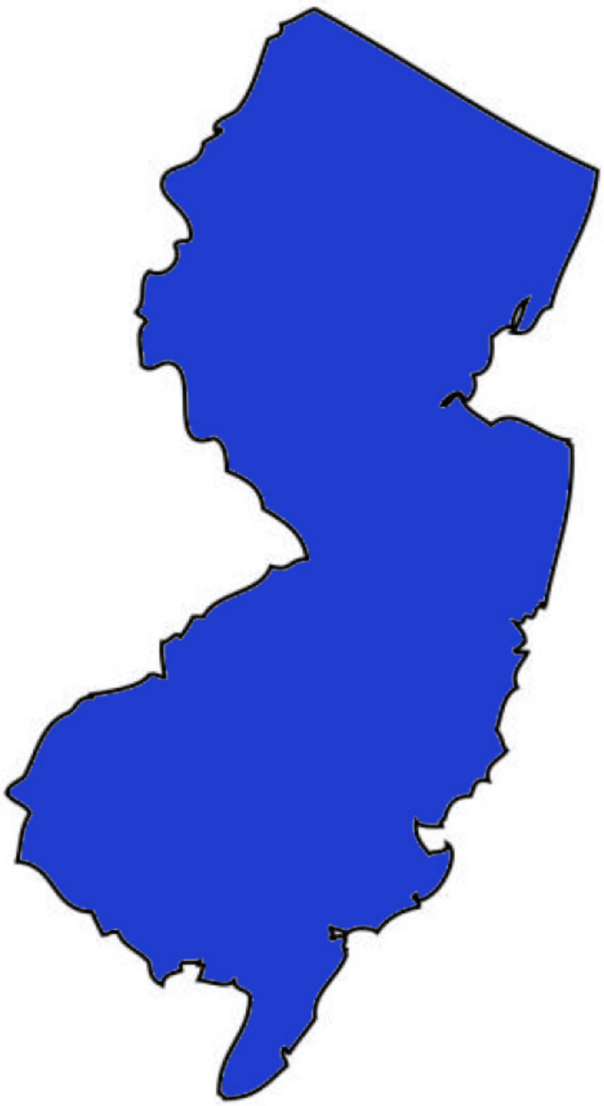
AMTNJ

ASSOCIATION OF MATHEMATICS TEACHERS OF NEW JERSEY

The New Jersey Mathematics Teacher



Vol. 82, No. 1

Summer 2025

The New Jersey Mathematics Teacher
Summer 2025

Association of Mathematics Teachers of New Jersey (AMTNJ)

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Call for Manuscripts

We publish articles of general interest as space is available. You may submit manuscripts on any topic that will appeal to the AMTNJ community. Remember that *The New Jersey Mathematics Teacher* articles foreground effective, creative and innovative mathematics classroom ideas and activities, including utilization of tools and technology, and contextualize it in sound research and theory.

General Guidelines for Manuscripts

• All manuscripts should be submitted electronically.
• Manuscripts vary in length. The preferred length is between 1500 and 2500 words.
• Manuscripts should include an abstract of 500 words or less.
• Manuscripts should be double-spaced with one-inch margins and 12-point Times New Roman font.
• References should be listed at the end of the manuscript in APA format. Do not use footnotes or headnotes.
• The abstract and reference should be single-spaced.
• Figures, tables, & graphs should appear embedded within the document. Figures must be accompanied by a full-sentence caption. NOTE: Should your manuscript be accepted for publication, images must be sent separately, with the final accepted manuscript.
• The author name, position, work address, telephone number, fax, email address and a brief biography must appear on a separate cover page. No author identification should appear on the manuscript after the cover sheet.

Mathematics Guidelines for Manuscripts

• All variables should be set in italics.
• Numbers, parentheses, and mathematical operators should not be set in italics.
• Points, as in "segment AB," should be italicized, and labels in geometric figures must also be italicized.
• Do **not** use MS Word Equation Editor for mathematical expressions.
• Most of the mathematics in AMTNJ can be keyed using Word's capability for subscripts and superscripts. Simple fractions can be entered as a/b, for example using a solidus (/), or inline fraction bar, unless part of a larger expression that requires MathType.
• Use MathType sparingly for expressions and equations that cannot be typed using the keyboard, such as since most expressions set in MathType will require to be retyped in manually, should be a guiding principle; if the expression is not important enough to be broken out on its own line, then do not use MathType.

Submissions are accepted via GoogleForms at bit.ly/AMTNJ-Journal

Questions? Email us at journal@amtnj.org

EDITORIAL

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As we commence the year 2025, we are clearly in uncharted territory. Education is at a crossroads, and we all need to be vigilant and purposeful in navigating these waters. As an old commercial for a New York City clothing company once attested, “an educated consumer is our best customer.” AMTNJ is well positioned to assist in achieving our goals in mathematics education. The pillars we need to embrace include equitable teaching practices, embracing diversity, including everyone in the teaching and learning process, and celebrating the role of history in shaping our discipline. Our organization closed 2024 with two remarkable conferences at Brookdale Community College that were split across grade bands. The first conference was convened on October 25 and focused on K-5 education while the second met on November 22 centering on education in the 6-12 arena. Both conferences enjoyed enrollments of more than six hundred, which was really heartening. In our one hundred eleventh year, we have much to celebrate with more to come. I could not be prouder of the people who lead our organization and know that during rocky times, leadership matters.

The accomplishments of our organization are numerous and varied. Our conferences for the 2024-2025 academic year began in September. Chicago IL hosted the NCTM Annual Meeting and Exposition from September 25-28, 2024, at the famous McCormick Place Convention Center. Presenters from AMTNJ included Melissa Pearson, our NCSM Liaison during the NCSM portion of the conference, President Cheryl Fricchione, President-Elect Kara Teehan, Past Presidents Mark Russo and John Kerrigan, co-treasurer Bhesh Mainali, and yours truly.

Our PreK-5 fall conference was convened on Friday, October 25, 2024, at Brookdale Community College in Lincroft and was titled PreK-5, "Mathematics in Action: Bringing the Math Practices.” There were five keynote speaker sessions convening one-hour sessions that were grouped into five categories focusing on a single relevant issue in the PreK-5 curriculum. The keynote speakers included the themes of Academic Language presented by Adrian Mendoza, author of *Teaching Math to English Learners*, Revision presented by Dr. Amanda (Mandy) Jansen, the author of *Rough Draft Math, Revising to Learn*, Coherence and Consistency delivered by Dr. Karen Karp, author of *The Math Pact*, Productive Struggle convened by NCTM Immediate Past President Kevin Dykema, featuring the work of John SanGiovanni titled *Productive Struggle*, and Fluency presented by John SanGiovanni using his work *Figuring Out Fluency in Mathematics*. Our participants in attendance were fully engaged throughout the day. Other speakers included Dr. Frank Gardella of Hunter College, Dr. Eric Milou of Rowan University, Dr. Debra Gulick of Rutgers University, and Diedre Richardson from The New Jersey Department of Education.

Our 6-12 fall conference was convened on Friday, November 22, 2024, at Brookdale Community College in Lincroft and was titled 6-12, "Mathematics in Action: Bringing the Math Practices.” Paul Battaglia and Eli Luberoff, the CEO of Desmos were two of our featured speakers. Some other notable presenters included Dr. Eric Milou of Rowan University, Professor Jim Matthews of Siena College, Kathleen Carter, Professor Emeritus Jim Carpenter of Iona University, AMTNJ President Elect Dr. Kara Teehan, and First Vice President, Dr. Serbay Zambak, both of Monmouth University, and Diedre Richardson from The New Jersey Department of Education. Steven Leinwand, co-author with Eric on *Reinvigorating High School Mathematics*, likewise delivered a must-see presentation. Robyn Poulsen from Texas Instruments was one of our vendor presenters. In addition, we were able to honor the 2024 Max Sobel Award winner and AMTNJ Past President Andrea Bean. Our participants in attendance were fully engaged in the teaching and learning process throughout the sessions. The

AMTNJ ladder with their vision and the sessions they offered were in no small measure responsible for two outstanding and memorable conferences.

