

Solutions to Problems from Previous Issues

Six-Pointed Triangle?

N-1 Proposed by Michael W. Ecker. Find all values of n for which the sum $1 + 2 + \dots + n$ is an integral power of 6. (Equivalently, which triangular numbers are powers of 6?)

Solution by Luke Paluso and Yvette Janecek (jointly), Blinn College, Brenham, TX. Before proceeding we prove the preliminary result (*) $3^j - 2^{j+1} > 1$ for $j \geq 3$. It is true for $j=3$ so it will suffice to show that the sequence $a_j = 3^j - 2^{j+1}$ is an increasing sequence. Indeed, $3^j > 2^j \Leftrightarrow 3^j \cdot 2 > 2^{j+1} \Leftrightarrow 3^j(3-1) > 2^{j+1}(2-1) \Leftrightarrow 3^{j+1} - 3^j > 2^{j+2} - 2^{j+1} \Leftrightarrow 3^{j+1} - 2^{j+2} > 3^j - 2^{j+1}$, so $a_{j+1} > a_j$ for $j \geq 3$, completing the proof.

Suppose now that a triangular number is a power of 6, that is $\frac{1}{2}n(n+1) = 6^j$ for some $j \in \mathbb{Z}$. If n is even then we infer from the fact that n and $n+1$ are relatively prime that $n+1 = 3^j$ and $n = 2^{j+1}$. Substitution yields $3^j - 2^{j+1} = 1$. By (*) this equation has no solutions for $j \geq 3$. Similarly, if n is odd then $n+1 = 2^{j+1}$ and $n = 3^j$. Substitution yields $3^j - 2^{j+1} = -1$ which has no solutions for $j \geq 3$. We conclude by individually checking cases with $j < 3$, resulting in solutions (n, j) of the form $(1, 0), (3, 1), (8, 2)$.

Also solved by Raymond Greenwell, Hofstra University, Hempstead, NY (using Catalan's conjecture); Ivan Retamoso, Borough of Manhattan Community College, New York, NY; Grace Kendall, Michelle Eversman, Mia Holzback, and Abby Smith (students, jointly), Taylor University, Upland, IN; and the Proposer. We received nearly correct solutions (forgetting the $n=1$ case) from Henry Ricardo, Westchester Area Math Circle, Purchase, NY; and Troy Williamson, Texas State Technical College, Abilene, TX; as well as several incorrect or partial solutions.

Maximal Box, Round 1

N-2 Proposed by Michael W. Ecker. We are building a rectangular solid box, and we wish it to have the largest possible volume. Each of two parallel faces costs $\$a$ per square foot, each of two other parallel faces costs $\$b$ per square foot, and each of the remaining two cost $\$c$ per square foot. We have a total budget of D dollars for the construction. Prove that the box of maximal volume satisfying these conditions is such that the subtotal cost for each of the three 'face directions' is one-third the total cost, or $D/3$ in each direction.

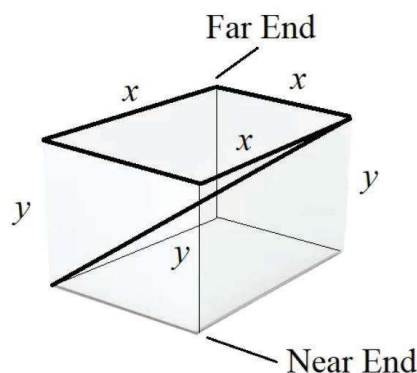
Solution by the Proposer, but similarly solved by Ivan Retamoso and Raymond Greenwell. We start with volume $V = xyz$ and use the given information about cost to get the constraint equation $2ayz + 2bxz + 2cxy = D(x, y, z) = D$, a constant. This reflects that the two faces that are y feet by z feet cost $a\$$ per square foot, thus contributing $2ayz$ to the total, and similarly for the other two pairs. (Aside: The mnemonic choice of letters is that unit cost a "covers the missing length" x . Similarly, b "covers" the y and c "covers" the z .) We use Lagrange multipliers. Set $\bar{\nabla}V = \lambda \bar{\nabla}D$. So $\frac{\partial V}{\partial x} = yz = \lambda(2bz + 2cy)$, $\frac{\partial V}{\partial y} = xz = \lambda(2az + 2cx)$, and $\frac{\partial V}{\partial z} = xy = \lambda(2ay + 2bx)$. Notice that $\lambda \neq 0$ as otherwise $V = xyz = 0$. Multiply these three equations by x, y, z respectively to get the following: $xyz = \lambda(2bxz + 2cxy)$, $xyz = \lambda(2ayz + 2cxy)$, and $xyz = \lambda(2ayz + 2bxz)$. Equating the first two shows that $2bxz = 2ayz$, while equating first and third shows that $2cxy = 2ayz$. Hence, $2cxy = 2bxz = 2ayz$. However, since $2ayz + 2bxz + 2cxy = D$, we finally conclude that $2cxy = 2bxz = 2ayz = D/3$. QED.

