

find intersection points by substituting $y = ax$ into original lines' equations to obtain $x = \frac{\pm b}{a^2 + 1}$, from which $y = ax = \frac{\pm ab}{a^2 + 1}$. The two closest points, which are equidistant from the origin, are $\left(\frac{b}{a^2 + 1}, \frac{ab}{a^2 + 1}\right)$ and $\left(\frac{-b}{a^2 + 1}, \frac{-ab}{a^2 + 1}\right)$.

Also solved by Raymond Greenwell (Emeritus), Hofstra University, Hempstead, NY; and Troy Williamson, who majestically treated the problem as dealing with a rotating conic section.

Even More Possibilities

O-2 Proposed by Michael W. Ecker. a) Recall that a function f is odd iff $f(-x) = -f(x)$, and it's even iff $f(-x) = f(x)$. Suppose $P(x)$ and $Q(x)$ are polynomials, $O(x)$ an odd function, and $E(x)$ an even function. Form the function $f(x) = P(x)O(x) + Q(x)E(x)$. (a) Find the largest possible class that you can that contains *all* such polynomials P, Q for which the resulting f must then be even. (b) Is it true that P must be odd and Q must be even?

Note: Problem J-2 in the Spring 2023 issue dealt with the special case having $O(x) = \sin(x)$, $E(x) = \cos(x)$. In this case it was proven that P must be odd and Q must be even.

Solution by the Proposer. (a) First note that it is *sufficient* that P be odd and Q be even, as then $f(-x) = P(-x)O(-x) + Q(-x)E(-x) = P(x)O(x) + Q(x)E(x) = f(x)$. To reverse this, we start by assuming f is even.

Given any real-valued function g of a real variable x with domain A , let's use Zg_A to denote the set of g 's zeros in A , and Ng_A the set of g 's nonzeros in A ; $Ng_A = \{x \in A : g(x) \neq 0\}$. Consider now specific $x \in ZO_A$, so $O(x) = 0$. By O 's oddness, $O(-x) = 0$. For such x , we have $f(x) = Q(x)E(x)$ and $f(-x) = Q(-x)E(-x) = Q(-x)E(x) = f(x)$. Hence, $Q(-x)E(x) = Q(x)E(x)$. If $E(x) \neq 0$, so $x \in NO_A$, then we can divide by $E(x)$ to get $Q(-x) = Q(x)$.

Hence, as long as $O(x) = 0$ and $E(x) \neq 0$, $Q(-x) = Q(x)$. Now define the difference function D by $D(x) = Q(x) - Q(-x)$. Note that D is a polynomial and therefore it is either identically zero, or has only a finite number of zeros. Suppose that the cardinality of set $ZO_A \cap NE_A$ is greater than the degree of Q , meaning that the number of values of x with $O(x) = 0$ and $E(x) \neq 0$ exceeds

the number of zeros of Q . Then we have a potential contradiction to the Fundamental Theorem of Algebra, because we then have $D(x) = Q(x) - Q(-x) = 0$ for more values of x than possible, *unless* D is identically zero, so Q is indeed even under our assumption. We now consider whether P is also odd.

We now have f, Q , and E even, and O odd. So, $f(x) = P(x)O(x) + Q(x)E(x)$, and $f(-x) = P(-x)O(-x) + Q(-x)E(-x) = -P(-x)O(x) + Q(x)E(x)$. Equate this to $f(x)$ to get $-P(-x)O(x) = P(x)O(x)$. Consider the added assumption now of specific $x \in NO_A$, so $O(x) \neq 0$. Now we can divide by $O(x)$ to get $P(-x) = -P(x)$ for all such x .

We conclude that, as long as $O(x) \neq 0$, $P(-x) = -P(x)$. Suppose now that the cardinality of set NO_A is greater than the degree of P , meaning that the number of values of x with $O(x) \neq 0$ exceeds the number of zeros of P . Then we again have a potential contradiction to the Fundamental Theorem of Algebra, this time applied to $d(x) = P(x) + P(-x)$. That is, P is indeed odd under our assumption.

Final Conclusion: *If there are more values of x with both $O(x) = 0$ and $E(x) \neq 0$ than the degree of Q , and also more values of x with $O(x) \neq 0$ than the degree of P , then we conclude that this is sufficient to prove P to be odd and Q to be even.*

(b) The answer is NO, as illustrated by the counterexample $O(x) = 0 = E(x)$, in which case P and Q can be any functions whatsoever, not just polynomials, and yet $f(x) = 0$ is even.

Other partial solutions: Ivan Retamoso considered polynomials O, E . As in the solution above, he first shows f is even. Consider now the special case where $O(x)$ and $E(x)$ are odd and even *polynomials* respectively. With k be any real number, let $ODD(x)$ and $EVEN(x)$ be polynomials that are odd and even, respectively. Construct polynomials $P(x) = kE(x) + ODD(x)$ and $Q(x) = -kO(x) + EVEN(x)$, resulting in $f(x) = ODD(x)O(x) + EVEN(x)E(x)$, which is easily shown to be even. This shows part (b) is false. For example, with $O(x) = x^3$ and $E(x) = x^2$, set $k = 1$, so $ODD(x) = x$, and $EVEN(x) = x^2$. This yields $P(x) = x^2 + x$ and $Q(x) = -x^3 + x^2$, neither of which is even or odd, respectively, and yet $f(x) = (x^2 + x)(x^3) + (-x^3 + x^2)(x^2) = 2x^4$ is even.

Square Dancing

O-3 Proposed by Michael W. Ecker and Stephen L. Plett. If we reverse the digits of the numbers in the true equation $12^2 = 144$, we get the true equation $21^2 = 441$. Find, with