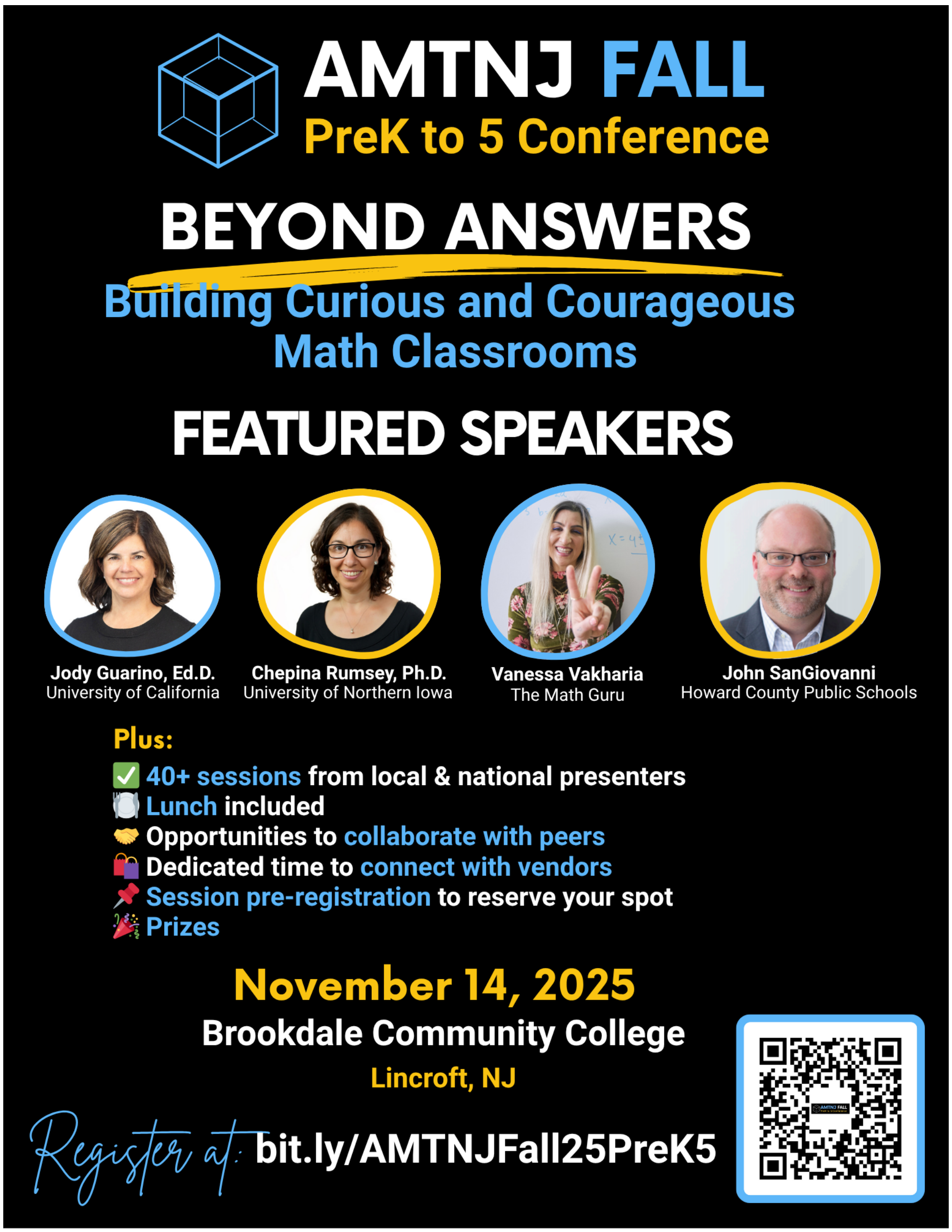
The current issue of The New Jersey Mathematics Teacher features six articles. Our compendium of excellent articles commences and concludes with two very energetic members of our AMTNJ Board of Trustees. In our first article, Dr. Pam Brett, our Executive Coordinator of Teacher Outreach and the founder of Blue Glasses Math provides us with a very timely piece titled *Emotions as Catalysts for Beliefs in Math Class*. This article is a must read; for beliefs and emotions shape our teaching and consequently student learning. The article additionally furnishes segments of student work and is nicely research based. Our readers will enjoy the ideas articulated. Our second article titled *Symptoms, Causes, and Consequences of Math Anxiety and Its Potential Remedies* by Dr. Harman Aryal from Stockton University nicely provides research-based evidence and intriguing strategies to overcome anxiety which often causes a pause in the learning of mathematics. Mathematics is challenging enough and adding barriers to the process is not productive. Our readers will be furnished with useful strategies to mitigate this hurdle. In our third article titled *High School Math in Action: A Trip to Bushkill Falls to Experience the Power of Indirect Trigonometric Measurement*, Jose Rodrigues and Nirmala Ramberran of Montclair State University furnish an exciting where students are actively involved in their own learning while exploring the beauty of mathematics and nature. Our readers will find such an application both novel and enjoyable while exploring mathematics in a different light. Dr. Ivan Retamoso from The Borough of Manhattan Community College of The City University of New York furnishes our fourth article that serves as a nice alternative to the traditional factoring of trinomials when the leading coefficient is different from one. His article titled *Proof of Why the Slide and Divide Method Works and its Modified Version Suitable for Educational Purposes* furnishes evidence to a factoring technique that is often taught as a rote procedure in the complete absence of any context whatsoever. Our fifth article titled *Using Excel to Learn Statistics Through Project Based Learning* is authored by Sun Ban of Merritt College and Jae Ki Lee, and Serine Ndiaye, both from The Borough of Manhattan Community College is nicely crafted and illustrates the value of the Excel program in statistical lore. Readers will embrace this platform. Our compendium of excellent articles concludes with Dr. Bhesh Mainali from Rider University who serves as our co-treasurer as well as one of my Associate Editors for our NJMT Journal along with his colleague Dr. Deependra Budhathoki from Defiance University in Ohio engages us with a very dynamic article on a subject that is of particular importance for all high school programs to partake in. The article is titled *Exploring Quantitative Reasoning Skills in School Mathematics*. This article is research based and features excellent examples on a course that is needed for every learned citizen and serves as a potential elective pathway for high school students not expecting to major in a STEM field during their collegiate careers. We thank all six authors and their coauthors where applicable for their fine contributions.

During the spring 2025 semester, AMTNJ sponsored two excellent professional development opportunities. The first professional development conference took place on April 8 from 8:00 A.M.-1:00 P.M. at the Rutgers-New Brunswick Lifelong Learning Center at the Division of Continuing Studies where AMTNJ hosted an intensive workshop on math intervention K-12, inclusion, meeting the needs of math students. Participants who attended partook in multiple focused learning sessions run by experts and engaged in an active afternoon session called an “implementation lab” where participants and facilitators worked to find solutions to current issues and discussed current trends. The second professional development opportunity occurred at Rutgers University, New Brunswick and was titled *Best Practices in Teaching Undergraduate Math Conference* on May 16th from 8:30 AM-4:30 PM. This was certainly relevant and open to teachers of math for any grade level as well. The location was The Rutgers Academic Building at 15 Seminary Place, New Brunswick, NJ 08901 on The College Avenue Campus. Our AMTNJ Past President John Kerrigan was one of the organizers for the conference.