Insight on Maximality: How do we know that volume is maximized, not minimized? To avoid tedious details, consider $a = b = c = 1$ and $D = 2$. The simplified constraint equation is now (*) $yz + xz + xy = 1$. Consider a box with $0 < y = z = \varepsilon < 1$ (so each of y, z is tiny). Solve for x in (*) to get $x = \frac{1 - \varepsilon^2}{2\varepsilon}$. Then volume can be made arbitrarily small,

$$\text{as we now have } V = xyz = \frac{(1 - \varepsilon^2)\varepsilon^2}{2\varepsilon} = \frac{(1 - \varepsilon^2)\varepsilon}{2} < \frac{\varepsilon}{2}.$$

Maximal Box, Round 2

N-3 Proposed by Albert Natian. A wire of length L is to be used in the construction of an antenna that runs along the top edges of a rectangular box with a square base and along a main diagonal. The antenna is illustrated in the figure (the bolded portion, five line segments).



In order to maximize the efficiency of the antenna, it is necessary that the box have maximum volume. Maximize the volume.

Solution by Ivan Retamoso. We seek to maximize $V = x^2 y$ subject to the constraint $L = 4x + \sqrt{2x^2 + y^2}$. Note that $L > 4x$, or $0 < x < \frac{L}{4}$. Solving this for y , we obtain

$L - 4x = \sqrt{2x^2 + y^2} \Rightarrow (L - 4x)^2 = 2x^2 + y^2$ or finally $y = \sqrt{14x^2 - 8Lx + L^2}$. We can substitute this to express V in

terms of x as $V = x^2 \sqrt{14x^2 - 8Lx + L^2}$. After taking the derivative and simplifying/factoring we obtain $V' = \frac{2x(7x - L)(3x - L)}{\sqrt{14x^2 - 8Lx + L^2}}$ which has critical values at

$x = 0, \frac{L}{7}, \frac{L}{3}$. Due to the restriction on x we consider only

$x = \frac{L}{7}$. We can verify by test values that if $0 < x < \frac{L}{7}$ then

$V'(x) > 0$, and if $\frac{L}{7} < x < \frac{L}{4}$ then $V'(x) < 0$, confirming by

the First Derivative Test that $x = \frac{L}{7}$ results in a maximum

with $V_{\max} = \frac{L^3}{49\sqrt{7}}$.

Also solved by Raymond Greenwell; Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC; and the Proposer. Nearly complete solution (with incomplete verification of a maximum) by Troy Williamson.

An Odd Requirement

N-4 Proposed by Michael W. Ecker. Suppose f is a polynomial and $\int_a^b f(x)dx = \int_a^b f(x)f'(x)dx$ for infinitely many choices of $a < b$. Must $f'(x) \equiv 1$?

Solution by Bryan Wilson, Solutions Editor. There is a trivial case where $f(x) \equiv 0$, resulting in equality with zero on both sides. For a more interesting example, suppose f is odd. It is easily proven that f' is even and $f \cdot f'$ is odd. Letting $b = -a$, we again get equality on both sides with an odd function integrated from $-a$ to a resulting in zero. We can achieve the same effect if f can be made into an odd function via a horizontal translation, ex. $f(x) = (x-1)^3$ integrated on any interval $(1-a, 1+a)$. Most generally, the problem statement is equivalent to $\int_a^b f(x)(1-f'(x))dx = 0$.

Let x be any value for which the integrand is zero and $f'(x) \neq 0$. With application of the Intermediate Value Theorem there are infinitely many values $a < x < b$ arbitrarily close to x such that the condition is satisfied.

Problem Editor's Note: It would have been better to have stated "Must $f(x) \equiv 0$ or $f'(x) \equiv 1$?" to avoid solving with simply the trivial case.

Also solved by Luke Paluso & Yvette Janecek (jointly); Ivan Retamoso; Raymond Greenwell; and the Proposer.

This Shirt Again?

N-5 Proposed by Stephen L. Plett. Prof. MMXXIV has $n > 3$ dress shirts and wears the top one from his neatly-folded stack each day Monday through Thursday. After the laundering of the four worn shirts on Saturday, they are randomly placed under the unworn ones. Let P = the probability that a particular shirt gets worn every week over a period of four weeks. Determine the value of $P = P(n)$.

Similar solutions by Kirsten Zimmerman, Luke Barnes, and Ava Callaway (students, jointly), Taylor University, Upland, IN; and Troy Williamson. When $n \geq 8$ it is impossible for any shirt be worn in consecutive weeks so now consider $n < 8$. The probability that the chosen shirt is worn in Week 1 is $\frac{4}{n}$ since 4 shirts are selected from n choices. In

subsequent weeks the top four slots are occupied by $n-4$ previously unworn shirts and $4-(n-4) = 8-n$ newly laundered shirts from the previous week. If the chosen shirt was laundered last week, it will appear in one of these slots with probability $\frac{8-n}{4}$. This must occur three times to be worn all four weeks, resulting in a final probability of

$$P(n) = \frac{4}{n} \left(\frac{8-n}{4} \right)^3, \quad 4 \leq n \leq 7, \quad \text{with } P(n) = 0, n \geq 8.$$

Also solved by the Proposer.

Divisor Decimal Duo

N-6 Proposed by Michael W. Ecker. For positive integer n , let $D(n) = \prod_{d|n} d$, the product of all the divisors of n . For example, $D(12) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 12 = 1728$. (a) Is there a value of n with $D(n) = 10$? b) More generally, find all values of n and k for which $D(n) = 10^k$, k a positive-integer.

Composite of solutions by Carl Libis, Southern New Hampshire University, Manchester, NH; Henry Ricardo; and the Proposer. In the course of solving (b) we will see there are no solutions for a). We first prove a preliminary