenuse $\sqrt{a^2 + b^2}$. Then $\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}}$, $\sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}$, and we now rewrite $a \cos(x) + b \sin(x) = \sqrt{a^2 + b^2} \cos(\theta) \cos(x) + \sqrt{a^2 + b^2} \sin(\theta) \sin(x) = \sqrt{a^2 + b^2} \cos(x - \theta)$, highlighting the amplitude being $\sqrt{a^2 + b^2}$. If one were using Calculus instead for the extreme values (setting derivative equal to zero, solving for x , and testing), one would be led to the conclusion that such a point of attainment x satisfies $\tan(x) = \frac{b}{a}$. Note how this lines up with our “mythical” θ for which we also had $\tan(\theta) = \frac{b}{a}$. And then using $x = \theta$ in $\sqrt{a^2 + b^2} \cos(x - \theta)$ directly results in $\sqrt{a^2 + b^2}$.

SQS-C By inspection, $x = 1$ is a solution. From $x^{1/x} = x^{1/2}$, we equate powers: $\frac{1}{x} = \frac{1}{2}$, giving $x = 2$.

SQS-D The sphere of radius r has volume $V = \frac{4}{3}\pi r^3$ and surface area $S = 4\pi r^2$, while the cylinder of radius r and height h has volume $V = \pi r^2 h$ and surface area $S = 2\pi r^2 + 2\pi r h$. Equating volumes and dividing by πr^2 yields $4r = 3h$, while equating surface areas, combining terms, and dividing by $2\pi r$ yields $r = h$. This contradicts $4r = 3h$, so we conclude that we can have equality of volumes or equality of surface areas, but we cannot have both.

QS-0 Never. The equivalent equation $ax^2 + bx + (c - k) = 0$ has discriminant $b^2 - 4a(c - k) = 0$ for a double root, or $b^2 = 4a(c - k)$. This is a contradiction, as left member is odd while the right one is even.

QS-1 Inductive Proof by the Problems Editor: This is trivial for $n = 1$, and for $n = 2$, each side equals 5. So, now assume the induction hypothesis that $\sum_{i=1}^n i \cdot i! = (n + 1)! - 1$. We must show

$$\sum_{i=1}^{n+1} i \cdot i! = (n + 2)! - 1. \text{ We split up } \sum_{i=1}^{n+1} i \cdot i! = \sum_{i=1}^n i \cdot i! + (n + 1) \cdot (n + 1)! \text{ By the inductive hypothesis,}$$

$$\sum_{i=1}^{n+1} i \cdot i! = (n + 1)! - 1 + (n + 1) \cdot (n + 1)! = (1 + (n + 1))(n + 1)! - 1 = (n + 2)(n + 1)! - 1 = (n + 2)! - 1,$$

where we common-factored out $(n + 1)!$ in the penultimate step and used $(n + 2)(n + 1)! = (n + 2)!$ to finish. “Identity Windmill” Extension: Even better and more profound, Contributing Editor Albert Natian offers a powerful generalization. The factor i next to the factorial $i!$ can be viewed to be a special function of i . Instead of having merely an i there, let’s start with a function $f(i)$. This then engenders another function $g(n)$ to appear as a multiplier to $(n + 1)!$ on the right side of the identity. Thus, $\sum_{i=1}^n f(i) \cdot i! = g(n) \cdot (n + 1)! + c$; the constant c depends on f . If we now *assume*

that this form is correct for unknown functions f and g , we are led by induction to the identity $f(n) = (n+1) \cdot g(n) - g(n-1)$. For example, we can let $g(n) = n$, in which case $f(n) = n^2 + 1$ and $c = 0$, so we have the identity $\sum_{i=1}^n (i^2 + 1) \cdot i! = n(n+1)!$. (Want more? Email DrMWEcker@aol.com and I'll send you Albert's full pdf.)

QS-2 Assume contrarily that $1 = 0$. For *any* element x of F , $0 \cdot x = 0$ and $1 \cdot x = x$. But since $1 = 0$, $0 \cdot x = 0 = 1 \cdot x = x$, proving that every element is zero, a contradiction to F having at least two members.

QS-3 We do a “reverse telescoping” and write the ratio as a product of ratios to get $\lim_{n \rightarrow \infty} \frac{F_{n+m}}{F_n} = \lim_{n \rightarrow \infty} \left(\frac{F_{n+m}}{F_{n+m-1}} \cdot \frac{F_{n+m-1}}{F_{n+m-2}} \cdot \frac{F_{n+m-2}}{F_{n+m-3}} \cdot \dots \cdot \frac{F_{n+2}}{F_{n+1}} \cdot \frac{F_{n+1}}{F_n} \right) = \phi \cdot \phi \cdot \phi \cdot \dots \cdot \phi \cdot \phi = \phi^m$ (using limit of a product).