A third conference was hosted by our sister organization, The New Jersey Association of Mathematics Teacher Educators (NJAMTE) convened on May 30 at The College of New Jersey in The Education Building, Room 113. The theme was *Empower, Engage, Retain: Investing in Teacher Excellence*. Several members on our AMTNJ Board will be presenting at this conference. Several of our fine AMTNJ Board of Trustees members presented at the conference including Pam Brett, Anne Paoletti Bayna, Serbay Zambak, and Nicole Panorkou. For more information, see their website at www.njamte.org.

This fall, AMTNJ will convene both a PreK-5 conference on November 14 and a 6-12 conference on November 21, both at Brookdale Community College in Lincroft. The theme for the PreK-5 Conference is titled Beyond Answers: Building Curious and Courageous Classrooms. The theme for the 6-12 conference is titled Beyond the Bell: Meaningful Math for Middle and High School. Please register for these must-see events. More details can be found on our AMTNJ Website.

Please feel free to reach out at any time. We are always seeking input from passionate and innovative educators as we navigate difficult times. Through persistence and perseverance, we will prevail!



Emotions as Catalysts for Beliefs in Math Class

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Abstract: Emotions play a pivotal role in learning and teaching mathematics and may act as catalysts in shaping students' beliefs about their abilities and the subject itself. Emotions are conceptualized as volatile and fleeting, interacting with students' cognition when doing mathematics. Beliefs are lenses through which students make meaning about their perceived mathematical competence and are stable and sustained over periods of time. This article describes an interactive cycle, modeling how emotions may catalyze student beliefs when they are doing mathematics. A vignette illustrates how this belief cycle may unfold in a snapshot of a class period. The discussion highlights the importance of teachers recognizing and leveraging students' emotions as an important way to foster productive beliefs.

Introduction

I recently visited a fifth-grade classroom in a culturally and economically diverse elementary school in New Jersey. This was my first time meeting the students in this particular class, and upon introducing myself as the new math coach, "Dr. Brett", they had questions. One student asked if I was a "psychological math doctor", which took me by surprise! I responded by posing a question to the class: "So…if I am a 'psychological math doctor' and psychology has to do with emotions, does that mean that there are emotions in math class?" The students responded with an eagerness that I was not expecting: some nodded their heads, others smiled. I asked them, "Like what kind of emotions? Do you experience emotions when doing math?" Without missing a beat, nearly every student's hand in the room shot up, offering to share. I recorded their responses on the whiteboard (See Fig. 1), and in those moments, every person in the room witnessed that doing math, in fact, can evoke powerful emotions.

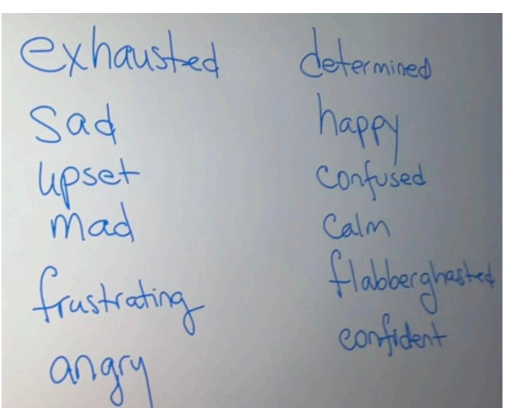

Figure 1

The spontaneity of this interaction reinforced, for me, that emotions do matter in the math classroom. And if fifth grade students exhibit such enthusiasm to share these powerful emotions when meeting a visitor to their classroom for the first time, we must consider how to acknowledge and leverage those emotions in math class.

Doing math is a deeply human endeavor, involving a synergistic landscape of emotional, cognitive, social, and behavioral experiences. Memories from math class can stimulate conversation in a way that we rarely encounter from any other school subject area. People often recall math class quite vividly, even decades later, recounting specific instances when they felt shame, embarrassment, excitement or joy. The powerful emotions that students experience during math class leave lasting residue forming their beliefs about the subject and about themselves as mathematical learners. These beliefs contribute to how students choose to engage with math and can have implications for informing teacher pedagogical decisions. In the sections that follow, I provide brief overviews of emotions and beliefs in the context of learning math. I present the Emotion-driven Belief Cycle as a framework to understand how emotions could be catalysts for students' beliefs about mathematics and about themselves as mathematical learners.

Emotions

Learning mathematics often evokes powerful emotions including uncertainty, frustration or anxiety as well as satisfaction, joy and elation. The emotions that students experience during math learning are not just side effects based on whether or not they enjoy math. Emotions interact with students' cognitive process when they are doing mathematics, often influencing the strategies that students use to solve math problems (Goldin, 2000, Quintanilla & Gallardo, 2022). A student's fear of making a mistake may influence their decision to play it safe and follow their classmates' lead, copying their steps to solve a math problem. Curiosity may drive a student to persist with a new or unique strategy to solve the problem. The literature suggests that emotions are fleeting, changing rapidly (DeBellis and Goldin, 2006) formed by the dynamic interplay of cognitive, physiological, and motivational processes (Pekrun, et. al, 2022). A wide range of emotions can make their way into math classes. A recent meta-analysis of 112 research studies identified 100 unique emotions relevant to mathematics learning (Schoenherr, Schukajlow, & Pekrun, 2025).

Emotions can be influenced by a variety of contexts in math learning, including the type of mathematical tasks; the messages students receive from teachers and parents about math; students' perceived or actual expectations about achievement in math (Pekrun, 2024). The social dynamics in the classroom also can invite emotions such as admiration of a peer's strategy or mathematical reasoning (Singh & Muis, 2021). Interactions with a teacher or peers in math class may bring relief when a solution path is validated by a classmate or defensiveness when an answer is challenged. (Brett, 2013, Pekrun et.al, 2002, Pekrun, 2024). Emotions can change moment to moment depending on a wide range of factors such as a students' misconception or breakthrough in understanding (Op 't Eynde, DeCorte, & Verschaffel, 2006).

Students who consistently experience emotions such as fear, discouragement and doubt when doing math tend to disengage or exhibit helplessness, potentially developing beliefs that they cannot be successful in math class. As DeBellis and Goldin (2006, p.134) explain: "A negative (affective) …may encode a search for safe procedures...When those procedures fail, frustration can turn to anxiety and despair. These emotional states may evoke avoidance, defense mechanisms, or reliance on authority, ultimately shaping global beliefs of mathematics- and self-hatred." While fleeting and volatile, repeated emotional patterns often have a lasting impact on the beliefs we hold about mathematics and about ourselves as mathematically capable individuals (Goldin, Roesken, Toerner, 2009). These emotions are what color our math stories. Regardless of students' beliefs about math, it is not the math itself that leaves these lasting impacts, it is the emotions they experience while learning math that they hold on to (Middleton, et.al, 2023).

Beliefs

Beliefs have been described as "lenses that affect one's view of some aspect of the world or as dispositions toward action" (Philipp, 2007, p. 259). Beliefs are considered to be more stable than emotions, they are developed over time and are deeply rooted with the individual's sense of self. Repeated emotional patterns can have a profound impact on the development of beliefs that students hold about themselves as mathematical learners (Goldin, Roesken, & Toerner, 2009). The consequences of strongly held beliefs are considered better predictors of human behavior than the outcome of the actions (Goldin, et.al., 2009). For example, students who have experienced success in math classrooms by following the lead of the teacher may filter learning through their beliefs, or "lenses," leading them to continue to follow the teachers' lead regardless of understanding. The

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example that comes to mind, and comes up often in my visits to middle school classrooms, is the "trick" for dividing fractions ("keep, change, flip"). This procedure, when followed precisely, directs students to the correct answer every time, students believe that it works and even when asked why the trick works, students will defend it as truth. Beliefs play a critical role in determining how learners approach challenges, persevere, and interpret their mathematical experiences. Bandura (1995) defined self-efficacy beliefs as cognitive in nature, stating: "Efficacy beliefs influence how people think, feel, motivate themselves, and act" (p. 2). Beliefs as lenses can also contribute to students' math identity, "dispositions and strongly held beliefs" (Aguirre, Mayfield-Ingram, & Martin, 2013, 2024). Math identity can profoundly impact students' sense of belonging in mathematics class and whether they can use mathematics meaningfully in their lives outside of school.

