The Problem Corner

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The Purpose of ***The Problem Corner*** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, problem solvers!

We are pleased to share that Problems 40 and 41 in *The Problem Corner* have received exceptional solutions, marked by both accuracy and originality. Highlighting these remarkable approaches aims to inspire others and foster a deeper appreciation for mathematical thinking worldwide.

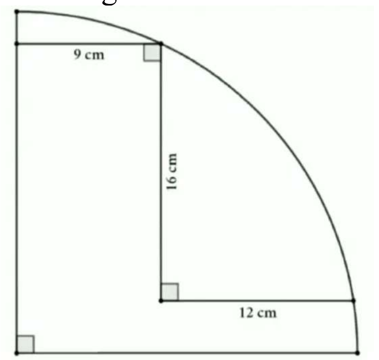
New Problems

Problem 42
Proposed by Ivan Retamoso, BMCC, USA.

A 5-foot-tall fence stands parallel to a tall building, 2 feet away from it. Determine the length of the shortest ladder that can extend from the ground, over the fence, to touch the wall of the building.

Problem 43
Proposed by Ivan Retamoso, BMCC, USA.

In the figure below find the radius of the circle.

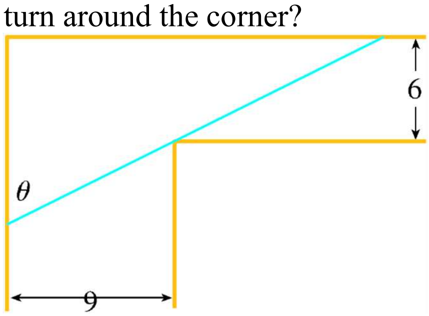


Solutions to **problems** from the previous issue.

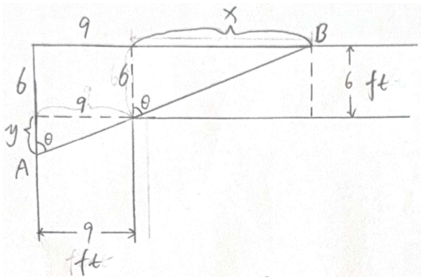
"Steel pipe" problem.

Problem 40
Proposed by Ivan Retamoso, BMCC, USA.

A steel pipe is being carried horizontally through a hallway that is 9 feet wide and turns at a right angle into a hallway that is 6 feet wide. What is the maximum length of pipe that can make the turn around the corner?



First solution to problem 40
By Stephen Chen, Borough of Manhattan CC, Fuzhou, China.
In this first solution, the solver provides a thorough explanation by first establishing a relationship between the length of the pipe and a clearly defined variable x. Through differentiation and a clever algebraic factorization, he identifies the critical point that leads to a well-justified solution.



$tan\theta=\frac{x}{b}$, $tan\theta=\frac{9}{y}$, length of pipe $L=AB$

$\frac{x}{b}=\frac{9}{y} \Rightarrow y=\frac{54}{x}$

$L^2=(x+9)^2+(y+6)^2 \Rightarrow L=\sqrt{(x+9)^2+(\frac{54}{x}+6)^2}$

Taking the derivative of L, setting it equal to zero, and solving for x, we find

$$L'=\frac{2(x+9)+2(\frac{54}{x}+6)(-\frac{54}{x^2})}{2\sqrt{(x+9)^2+(\frac{54}{x}+6)^2}}=0$$

$$2x^4+18x^3\text{-}648x\text{-}5832=0$$
$$(2x+18)(x^3\text{-}324)=0$$
$$2x+18=0 \text{ or } x^3\text{-}324=0$$

x=-9 or $x^3=324$ then $x=6.868\ ft$

Since we have minimized L, the value of x found above will lead to the maximum length of pipe that can make the turn around the corner.

$$L=\sqrt{(x+9)^2+(\frac{54}{x}+6)^2}=\sqrt{(6.868+9)^2+(\frac{54}{6.868}+6)^2}=21.07\ ft$$

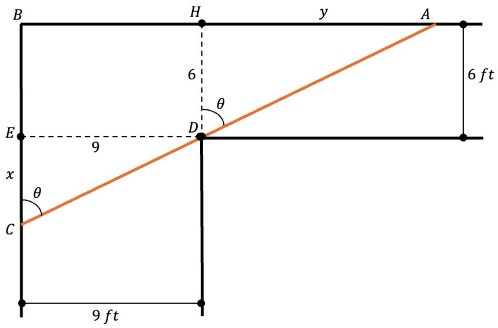
Second solution to problem 40

Dr. Abdullah Kurudirek, Al-Naji University, Baghdad.