QS-4 $f(x) + g(y) = \sin(x) + \cos(y)$. It would be tempting to believe that f is sine and g is cosine. If $x = 0$ then $f(0) + g(y) = \sin(0) + \cos(y) = \cos(y)$. Re-label to get $g(x) = \cos(x) - f(0) = \cos(x) + c$, $c = -f(0)$. Let $y = \frac{\pi}{2}$ to get $f(x) + g\left(\frac{\pi}{2}\right) = \sin(x) + \cos\left(\frac{\pi}{2}\right) = \sin(x)$, so then $f(x) = \sin(x) - g\left(\frac{\pi}{2}\right)$. But recall $g(x) = \cos(x) + c$, so $g\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + c = c$, yielding $f(x) = \sin(x) - c$.

Summary: $f(x) = \sin(x) - c$, $g(x) = \cos(x) + c$. **Note:** We could restart and show right away that g must be even. Indeed, the unique even part of the sum function $f(x) + g(x) = \sin(x) + \cos(x)$ is precisely $g(x) = \cos(x) + c$ regardless of the value of c , but the unique odd part is *not* $f(x) = \sin(x) - c$ (unless the constant is zero), which might explain why I could not similarly prove that f must be odd.

QS-5 (a) $xy = 0$, from $xy = 0 \Leftrightarrow [x = 0 \text{ or } y = 0] \Leftrightarrow$ point is on y -axis or on x -axis. (b) $\sqrt{xy} = \sqrt{x}\sqrt{y}$.

QS-6 Write $\begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ t & u & 1 \end{bmatrix} \begin{bmatrix} 1 & v & w \\ 0 & x & y \\ 0 & 0 & z \end{bmatrix} = \begin{bmatrix} 1 & a & b \\ c & d & e \\ f & g & h \end{bmatrix}$ and notice that as long as $d \neq ac$, we can now

solve a system of equations as we “multiply out” the left side, then equate corresponding entries. Using the first row of L gives $1 = 1$, $v = a$, and $w = b$. Using the second row of L we see that $s = c$,

$sv + x = d$ or $x = d - ac$, and $sw + y = e$ which yields $y = e - bc$. Row three produces $t = f$, $tv + ux = g$ or $u = \frac{g-af}{d-ac}$, and $tw + uy + z = h$ resulting in $z = h - bf - \frac{g-af}{d-ac}(e - bc)$.

Answer to Bonus: $e^\pi - \pi \approx 20$. In fact, the value begins 19.999 (to the third decimal place).

Solutions to Problems from Previous Issues

(Note: Solutions to the regular problems follow the solution to this special problem.)

SPECIAL PROBLEM 6 Proposed by Michael W. Ecker. *An idealized ladder is leaning against a vertical wall. The top of the ladder is constrained to a vertical groove along the wall to prevent the ladder being pulled away from the wall. The bottom of the ladder is attached to a little cart that is constrained to pull the ladder horizontally away from the wall, always at instantaneous rate of exactly 1 foot per second. This starts at time $t = 0$. Sensational Question: How high is the top of the ladder when it is falling at twice the speed of light?*

Well, I unfairly loaded up the question with that vertical groove. Frankly, I should have omitted that, so let's pretend I did not say it. The ladder is L feet long and at time t the bottom is $x = x(t)$ from the wall and the top is $y = y(t)$ above the ground. At each time t we have a right triangle so that $x^2 + y^2 = L^2$. Differentiate with respect to t to get $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$. Since $\frac{dx}{dt} \equiv 1$ and $\frac{dy}{dt} = -2c$, we solve to get $y = -x / \frac{dy}{dt} = \frac{x}{2c}$. From this equation, the size of c (the speed of light), and x nearly L , we can see that y would have to be tiny. (How tiny? We'd assume perfection since y 's size here may be sub-atomic!)

If we wish to continue, recall $x^2 + y^2 = L^2$ to get a single equation in just y now, to nail down the value of y : $y = \frac{L}{\sqrt{4c^2 + 1}}$. If the ladder length L is 10 feet long, then $y \approx 10^{-9}$, a billionth of a foot!

However, this farce has gone on long enough. There was an article that mentioned an experiment to try this problem. Close camera work showed that as the ladder was on its way to falling down, the ladder actually got pulled from the wall. Ergo, no more right triangle, so the two equations no longer apply.