Emotions as Catalysts for Beliefs

"People won't remember what you said or did, they will remember how you made them feel." - Maya Angelou

Beliefs about mathematics develop over time, as by-products of repeated emotions experienced while doing mathematics (Goldin, 2000). Students' motivation to engage with mathematical learning is driven by their beliefs (Goldin, et. al, 2011), influencing the way students act and interact in math class (Philipp, 2007).

Throughout my 25 years working in math education, I have experienced the synergy of student emotions and beliefs in elementary and middle school classrooms. Whether serving as a math teacher across grades three through eight, a research fellow or as a math coach for elementary and middle school teachers, I have witnessed how emotions can reinforce or shift students' deeply rooted beliefs about mathematics. I have seen first graders approach math with curiosity and openness, not afraid to make mistakes and start over. Just like building and knocking down a tower of blocks, math can always be rebuilt. My eighth grade students entered my class expecting me to lead them through each procedure, helping them to safely land at the right answer. They revolted when I did not follow their script, especially when I asked them to explain their reasoning. These students thought that I did not know how to do math, because no teacher had ever asked them to justify their reasoning. These beliefs can be held as truth to the learner, determining how students use the knowledge and skills that they bring to new learning (Bandura, 1994). Beliefs can be difficult, but not impossible to shift, especially when the teacher makes room for emotions in math class.

The fleeting and contextual nature of emotions position them as powerful catalysts for shaping and re-shaping students' beliefs about mathematics and about themselves as mathematical learners. For example, a moment of success can surprise a student with pride, perhaps shifting a students' belief that they are capable of doing mathematics. Repeated frustration and despair may lead a student to question their competence in math, unless that frustration is positioned as an acceptable and necessary ingredient for grappling with mathematical problems (DeBellis and Goldin, 2006). Students who continually experience emotions such as curiosity, excitement and pride develop beliefs that bolster their self-determination as capable, independent problem solvers (Schoenherr, Schukajlow, & Pekrun, 2025).

The "Emotion-driven Belief Cycle" introduced in Figure 2, is driven by the emotions that learners experience when they are doing mathematics. I consider this cycle an extension of Goldin's (2000) affective pathways, which describes the interplay between local affect (i.e., emotions) and cognition when students are grappling with conceptually challenging mathematical concepts. In the Emotion-driven Belief Cycle emotions act as catalysts between the four stages, **belief, thought, action, result**. This process is iterative as students' deeply held beliefs about mathematics may take multiple experiences to shift their beliefs about math.

In this section, I offer two hypothetical examples that I have frequently seen in my work to illustrate how emotions act as catalysts for beliefs. I am using bold text to emphasize each of the four phases in the cycle, **belief, thought, action, result**. I invite you to reflect on your own experiences as a math learner/teacher or consider one of your current students' experiences doing math in the context of the belief cycle

A student enters a math class with a **belief**, holding the truth that he is not good at math, evidenced by disappointing grades, or frustration doing math homework each night. This belief leads to an internal dialogue when faced with a new math concept, a **thought**: "This problem is too hard for me, I will need help or I will mess up." These thoughts trigger another emotional response - perhaps hesitation or anxiety which influences his **actions**, such as avoiding the problem, rushing through it or giving up altogether. These actions lead to **results** which may include solving the problem with a large reliance on the teacher's expertise.

Conversely, a different student may hold a **belief** that math is easy for her, which carries a different emotional charge, often satisfaction or confidence. This belief leads to an internal dialogue, a **thought**: "I can do this problem." Then this thought triggers positive emotions, like eagerness or joy, feeling in control and valuing the process of solving the problem, which influence productive **actions**, such as engaging fully with the problem or exploring new methods. The resulting success confirms her satisfaction and reinforces her **belief**, further solidifying this student seeing herself as capable in mathematics.

Figure 2

The vignette below illustrates the Emotion-driven Belief cycle in action, during my interaction with a 5th grade special education student. I was hired by the school principal to provide mathematics coaching for the teachers in this elementary school. The interaction describes what happened when I was invited into a classroom by the teacher who was seeking my input on how she was teaching her multi-grade, multi-ability math class. As I entered the classroom, the teacher was working with a small group of students and I was beckoned by a student who was sitting by himself. The teacher looked at me, and nodded toward the student, affirming that I could respond to his request. I will refer to this student as "Kai", a pseudonym. The excerpt was recorded as field notes after the interaction.

Vignette - Kai's Belief Breakthrough

The student, Kai (name is a pseudonym), is working to complete a worksheet of single digit multiplication problems. The teacher has told Kai to practice his multiplication facts by completing all the problems on the worksheet. I (A) noticed Kai (K), picking at his pencil eraser, looking out the window, seemingly disengaged from the task.

(K) I don't know my multiplication facts.

(A): Hi Kai, what are you working on?

(K): Oh. These times [multiplication] problems. I don't know my facts…. *(perceived disappointment in his tone of voice and facial expression)*.

Well, I know some of them. But that's all I know.

I notice that Kai has completed all the "fives" facts, i.e., 3x5, 4x5...approximately 10 problems on a worksheet of 50 single digit multiplication problems.

(A): Hmmm…ok, well, I can see that you answered some of the questions already…

(K): Well, that's because they are the fives. I know my fives.

(A): Gotcha…so can you show me how you figured out 3x5?

I give Kai a blank piece of paper, and pointed to the paper inviting Kai to represent 3 x 5.

Kai draws 3 circles and places 5 dots in each circle.

(K): pointing at each circle, that's easy, 5, 10, 15…I know how to count by 5s.

(A): Hmmm. Ok. That makes sense to me. I wonder if you could use that picture that you drew to figure 3 x 6?

Kai taps his pencil on his forehead, looking at the paper.

(A) How could you change that picture *(I hand Kai a different color pen)* so that it would show 3x6 instead of 3x 5?

Kai places one more dot in each of the 3 circles - gasp! (perceived relief in his body language, tone of voice and facial expression)

(K): Wait. So, 3 x 6 is just 3 more? Because it's an extra number! 6 is 3 more than 5!

(A): What do you think?

(K): Can we try that for allllll of the facts I don't know? *(Kai seems joyful and expresses his curiosity about using this approach to answering the multiplication facts he doesn't know from memory.)*

Kai proceeded to search for "sixes" facts to apply his new understanding while I sat silently beside him, offering my physical presence, but not intervening in his thought process.

This anecdote highlights how emotions might work as catalysts within the belief cycle, shifting Kai's beliefs about mathematics and his **beliefs** in his ability to recite his multiplication facts. Likely shaped by his prior experiences and reinforced by being assigned the task of "practicing" his facts on the worksheet while the teacher worked with other students, Kai articulated this belief, by stating, "I am not good at multiplication." These emotions prompted a **thought**, "I don't know these facts, so I can't do this." Kai expresses his overwhelm at the perceived enormity of the task in front of him, he is disengaged from the task, lacking initiative to complete the worksheet, picking at his pencil eraser, looking out the window. This **"action"** is perceived to be Kai's way of coping in the moment with his uncomfortable emotions. The result of this action is that only a portion of the worksheet is completed, reinforcing Kai's **belief** that he does not know his facts.