In this second solution, the solver presents two approaches: one based on the given angle and the other on a properly defined distance x. Although some computational details are omitted, the work shown leads to the correct result for both distinct yet similar methods.

First, let's identify some points on the given diagram and form a right-angled triangle ABC. Accordingly, the maximum length of the pipe that can make the turn will be equal to the length of the hypotenuse of this triangle, which is $AC=L$. Then, to help us find this, let's draw some auxiliary lines to simplify the calculations, for example, lines like $DH=6\ ft$ and $DE=9\ ft$. Next, let's define $DA=L_1$ and $DC=L_2$, so that

L=L_1+L_2 . From this point, we can proceed in two different ways. I would like to briefly show the first method and then continue with the second one.



Method 1: $L=\frac{6}{cos\theta}+\frac{9}{sin\theta}$ by finding the minimum possible value of this expression with the appropriate value of θ, we determine the maximum pipe length L that can make the turn. So, we will proceed by setting $\frac{dL}{d\theta}$=0.

$$\frac{dL}{d\theta}=\frac{6sin\theta}{cos^2\theta}-\frac{9cos\theta}{sin^2\theta}=0$$

After performing the necessary algebraic operations, we express $tan\theta=\sqrt[3]{\frac{3}{2}}$. To simplify our calculations, let's set $tan\theta=\sqrt[3]{\frac{3}{2}}=a$. Then, if we continue our work by expressing $sin\theta=\frac{a}{\sqrt{1+a^2}}$ and $cos\theta=\frac{1}{\sqrt{1+a^2}}$ in terms of a, we proceed as follows.

$L=\frac{6}{cos\theta}+\frac{9}{sin\theta}=6\sqrt{1+a^2}+9\frac{\sqrt{1+a^2}}{a}$. Now substitute $a=\sqrt[3]{\frac{3}{2}}$ and get $L \approx 21.07\ ft$.

Method 2: In this method, we will proceed by using trigonometry and the concept of similarity in triangles, and by writing the following relationships.

From $tan\theta=\frac{y}{6}=\frac{9}{x}$, we can deduce that xy=54 , and from there, we can write $x=\frac{54}{y}$.

Now, if we apply the Pythagorean Theorem in the right triangle ABC

$$AC=L=\sqrt{(x+6)^2+(y+9)^2}$$

After substituting $x=\frac{54}{y}$ here, we find that $L=\sqrt{(\frac{54}{y}+6)^2+(y+9)^2}$

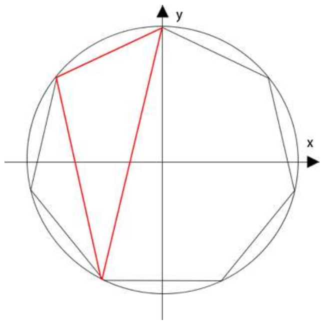
then set $\frac{dL}{dy}$=0 as we did in the first method by finding the minimum possible value of this expression with the appropriate value of y which is $y \approx 6.87$ (*attention*, not y=-9), we determine the maximum pipe length L that can make the turn. *Finally, the maximum length of the pipe that can be carried horizontally around the corner is approximately as follows:* $L \approx 21.07\ ft$

"Heptagon" problem.

Problem 41

Proposed by Ivan Retamoso, BMCC, USA.

A regular heptagon is inscribed in a unit circle with a radius of 1 cm. Calculate the area of the red triangle.



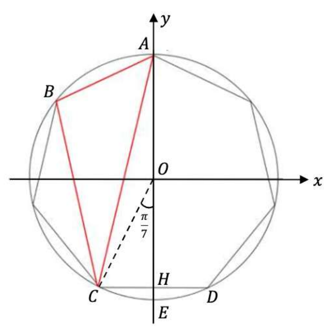
First Solution to problem 41
Dr. Abdullah Kurudirek, Al-Naji University, Baghdad.

Our solver has presented two detailed and well-structured solutions, each employing different trigonometric identities and formulas for expressing the area of a triangle in terms of the angle between its sides.

Method 1:
Let us first identify some points on the given figure and draw an auxiliary line. Using the concept of power of a point with respect to circles, we can write:

$$CH.HD=AH.HE$$

Here, let $HE=x$ and $AH=2\text{-}x$. Also, since a radius perpendicular to a chord bisects the chord, we know that $CH=HD$.



$$CH^2=(2\text{-}x)x$$

Now, in the right triangle OCH, we can use the sine theorem (law of sines) to write the following equality:

$$\frac{CH}{sin\frac{\pi}{7}}=\frac{OC}{sin\frac{\pi}{2}}$$

From here, we obtain: $CH=sin\frac{\pi}{7}$. If we substitute this into equation (1), which was: $(sin\frac{\pi}{7})^2=2x\text{-}x^2$ and by using the discriminant formula, we obtain $x=1\pm cos\frac{\pi}{7}$. However, if we examine the given graph closely, it is evident that $HE<1$, so $x=1\text{-}cos\frac{\pi}{7}$.