Which brings us to my premature introduction of the vertical groove. I added that precisely in anticipation and knowledge of this experiment's result. My hunch is that the physics of the situation would pose a different paradox, specifically an infinite amount of energy to keep the top of the ladder attached to the wall *and* the ladder's bottom maintaining that constant speed of 1 ft/sec.

Finally, the elephant in the room remains: my absurd assumption in which the speed of light is violated!

Above solution is by the Proposer, with similar solution by Troy Williamson.

Differentially Compatible

P-1 Proposed by Michael W. Ecker. Call two differentiable functions f, g *differentially compatible* (DC) iff there exist real values a, b with $f'(a) = g'(b)$. (a) Prove that every two non-invertible differentiable functions are DC. (b) Give two different examples of two polynomials that are not DC. (c) If one function is invertible and the other is not, must this pair be DC? (d) Is DC an equivalence relation?

Solution to (a) by Canyon Sandel and Luke Paluso (jointly), Blinn College, Brenham Texas. If f and g are non-invertible then they fail the horizontal line test. By Rolle's Theorem there exist real numbers a and b such that $f'(a) = 0 = g'(b)$. (Solutions Editor's Note: Some submitted solutions relied on the Intermediate Value Theorem applying to f' and g' . The problem's conditions do not guarantee that these derivatives are continuous, but Darboux's Theorem still ensures that the conclusion of the Intermediate Value Theorem applies to these functions obtained through differentiation.)

Solution to (b) by Raymond N. Greenwell, Hofstra University (Emeritus), Hempstead, NY. For $c \geq 1$ let $f(x) = x^3 + cx$ and $g(x) = -x^3 - cx$. This yields infinitely many pairs of polynomials f, g which are not DC because $f'(a) \geq c \geq 1$ and $g'(b) \leq -c \leq -1$. (Solutions Editor's Note: This solution idea can be generalized by finding polynomials f' bounded below by M and polynomial g' bounded above by $N < M$. Then integrate f' and g' to create the desired non-DC polynomial pair f, g .)

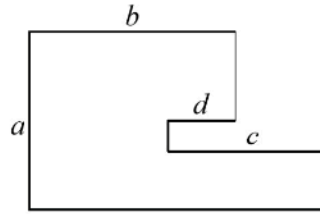
Solution to (c) by Adam Moreno, West Shore Community College, Scottville, MI. The answer is no. Let $f(x) \equiv 1$ (non-invertible) and $g(x) = x$ (invertible). These are not DC as $f'(x) \equiv 0$ and $g'(x) \equiv 1$.

Solution to (d) by Michael S. McClendon, University of Central Oklahoma, Edmond, OK. No, DC is not an equivalence relation. Let $f(x) = x^2$, $g(x) = x^3$, and $h(x) = -x$. We find that f and g are DC because $f'(0) = g'(0) = 0$. We also find that f and h are DC because $f'(-1/2) = h'(0) = -1$. However, g and h are not DC because $g'(x) = 3x^2 \geq 0$, while $h'(x) \equiv -1 < 0$.

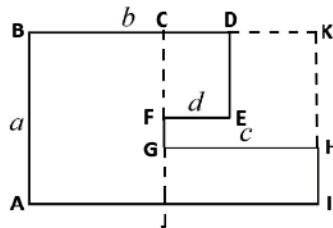
Also solved fully by the Proposer. All solvers above solved all four parts.

Admire The Shape!

P-2 Proposed by Michael W. Ecker. (a) Given this faithfully represented rectilinear shape with a, b, c, d as the only known lengths (see next page), find its perimeter. (b) Find the greatest lower bound and the least upper bound for the area of this shape.



Solution by Ivan Retamoso. We label some additional points and sides of the diagram to facilitate explaining our approach.



(a) The perimeter can be expressed as $\text{Perimeter} = a + b + c + d + (DE + FG + HI) + (AI) = a + b + c + d + (a) + (b - d + c) = 2a + 2b + 2c$.

(b) Note that \overline{AI} , which depends on b , d , and c , has a fixed length. Only \overline{DE} , \overline{FG} , and \overline{HI} can be changed to modify the area. Any shape we create must be contained in the rectangle $ABKI$, which has area $a(b + c - d)$. The area of our figure can be made as close to this as possible by letting $DE \rightarrow 0$, $FG \rightarrow 0$, and $HI \rightarrow a$. Similarly, any shape we create must contain the rectangle $ABCJ$, which has area $a(b - d)$. The area of our figure can be made as close to this as possible by letting $DE \rightarrow 0$, $FG \rightarrow a$, and $HI \rightarrow 0$. Therefore, the greatest lower bound for the area of the shape is $a(b - d)$ and the least upper bound for the area is $a(b + c - d)$.