Kai experienced a shift in his perspective about his ability to do math during our interaction, first expressing his self-doubt about completing all of the problems on the page. Through exploratory action-taking, Kai used what he knew about 3 x 5 using three groups of five dots to determine the product, demonstrating that the "fives" are easy to skip count, "5, 10, 15". When I asked Kai, "how could you change that picture…", he took **action**, built on this model, adding "one more" in each group to determine the product of 3 x 6. In this moment, Kai experienced curiosity and joy when he noticed "...3 x 6 is just 3 more?" and asked, "Can we try that for allllll of the facts I don't know?". This shift prompted Kai's to take **action** that led to completing more of the assignment. Kai experienced an emotional reaction (relief) when he realized that 3 x 6 is simply three more than 3 x 5. Next, and perhaps most importantly, Kai expressed his curiosity in having the power to use this strategy with other problems that he does not know, specifically, the "sixes."

Kai's experience illustrates the dynamic interaction between the components of the belief cycle - beliefs, thoughts, actions and results - driven by the emotions Kai experienced, self-doubt, curiosity, relief. This vignette demonstrates how leveraging students' emotions, even negative thoughts such as self-doubt can empower students to take action and witness empowering results. In this case, Kai's breakthrough in understanding resulted in his new conception of using related facts and gave him the confidence to practice this new strategy.

Implications for Teachers

When students enter a math classroom, they bring supplies to support their learning including pencils, notebooks, calculators and folders. Students also bring a collection of invisible supplies; emotions and beliefs they have about mathematics and about themselves as mathematical learners. Though unseen, emotions and beliefs can have a profound impact on students' willingness to take risks, participate in discourse about mathematics and even their overall achievement. Effective and equitable math teaching calls for students to engage in productive struggle, grapple with mathematical ideas (NCTM, 2014) and go deep with mathematics (Aguirre, et. al., 2013, 2024). In these settings, effective teachers are empahetic teachers, equipped to respond in ways that cultivate productive beliefs about mathematics and about themselves as mathematical problem solvers.

Three things teachers can do to cultivate productive beliefs in math class:

I. Do Math

Teachers need to do math to empathize with students' emotions. When teachers grapple with mathematical ideas, they learn how emotions arise when learning math. By experiencing how their own emotions interact with their beliefs about mathematics, teachers can learn to recognize what prompts the emotions (Schoenherr, Schukajlow, & Pekrun, 2025). These experiences help teachers to develop empathy for their students, and also recognize the importance of helping students to recognize emotions as natural and necessary to the learning process (Op 't Eynde, De Corte, & Verschaffel, 2006, p. 204).

II. Plan for Instruction

Teachers can carefully select, create or modify "rich" mathematical tasks and routines that consider the cognitive and affective aspects of learning mathematics (Radmehr, 2023). For example, tasks found on nrich.maths.org, YouCubed, Math for Love, and Building Thinking Classrooms (Liljedahl, 2020). Routines that evoke curiosity and puzzlement that support mathematical thinking include number talks, Parrish (2014), Steve Wyborney's Estimysteries, Splat and Open Middle. These routines invite students to take risks, revise thinking and feel the feelings that come along with doing mathematics.

III. Teach with Empathy

Teachers can disrupt negative belief cycles by fostering classrooms where students feel safe to make mistakes and revise their thinking (Jansen, 2020). By skillfully knowing when to intervene, step back, or offer encouragement, teachers help students leverage their emotions as essential tools for learning mathematics.

As we consider next steps, teacher preparation and professional development should include elements that help teachers recognize the antecedents of students' emotions, such as prior experiences, task demands and classroom norms. By integrating emotional awareness, we can move closer to creating classrooms that support students' academic progress while transforming beliefs that all students are capable of learning mathematics.

Resources That Can Transform Your Students' Beliefs

NRICH – Rich math tasks that foster deep thinking.

YouCubed – Inspiring research-based math tasks and professional learning.

Math for Love – Games and lessons that spark joyful math learning.

Building Thinking Classrooms – Tasks and practices that foster classrooms where students do the thinking.

Number Talks – Routines for developing number sense and mental math (see Number Talks by Sherry D. Parrish).

Steve Wyborney's Esti-Mysteries and Splat! – Free interactive estimation routines and visual math puzzles.

Open Middle – Challenging math problems with multiple ways to approach and solve.

Struggly – Digital platform with visual, standards-aligned tasks that emphasize the value of purposeful struggle in math.

Lesson Openers on Polypad by Amplify – Short activities that get students thinking and discussing mathematics.

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Symptoms, Causes, and Consequences of Math Anxiety and Its Potential Remedies

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Abstract: Math anxiety affects individuals across ages and educational levels, causing apprehension, avoidance, and stress when faced with mathematics. This review article explores the factors associated with math anxiety and discusses effective strategies for its reduction, such as collaborative learning, a supportive environment, and technology use, including AI tools like ChatGPT. These strategies are crucial for alleviating math anxiety and promoting positive attitudes toward mathematics. Overall, this article provides one-stop-shopping for educators, students, and researchers who intend to learn about math anxiety and implement some strategies to overcome the challenges associated with it.

Introduction

Math anxiety has been a part of the human experience for centuries. The verse, "Multiplication is vexation ... and practice drives me mad," goes back at least to the 16th century (Dowker et al., 2016, p. 2). In 1954, schoolteacher Mary Gough documented that most of her secondary-level students experienced anxiety, especially when thinking of failure in math. Later, in 1957, Dreger and Aiken introduced the concept of "number anxiety," and math anxiety received increasing attention thereafter. In 1972, Richardson and Suinn formally defined math anxiety as "feelings of tension and anxiety that interface with the manipulation of numbers and solving of mathematical problems" (pp. 551–552)." Tobias and Weissbrod (1980) defined math anxiety as "panic, paralysis, and mental disorientation" when solving problems (p. 65). Due to its aversive impact, there is a need for a shared responsibility among stakeholders to overcome it.

This article aims to provide knowledge about math anxiety and offers practical, research-based strategies for its reduction. By exploring the environmental, intellectual, and cognitive factors of math anxiety, the article provides actionable insights to further educational practices that can foster better academic and emotional outcomes for students.

Symptoms and Existence of Math Anxiety

Symptoms of math anxiety vary among individuals, but it can be narrowed down into three major categories—physical, mental, and emotional. Physical symptoms include nausea, sweaty palms, restlessness, blurred vision, queasiness, nail-biting, and increased heart rate (Aryal, 2022; Ashcraft, 2002; Chang & Beilock, 2016). Mental symptoms include intrusive thoughts, inability to concentrate, jitteriness, and mind-blanking (Aryal, 2022; Plaisance, 2009; Ruffins, 2007). Emotional symptoms include feelings of helplessness, low confidence, and extreme nervousness (Chang & Beilock, 2016; Kitchens, 1995; Mattarella-Micke et al., 2011). Math-anxious individuals may experience these symptoms while learning math, taking exams, doing homework assignments, or when confronted with math or math-related situations.

Math-anxious individuals often carry a range of personal and emotional experiences. For some, even hearing the word mathematics evokes apprehension, frustration, or discomfort. Others perceive it as a foreign or meaningless subject—a collection of symbols and abstract rules disconnected from everyday lives (Luitel, 2009). Yet, others come to believe, often from an early age, that only those individuals who are "born with math genes" can succeed in mathematics (Devlin,

In the U. S., around 17% of the population suffers from high math anxiety (Ashcraft & Moore, 2009), and about 93% of adults experience some level of it (Blazer, 2011).