Now, in the right-angled triangle AHC, we can use the Pythagorean Theorem to write the following:

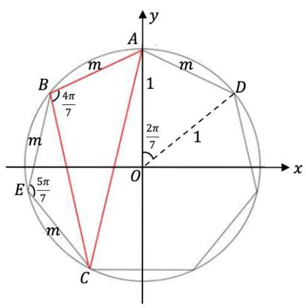
$$AH^2+CH^2=AC^2 \text{ then } (2\text{-}x)^2+(sin\frac{\pi}{7})^2=AC^2 \text{ where } x=1\text{-}cos\frac{\pi}{7}$$

Hence, $AC=\sqrt{(1+cos\frac{\pi}{7})^2+(sin\frac{\pi}{7})^2}$. Knowing that $<BAC=\frac{2\pi}{7}$ and $AB=2CH=2sin\frac{\pi}{7}$

Now, we can calculate the area of the desired red triangle using the sine area formula for triangles. Accordingly:

$$A(\Delta ABC)=\frac{AB.AC.sin\,(<BAC)}{2}=\frac{2sin\frac{\pi}{7}\sqrt{(1+cos\frac{\pi}{7})^2+(sin\frac{\pi}{7})^2}.sin\frac{2\pi}{7}}{2}\approx 0.661$$

Method 2: Again, let us first identify some points on the given figure and draw an auxiliary line. Next, let's denote one side of our regular heptagon as m; it was already given that the radius of the circle is 1 cm.



If we apply the law of cosines in triangle ΔADO

$m^2=1^2+1^2\text{-}2.1.1.cos\frac{2\pi}{7}$ then we obtain $m=\sqrt{2\text{-}2cos\frac{2\pi}{7}}$

Similarly, if we apply the law of cosines in triangle ΔBEC as well

$BC^2=m^2+m^2\text{-}2.m.m.cos\frac{5\pi}{7}$ and if we substitute the value of m here, we obtain

$BC=2sin\frac{2\pi}{7}$.

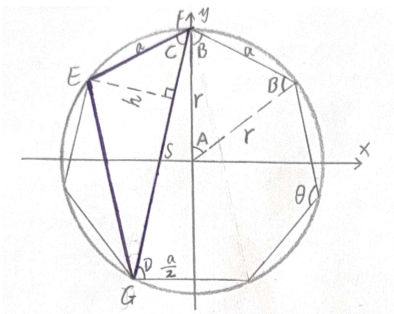
Now, we can calculate the area of the desired red triangle using the sine area formula for triangles. Accordingly:

$$A(\Delta ABC)=\frac{AB.BC.sin\,(<ABC)}{2}=\frac{\sqrt{2\text{-}2cos\frac{2\pi}{7}}.\,2sin2\frac{\pi}{7}.sin\frac{4\pi}{7}}{2}\approx 0.661$$

Second Solution to problem 41

By Stephen Chen, Borough of Manhattan CC, Fuzhou, China.

This alternative solution employs a clever combination of geometric and trigonometric formulas to determine the required area of the triangle. The detailed computations are clearly presented, making it easier for readers to follow.



Each angle of the heptagon $\theta=\frac{180(7\text{-}2)}{7}$=128.57°, r=1cm

$LB=\frac{\theta}{2}$=64.285°, LA=180°-2LB=180-2(64.285°)=51.43°

$$\frac{sinA}{a}=\frac{sinB}{1}$$

Then $a=\frac{sin\,(51.43°)}{sin\,(64.285°)}$

$$LC=\frac{2\theta}{5}=51.43°$$

$h=asinC=\frac{sin^2(51.43°)}{sin\,(64.285°)}$

$$LD=\theta\text{-}\frac{2\theta}{5}=77.14°$$

$$S=\frac{\frac{a}{2}}{cosD}=\frac{\frac{sin\,(51.43°)}{2sin\,(64.285°)}}{cos\,(77.14°)}$$

$$A_{EFG}=\frac{1}{2}sh=\frac{1}{2}\frac{sin\,(51.43°)}{2sin\,(64.285°)cos\,(77.14°)}\frac{sin^2(51.43°)}{sin\,(64.285°)}=\frac{sin^3(51.43)}{4sin^2(64.285)cos\,(77.14°)}=0.66\ cm^2$$