Also solved by Raymond Greenwell; jointly by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC, and students Vigneswaran Madappan Chinnasami and Nishant Chinnasami of University of South Carolina, Columbia, SC; Troy Williamson, Texas State Technical College, Abilene, TX; and the Proposer.

It's Okay If It's Bi!

P-3 Stephen L. Plett. Let $R_0 = \{x \in \mathbb{R} : x \neq 0\}$. Pick any c in $(0, 1)$, and let $S = \{x \in (0, 1) : x \neq c\}$. Is there a bijective function from S to R_0 ? Explain fully.

Solution I by Raymond Greenwell. There is such a function. Let $f(x) = \frac{x-c}{x^2-x}$. It is easily verified that f is strictly decreasing on $(0, 1)$, going to infinity as x approaches 0 from the right, and going to negative infinity as x approaches 1 from the left. Thus, f is injective and by the Intermediate Value Theorem, the image of $(0, 1)$ through f is \mathbb{R} . Furthermore, f achieves the value of 0 only at $x = c$, which is not in S .

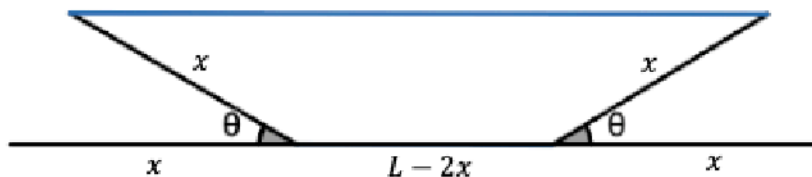
Solution II by Adam Moreno. We construct such a function f using transformations of $f_0(x) = \tan(x)$ restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. On this interval, f_0 is one-to-one and onto \mathbb{R} . We begin by scaling horizontally to diameter of 1 with $6x$. Then translate horizontally to get $f_2(x) = \tan(\pi(x-1/2))$, which has domain $(0, 1)$. We now ensure that $f(c) = 0$ by shifting f_2 vertically by the value it takes at c . The final function $f(x) = \tan(\pi(x-1/2)) - \tan(\pi(c-1/2))$ satisfies all of the conditions upon restricting the domain to S .

Also solved by Ivan Retamoso and Michael McClendon.

The Best Gutter

P-4 Proposed by Albert Natian. Given a rod of length L , fold up x units of length from each end to form a trapezoid with a side missing. What is its greatest possible area?

Solution by Ivan Retamoso. Fold up x units of length from each end at an angle of θ as shown in the figure.



We must have $0 < \theta < \frac{\pi}{2}$ because if $\theta > \frac{\pi}{2}$, the resulting trapezoid can be embedded inside a larger one with a corresponding acute angle. The area A of the trapezoid is a sum involving two triangles and a rectangle, $A = 2\left(\frac{x \sin(\theta)x \cos(\theta)}{2}\right) + (L - 2x)x \sin \theta$. So, $A = \sin \theta(\cos \theta - 2)x^2 + L \sin(\theta)x$.

Completing the square yields $A = -\sin \theta(2 - \cos \theta)\left(x - \frac{L}{2(2 - \cos \theta)}\right)^2 + \frac{L^2 \sin \theta}{4(2 - \cos \theta)}$. Observe

that for $0 < \theta < \frac{\pi}{2}$, $\sin \theta(2 - \cos \theta)\left(x - \frac{L}{2(2 - \cos \theta)}\right)^2 \geq 0$ and $\frac{L^2 \sin \theta}{4(2 - \cos \theta)} > 0$. Thus, for any

fixed θ , A will achieve its maximum when $x = \frac{L}{2(2 - \cos \theta)}$. In this case, we obtain a simpler

equation $A(\theta) = \frac{L^2 \sin \theta}{4(2 - \cos \theta)}$. Then $A'(\theta) = \frac{L^2(2 \cos \theta - 1)}{4(2 - \cos \theta)^2}$, with zero at $\theta = \frac{\pi}{3}$. The first-

derivative test confirms maximality, and this choice of θ results in $x = \frac{L}{3}$ and

$$A = \frac{L^2 \sin\left(\frac{\pi}{3}\right)}{4\left(2 - \cos\left(\frac{\pi}{3}\right)\right)} = \frac{L^2 \sqrt{3}}{12}.$$

Also solved using partial derivatives by Raymond Greenwell; Troy Williamson; and the Proposer.