2000). These beliefs demotivate people from engaging in math-related activities, leading to fear, aversion, and even phobia toward the subject.

Math anxiety is an issue among various age groups. Program for International Student Assessment (PISA) found that 59% of 15-year-olds across 65 countries often worried about math classes; 33% felt tense doing math homework; and 31% were nervous solving problems (Organization for Economic Co-operation and Development [OECD], 2012). In the United States, around 17% of the population suffers from high math anxiety (Ashcraft & Moore, 2009), and about 93% of adults experience some level of it (Blazer, 2011). Likewise, only 7% of US elementary preservice teachers had positive experiences with learning math in their school; the rest 93% reported feeling anxious, especially during Grades 3–4, Grades 9–10, and in the first year of college (Jackson & Leffingwell, 1999). In addition, Yeager (2012) found that about 25% of university and 80% of community college students greatly suffer from math anxiety. Regardless of age group, math anxiety can impact people's everyday numeracy and higher-level math learning throughout their lives (Oxford & Vordick, 2006).

Causes of Math Anxiety

Substantial research has identified numerous factors associated with students' learning experiences and math anxiety. This article summarizes some major causes that significantly contribute to or exacerbate anxiety in students as they engage with mathematics.

Students' Negative Experiences

Students' negative experiences, such as failing exams, receiving poor grades, or suddenly being called on for a response in front of their peers, can create a lasting aversion to math (cf. Aryal, 2022; Barrows et al., 2013; Finlayson, 2014). Harsh criticism from teachers or parents can deteriorate a student's self-esteem and foster a fear of making mistakes. When mistakes are punished rather than used as a learning opportunity, students may gradually develop a negative perception and attitude towards math (Ashcraft & Krause, 2007).

Lack of Confidence

A lack of confidence in one's mathematical abilities can make even simple tasks seem daunting. This self-doubt can create a cycle where anxiety prevents from effective learning, which causes poor performance, ultimately reinforcing math anxiety. A study by Inferio et al. (2024) found that students' low self-efficacy and their negative views about mathematics can contribute to anxiety, which in turn disrupts their ability to engage with the subject and perform well. These students are more likely to avoid solving mathematics problems and shy away from mathematics-related activities. Students who frequently compare themselves to their peers who seem to grasp concepts more quickly may feel incompetent, further losing their confidence.

High Expectations

High expectations can bring immense pressure on students to perform well. Parents with high expectations can put pressure on their children to opt for STEM majors in college, which require sophisticated mathematics proficiency. This causes students to feel stressed and frustrated when forced to choose such majors rather than given permission to decide for themselves. Moreover, some students set unrealistic goals for themselves. When students do not meet their expectations, they tend to experience chronic stress and anxiety. Beilock et al. (2010) found that high levels of pressure, whether external or self-imposed, are strongly associated with increased math anxiety and performance deficits.

Attitudes and Beliefs

Parents' and teachers' attitudes towards mathematics affect students' math anxiety and performance (Beilock et al., 2010; Chang & Beilock, 2016; Scarpello, 2007; Sun & Pyzdrowski, 2009). If teachers and parents themselves experience math anxiety during their study, it is highly possible that they might inadvertently transmit that anxiety to their students and children (Dagaylo & Tancinco, 2016). Similarly, teachers' beliefs and their negative experiences with mathematics affect their students' math anxiety. For instance, female teachers' beliefs about mathematics anxiety were found to impact girls' math achievement (Beilock et al., 2010).

Teaching Methods

Most mathematics classes are still teacher-centered, where students have little to no opportunity to share their ideas and experiences with their peers or instructors. Teacher-centered approaches emphasize memorization over understanding. Jackson and Leffingwell (1999) found that teaching methods that do not foster a deep understanding of mathematical concepts are critical to developing math anxiety. Moreover, rigid and fast-paced teaching methods that do not accommodate diverse learning needs can lead to frustration and anxiety in some students. Ramirez et al. (2013) indicate that teaching methods that are not adaptive to student needs can increase math anxiety, especially if they do not foster understanding and confidence.

Assessments and Tests

Assessments and tests, especially high-stakes tests, standardized tests, and timed tests, intimidate students as they approach closer to the testing dates. High-stakes tests, such as the exit examination for high school graduation, and standardized tests, such as The Scholastic Aptitude Test (SAT) and The American College Test (ACT), create immense pressure on students, affecting their performance. This is aligned with the findings of Yasmeen et al. (2016), in which high math-anxious students scored lower overall on the SAT than low math-anxious students. These scores are not only used to assess students' performance, but also used to evaluate teachers, schools, and school districts, raising anxiety among stakeholders. Additionally, timed tests add extra pressure on students, causes them to rush through the problems, that can lead to mistakes and reinforce negative beliefs about their abilities, further worsening their test anxiety.

Low Self-Efficacy

Self-efficacy refers to an individual's belief in their ability to succeed in specific tasks or goals. People with low self-efficacy in math often perceive mathematics tasks as more difficult than they are, increasing the likelihood of experiencing failure and feelings of inadequacy and anxiety. Pajares and Graham (1999) found that students with low self-efficacy are more likely to experience math anxiety due to negative perceptions of their math abilities, which can prevent skill development and increase anxiety. In addition, individuals with low self-efficacy are more likely to give up quickly, believing that they are not capable of solving a problem. Schunk and Pajares (2002) highlight that persistence is strongly linked to self-efficacy—students with higher self-efficacy are more likely to persevere and succeed.

Introversion

Introversion, a personality trait characterized by a preference for solitary activities and a lower level of social interaction, can contribute to math anxiety in several ways. Introverted students usually prefer to avoid communicating with their peers and participating in many academic and social growth activities (Ibaishwa, 2014), which can be challenging in collaborative classroom settings. Research by Anderson and Harvey (1988) shows that introverted students are much more likely to internalize problems such as depression and anxiety disorders (Anderson & Harvey, 1988). Introverts may also feel uncomfortable performing in front of their peers, such as solving math problems on the board, explaining their solution strategies in front of the class, and interacting with instructors and peers. Anderson and Harvey (1998) reported that the students, especially those who were shy in nature, experienced anxiety in asking questions of their instructor and responding to them in front of their peers.

Cognitive Impairment

Cognitive factors are associated with working memory. Working memory, also known as a limited short-term memory system, generally draws an individual's attention to the relevant task at hand while inhibiting unwanted information (Miyake & Shah, 1999). It is believed that worries and intrusive thoughts associated with math anxiety reduce the efficacy of working memory needed to engage with mathematical tasks that require a high level of cognitive ability (Chang & Beilock, 2016). Moreover, when highly math-anxious individuals confront a mathematical task, their brains exhibit increased activity in the dorsal-posterior insula area, indicating visceral pain in that area (Lyons & Beilock, 2012). It is also found that high levels of math anxiety are highly correlated with increased salivary cortisol concentration, which might predict poor math performance (Lyons & Beilock, 2012).

Consequences of Math Anxiety

Several studies have highlighted the consequences of math anxiety, where the most widely discussed consequences include mathematics avoidance, poor mathematics performance, and deficiency in working memory. Math-anxious individuals begin to take fewer mathematics courses and tend to avoid mathematics, mathematics-related situations, and even STEM careers (Hembree, 1990). Likewise, math-anxious individuals avoid exerting effort in mathematics, feeling a need to escape as a spider-phobic would avoid spiders (Choe et al., 2019). This avoidance perhaps stems from their past experience of failure, humiliation, or fear of mathematics. Choe and colleagues also found that math-anxious individuals often avoid high-effort mathematics tasks, believing the effort required outweighs the potential benefits.

