

Solutions to Problems from Previous Issues

Six-Pointed Triangle?

N-1 Proposed by Michael W. Ecker. Find all values of n for which the sum $1 + 2 + \dots + n$ is an integral power of 6. (Equivalently, which triangular numbers are powers of 6?)

Solution by Luke Paluso and Yvette Janecek (jointly), Blinn College, Brenham, TX. Before proceeding we prove the preliminary result (*) $3^j - 2^{j+1} > 1$ for $j \geq 3$. It is true for $j=3$ so it will suffice to show that the sequence $a_j = 3^j - 2^{j+1}$ is an increasing sequence. Indeed, $3^j > 2^j \Leftrightarrow 3^j \cdot 2 > 2^{j+1} \Leftrightarrow 3^j(3-1) > 2^{j+1}(2-1) \Leftrightarrow 3^{j+1} - 3^j > 2^{j+2} - 2^{j+1} \Leftrightarrow 3^{j+1} - 2^{j+2} > 3^j - 2^{j+1}$, so $a_{j+1} > a_j$ for $j \geq 3$, completing the proof.

Suppose now that a triangular number is a power of 6, that is $\frac{1}{2}n(n+1) = 6^j$ for some $j \in \mathbb{Z}$. If n is even then we infer from the fact that n and $n+1$ are relatively prime that $n+1 = 3^j$ and $n = 2^{j+1}$. Substitution yields $3^j - 2^{j+1} = 1$. By (*) this equation has no solutions for $j \geq 3$. Similarly, if n is odd then $n+1 = 2^{j+1}$ and $n = 3^j$. Substitution yields $3^j - 2^{j+1} = -1$ which has no solutions for $j \geq 3$. We conclude by individually checking cases with $j < 3$, resulting in solutions (n, j) of the form $(1, 0), (3, 1), (8, 2)$.

Also solved by Raymond Greenwell, Hofstra University, Hempstead, NY (using Catalan's conjecture); Ivan Retamoso, Borough of Manhattan Community College, New York, NY; Grace Kendall, Michelle Eversman, Mia Holzback, and Abby Smith (students, jointly), Taylor University, Upland, IN; and the Proposer. We received nearly correct solutions (forgetting the $n=1$ case) from Henry Ricardo, Westchester Area Math Circle, Purchase, NY; and Troy Williamson, Texas State Technical College, Abilene, TX; as well as several incorrect or partial solutions.

Maximal Box, Round 1

N-2 Proposed by Michael W. Ecker. We are building a rectangular solid box, and we wish it to have the largest possible volume. Each of two parallel faces costs $\$a$ per square foot, each of two other parallel faces costs $\$b$ per square foot, and each of the remaining two cost $\$c$ per square foot. We have a total budget of D dollars for the construction. Prove that the box of maximal volume satisfying these conditions is such that the subtotal cost for each of the three 'face directions' is one-third the total cost, or $D/3$ in each direction.

Solution by the Proposer, but similarly solved by Ivan Retamoso and Raymond Greenwell. We start with volume $V = xyz$ and use the given information about cost to get the constraint equation $2ayz + 2bxz + 2cxy = D(x, y, z) = D$, a constant. This reflects that the two faces that are y feet by z feet cost $a\$$ per square foot, thus contributing $2ayz$ to the total, and similarly for the other two pairs. (Aside: The mnemonic choice of letters is that unit cost a "covers the missing length" x . Similarly, b "covers" the y and c "covers" the z .) We use Lagrange multipliers. Set $\bar{\nabla}V = \lambda \bar{\nabla}D$. So $\frac{\partial V}{\partial x} = yz = \lambda(2bz + 2cy)$, $\frac{\partial V}{\partial y} = xz = \lambda(2az + 2cx)$, and $\frac{\partial V}{\partial z} = xy = \lambda(2ay + 2bx)$. Notice that $\lambda \neq 0$ as otherwise $V = xyz = 0$. Multiply these three equations by x, y, z respectively to get the following: $xyz = \lambda(2bxz + 2cxy)$, $xyz = \lambda(2ayz + 2cxy)$, and $xyz = \lambda(2ayz + 2bxz)$. Equating the first two shows that $2bxz = 2ayz$, while equating first and third shows that $2cxy = 2ayz$. Hence, $2cxy = 2bxz = 2ayz$. However, since $2ayz + 2bxz + 2cxy = D$, we finally conclude that $2cxy = 2bxz = 2ayz = D/3$. QED.

Insight on Maximality: How do we know that volume is maximized, not minimized? To avoid tedious details, consider $a = b = c = 1$ and $D = 2$. The simplified constraint equation is now (*) $yz + xz + xy = 1$. Consider a box with $0 < y = z = \varepsilon < 1$ (so each of y, z is tiny). Solve for x in (*) to get $x = \frac{1 - \varepsilon^2}{2\varepsilon}$. Then volume can be made arbitrarily small,

$$\text{as we now have } V = xyz = \frac{(1 - \varepsilon^2)\varepsilon^2}{2\varepsilon} = \frac{(1 - \varepsilon^2)\varepsilon}{2} < \frac{\varepsilon}{2}.$$

Maximal Box, Round 2

N-3 Proposed by Albert Natian. A wire of length L is to be used in the construction of an antenna that runs along the top edges of a rectangular box with a square base and along a main diagonal. The antenna is illustrated in the figure (the bolded portion, five line segments).