Tangent Product Is Just A Tangent Sum

P-5 Proposed by Michael W. Ecker. (a) Of all possible triangles, is there one that has a smallest possible product of the tangents of the angles? What about a largest possible product? (b) What if we restricted ourselves to acute triangles only?

Solution by the Proposer. We first prove that for any triangle ABC, the product of the tangents of the angles equals their sum: $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. *Proof:* $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, so $\tan A + \tan B = \tan(A+B) - \tan A \tan B \tan(A+B)$. However, $\tan C = \tan(\pi - (A+B)) = -\tan(A+B)$, so $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. Therefore, we will work with the sum of the tangents instead of with the product.

(a) There is no smallest sum of tangents, S ; e.g., take angle C to be a bit *over a right angle*, as in $A = 90.001^\circ$ and $B = C = 44.9995^\circ$, so $S \approx -62,585$. There is also no largest sum S ; e.g., take angle C to be a bit *under a right angle*, as in $A = 89.999^\circ$ and $B = C = 45.0005^\circ$, so $S \approx 52,830$.
 (b) The second example in Part (a) was acute so there is still no maximum. We seek to minimize the sum $S = \tan A + \tan B + \tan C$ subject to the condition $g(A, B, C) = A + B + C = \pi$, where $0 < A, B, C < \pi/2$. We can work with three variables or two. An outline for the three-variable case is to take the gradient of S and set it equal to a constant multiple of the gradient of g . Or we could work with just two variables, A and B , as $C = \pi - A - B$. Either method leads to the same conclusion, $\sec^2 A = \sec^2 B = \sec^2 C$. Hence, the cosine of any one acute angle is equal to the cosine of any other acute angle, and therefore, all angles are equal, as cosine is one-to-one on $[0, \pi]$. The equilateral triangle's sum of tangents is $3\sqrt{3}$. That this is a minimum can be confirmed by using two variables and performing the second-partials test.

Several partial or incorrect solutions were received.

Black and White Going Together

P-6 Proposed by Dianne Marquart (retired), Jefferson College, Hillsboro, MO. A soccer ball cover is sewn from 32 leather panels: white hexagonal panels and black pentagonal panels. Every black panel is surrounded by 5 white panels, and each white panel is surrounded by 3 white and 3 black panels. At each vertex there are three angles. How many black panels and white panels are there?

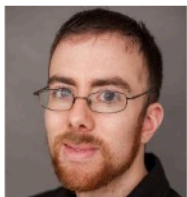
Solution I by Bryan Wilson, Solutions Editor. Let W be the number of white hexagons and B be the number of black pentagons. The black pentagons produce a total of $5B$ edges, all of which must have a white neighbor. The white hexagons produce a total of $6W$ edges, $3W$ of which must border a black pentagon. Equating these shared white-black edges, we obtain $5B = 3W$, resulting in $W + \frac{3}{5}W = 32$, with solution $W = 20$ and $B = 12$.

Solution II by Michael McClendon. Let W and B be as in Solution I. By Euler's Formula for Polyhedra, $V - E + F = 2$, where V is the number of vertices, E the number of edges, and F the number of faces. The number of vertices contributed by the panels is $V = \frac{1}{3}(5B + 6W)$, where division by 3 is required since each vertex is shared by 3 panels. Similarly, $E = \frac{1}{2}(5B + 6W)$, with each edge shared by 2 panels. Substituting into Euler's Formula yields $\frac{1}{3}(5B + 6W) - \frac{1}{2}(5B + 6W) + B + W = 2$, which reduces to $\frac{1}{6}B = 2$, giving $B = 12$ and $W = 20$.

Also solved by Raymond Greenwell (using angle sums) and the Proposer. ■



Michael W. Ecker had a 45-year career as a mathematics professor, most of it at Pennsylvania State University's Wilkes-Barre campus. He retired from teaching in 2016. His PhD in mathematics was from the City University of New York (1978). Published 500+ times as a mathematician or computer journalist, Mike also served on national committees responsible for creating competitive national exams, and was the Founding Problem Section Editor of *The AMATYC Review* (1981-1997). As a recreational mathematician, he published his own newsletter, *Recreational & Educational Computing* (1986-2007). For free PDF copies of REC, visit <https://dr-michael-ecker.weebly.com>.



Bryan Wilson is an Associate Professor at Salt Lake Community College and has been teaching there since 2015. He holds a master's degree in mathematics from University of Utah (2015). He has extensive experience in math competitions, including being a multiple year Idaho state champion in high school and placing in the top 200 nationally in the 2006 Putnam Exam. In his free time, Bryan enjoys coding challenges such as his college sports ranking algorithm and programming bots to play various board and card games.