Another widely investigated topic in this area is the relationship between math anxiety and math performance. Numerous studies have found a negative association between the two (Ashcraft & Kirk, 2001; Chang & Beilock, 2016; Dagyalo & Tancinco, 2016). That is, high-achieving students showed low levels of math anxiety, and low-achieving showed high levels of stress (Dagaylo & Tancinco, 2016). Consequently, low math-anxious students scored significantly higher than moderately or highly anxious students, and moderate math-anxious students scored significantly higher than high math-anxious students (Zakaria & Nordin, 2008).

The cognitive effects of math anxiety are well-studied area in psychophysiology and neurology, particularly in relation to working memory. Ashcraft and Kirk (2001) found that math anxiety disrupts working memory when individuals face arithmetic or mathematics-related tasks. Young et al. (2012) among 7–9-year-olds found an association between hyperactivity and abnormal activity of the amygdala, a brain region associated with fear and negative emotions. As a result, highly math-anxious individuals often suffer from various disconcerting brain activities that reduce their efficacy when faced with challenging mathematical tasks.

Remedies of Math Anxiety

Due to the negative impact of math anxiety on students' lives and academic careers, it is essential to take meaningful steps to prevent and reduce it. The National Council of Teachers of Mathematics (1989) recommended teachers implement various strategies, such as using diverse learning methods, welcoming students' mistakes, connecting mathematics to real life, and focusing on reasoning over memorization. Addressing math anxiety requires a comprehensive approach that considers the environmental, emotional, and cognitive factors associated with math anxiety. The following sections describe some strategies, supported by research, that have been effective in alleviating math anxiety.

Providing Positive Reinforcement

Positive reinforcement can be a powerful tool in reducing math anxiety, as it encourages and rewards desired behaviors and efforts. Acknowledging and providing immediate praise for students' thoughtful, correct, or incorrect attempts can create a supportive and motivating environment. Positive reinforcement increases confidence, enhancing their willingness to engage with the subject and valuing the effort in the learning process over perfection. Additionally, verbal appreciation, such as great work, very impressive, well done, good try, and non-verbal rewards, such as small incentives, can be a milestone for further motivating students to participate and try their best. Positive reinforcement boosts self-esteem and fosters a growth mindset, where students believe their abilities can be improved with their efforts, thus reducing fears and anxiety associated with mathematics. (Dweck, 2006).

Creating Supportive Environment

A supportive environment is essential for reducing math anxiety by fostering a safe and encouraging space for learning. When students feel supported by teachers and peers, they are more likely to take risks, ask questions, and engage with math without fear of judgment. Teacher support and a positive classroom environment have been shown to lower anxiety and improve attitudes toward math (Hemmings & Kay, 2010). In such an environment, open communication is encouraged, and mistakes are seen as natural and valuable learning opportunities rather than failures. This reduces the pressure for perfection, helping students to approach the subject with greater confidence and less anxiety (Boaler, 2016; Dweck, 2006).

Focusing on Conceptual Understanding

Building conceptual understanding in mathematics is vital for reducing math anxiety, as it helps students grasp the underlying principles and relationships behind mathematical concepts rather than merely memorizing procedures. With a strong conceptual foundation, students approach new problems with more confidence, as they are less dependent on recalling the memorized algorithms. This understanding also enables them to transfer the learned skills across disciplines, developing confidence and reducing anxiety. Educators who emphasize conceptual learning foster a more positive and secure attitude toward math and help reduce the intimidation often associated with the subject (Hiebert & Grouws, 2007).

Incorporating Formative Assessments

Incorporating formative assessments can reduce math anxiety by offering continuous feedback that guides students on their progress and improvement. Unlike summative assessments that evaluate learning at the end of the course, formative assessments are typically ungraded or minimally weighted, which reduces performance anxiety, allowing them to focus on learning. Techniques like questioning, collaborative activities, short quizzes, exit tickets, and reflective journals provide immediate feedback, helping students reflect on their misunderstandings early before they escalate. (Hattie &Timperley, 2007). Formative assessments also enable teachers to adjust their instruction based on student needs. When students' needs are addressed quickly, they feel more confident and motivated toward mathematics learning.

Use of Technology

Technology offers numerous tools that can help reduce math anxiety by providing personalized learning, interactive engagement, and immediate feedback. These features allow students to learn at their own pace without feeling pressure to keep up with their peers. A meta-analysis of more than 40 articles revealed that developmental software, including online discussion boards and applications, reduces anxiety by providing a collaborative platform to express their feelings and receive quick feedback (Sun & Pyzdrowski, 2009). Dynamic and interactive tools, such as Desmos, GeoGebra, and educational games, can also make mathematics learning more engaging, enjoyable, and conceptual.

Artificial intelligence (AI), such as ChatGPT, can help reduce math anxiety by providing personalized and non-judgmental support for students. AIs are easily available and accessible and can provide adapted examples, explanations, and problem-solving techniques based on individual learning pace. Pane et al. (2015) stated that tech-based personalized learning can improve educational outcomes by addressing students' unique needs and reducing anxiety caused by one-size-fits-all approaches. Additionally, AI tools provide a safe learning environment where students can ask questions without time restrictions and fear of evaluation—a key factor in reducing math anxiety (Ashcraft & Ridley, 2005).

Journal Writing

Some studies have shown that asking students to write about their feelings can reduce their math anxiety. Providing opportunities for students to write short expressive journals minimizes the worries about test performance, especially for math-anxious students (Beilock, 2008). For example, a study by Ramirez and Beilock (2011) found that individuals who were asked to write a 10–minute note about their feelings and thoughts just before the exam performed significantly better than those who did not. This shows that writing about traumatic or negative contexts and experiences benefits math-anxious individuals (Klein & Boals, 2001). Engaging individuals in writing about their performance worries may free up working memory resources, helping them better identify, differentiate, and understand their emotional experiences (Gohm & Clore, 2000). Thus, allowing students to express their fear of mathematics performance, tests, or problem-solving can reduce their math anxiety.

Active Learning

Active learning, an instructional approach that actively engages students in the learning process through collaboration, discussion, problem-solving, case studies, and other hands-on activities, can be very effective in reducing math anxiety. There are various forms of active learning, such as inquiry-based learning (IBL), flipped classroom, discourse learning, and project-based learning. When students are engaged actively, they are more likely to participate in the tasks and activities without feelings of stress and anxiety. Freeman et al. (2014) found that active learning increases student performance and engagement,by in which students are engaged through deep engagement with mathematics and collaboration with peers (Yoshinobu & Jones, 2013). Students in IBL classes experience a higher level of growth in understanding mathematics concepts and an increased positive attitude towards mathematics (Laursen et al., 2011). IBL also enhances students' learning by empowering them with reasoning, critical thinking, and problem-solving abilities (Aryal, 2022). It also encourages active engagement and curiosity, which are critical factors in reducing anxiety and increasing motivation (Cakir, 2008).

Conclusion

Math anxiety is a multifaceted issue influenced by various factors, including personality traits, intellectual abilities, environmental contexts, and specific educational practices. Some factors, such as high-stakes testing, a lack of supportive learning environments, and low self-efficacy, contribute significantly to the development and continuation of math anxiety. In contrast, several strategies can mitigate anxiety, including active learning, positive reinforcement, a supportive environment, and technology, such as AI tools like ChatGPT. These approaches emphasize engagement, motivation, personalized support, and conceptual understanding, which collectively foster anxiety-less educational experiences. Also, by providing timely feedback and encouraging a supportive and collaborative learning environment, educators can help students overcome their math anxiety and enhance their overall mathematical competence and performance.

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High School Math in Action: A Trip to Bushkill Falls to Experience the Power of Indirect Trigonometric Measurement

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Abstract: A ubiquitous question heard in math classrooms across the country is "when will we ever use this". Particularly in geometry class, where abstraction seems to take precedence over reality, students would greatly benefit from interactive experiences to apply the concepts they are learning. To this end, here we propose a field trip for north Jersey schools in which students can indirectly measure the heights of waterfalls at Bushkill Falls, Pennsylvania by using right triangle trigonometry. By constructing clinometers and measuring horizontal distances, they can estimate the height of the falls to a reasonable accuracy. We provide a framing for the field trip, logistics of the site itself, requisite background knowledge, an overview of a lesson plan, along with possible interdisciplinary connections.

Introduction

Every mathematics teacher has surely heard "when will I ever use this" from a student at least once. Though we often think students use it as an excuse for not taking initiative in their own learning, there's no doubt that most of the math we teach our students is never presented to them through genuine applications. In his seminal lament, mathematician Paul Lockhart loathingly admits "There is nothing quite so vexing to the author of a scathing indictment as having the primary target of his venom offered up in his support. And never was a wolf in sheep's clothing as insidious, nor a false friend as treacherous, as high school geometry" (Lockhart, 2009, p. 67). A forced textbook word problem isn't the same as engaging with mathematics in an authentic setting. Given the ubiquity of this issue, it should be surprising then that math classes rarely go on field trips. In this paper, we propose a field trip to Bushkill Falls, Pennsylvania, which serves to illustrate the applications of indirect trigonometric measurement in a high school geometry, algebra 2, or precalculus class, with an overarching theme of estimation. We examine the framing of the field trip, the site logistics, the background knowledge students need, the lesson plan, and possible interdisciplinary connections.

Framing of the Field Trip

For the purposes of this trip, we have chosen an overarching theme of estimation, but a focus on indirect measurement for the day of the field trip. In this way, we hope that students have an example of framing strong estimation questions and carrying out all the work necessary to answer these questions. Simply asking students to write a few questions involving the topic of estimation and then answer them is too open-ended to elicit good work from students without sufficient scaffolding or prior exposure (Pólya, 1945). By having a very specific task to complete during the field trip, the educator can then have students explore their own estimation inspired project ideas offsite. In this way, we get the most value out of our time and money.

For the field trip, students will first take a hike along the waterfall trail at Bushkill Falls. The trail features eight waterfalls that are impossible to measure directly thanks to their size, slope, and distance from the trail. Students will be given a worksheet to document all the things that they see and wonder. Then, they will each be asked to select a waterfall and, using the worksheet as a scaffold, estimate the height of their waterfall by using basic right triangle trigonometry. We should note that while students will have seen similar problems in class, they will ultimately need to decide what specific measurements and variables they must account for in this real world situation, which educator Jo Boaler emphasizes is the real value to these sort of authentic tasks compared to the pseudo context word problems students generally encounter in class (Boaler, 2016, p.53). As a reflection, students will compare their estimated height to the true height and think of possible sources of error. The hope is that they can then take their own "I notice, I wonder" questions and answer them through this framework.

Logistics of the Site

Bushkill Falls is a privately owned site located one hour and fifteen minutes from Montclair, New Jersey, making it an accessible day trip for most North Jersey schools. That it is a privately owned natural world might be a concern for some (and could make for a good debate), but it comes with the advantage of being well maintained with walkways and barriers that keep visitors from slipping and falling. Ticket prices for children thirteen and above are $15, which is a feasible price once transportation costs are factored in. Reservations for group trips are required at least two weeks in advance, and group sizes must be between 25 and 50, fitting the usual size of a class field trip. For a group of up to 50, the two chaperones are admitted free of charge. What's especially useful about this site is that there are eight waterfalls in a relatively small area, so students will get a chance to measure a different waterfall instead of having everyone crowded around one. Cell phone connection is weak at the site, which will force students to be present in the moment and focus on the experience (Bushkill Falls, 2008).

We recommended that this field trip be staged at the end of the school year, ideally May, because the site is closed during the winter season. Additionally, the hope is that students can create their own projects after attending the site, and by the end of the year students will have learned a great deal of content that probably hasn't been presented in its authentic settings. Finally, the end of the year is rife with activities that disrupt regular class sessions, so a project is a good way to have students remain engaged without missing out on new material. In this way, students can do "real" mathematics by posing and answering their own questions (Boaler, 2016, p. 27).

If the reader is interested in this idea but not the site itself, here are some other waterfall sites in the tristate area: Paterson Falls (wheelchair accessible, with assistance), Dingmans Falls (handicap accessible), and Ricketts Glen State Park (further in Pennsylvania, more waterfalls, but no barriers and slippery stone steps). It should also be noted that this indirect measurement method can be applied to other objects. For example, it might make more sense for students in a particular school to measure the height of man-made landmarks or buildings, like the High Point state monument. Measuring the heights of trees might be cliché, but students in California measuring the heights of redwood trees would see an excellent demonstration of the power of indirect measurement.

Background Knowledge

There are several methods to calculate waterfall heights, but as the field trip is currently framed, students need to understand basic right triangle trigonometry (Cheng, 2015). It is sufficient to know the SOHCAHTOA ratios, be able to work with degree measurements, and understand angles of elevation and depression (Bragg, p.24). Prior exposure to indirect measurement problems would also be advised, to ensure that students have the requisite algebra skills such as being able to solve proportions. A detailed description of measuring heights is presented in the Don Bragg article cited in the sources. This article might be beneficial to assign before or after the field trip, de-

Making the clinometers is a great way to get students to talk about angles of elevation and depression and complementary angles, since the angles they see on their clinometers are not necessarily the ones they will use in their diagrams or calculations

pending on how much "discovering" the teacher wants their students to do.

For this trip, students are also required to measure the angle of elevation and depression using a clinometer. These clinometers are easy to make with a straw, protractor, string, and weight, but it is advised that they are made and tested before arriving at the site. Making the clinometers is a great way to get students to talk about angles of elevation and depression and complementary angles, since the angles they see on their clinometers are not necessarily the ones they will use in their diagrams or calculations. It would be beneficial to have students read through the clinometer directions posted on Cambridge's NRICH project site since it gives them practice in following directions, visualizing in three dimensions, and persevering without unnecessary teacher input (Making Maths: Clinometer).

Finally, for the "I see, I wonder" component of the lesson, exposure to Dan Meyer's 3-act tasks or similar warmups would be suggested (Meyer, 2018). Students often have difficulty coming up with their own mathematically interesting questions since educators and textbooks do that for them. 20th century education critic John Holt noted students' awareness of this matter even in his day, quoting one who said "If people give me the questions I can remember most of the answers, but I can never remember the questions" (Holt, 1995, p. 146). Question generation is a task that most students just haven't had practice with, and so using these sorts of warm ups on a regular basis will make the field trip an extension of what they are doing in class.

Lesson Plan Overview

Now, we will summarize the lesson plan for this field trip assignment. We have divided this lesson into a warmup, a task, and a reflection. Prior to arriving at the site, students will be placed in groups of three and each will be given a job–measurer, photographer, and note taker–and provided with the necessary materials: phone, tape measure, clinometer, clipboard, and worksheet. Upon arrival at the site, students will begin with the warmup activity, which asks them to spend about an hour walking the trails and recording what they notice and what they wonder. The note taker should record their group's thoughts, and students should attempt to write some mathematically leaning questions about what they see. For example, students might write "I wonder how much water is flowing down that waterfall per minute." NCTM has an excellent list of resources in the Notice and Wonder section of their website that teachers can use throughout the school year. Students should also take photos so that they have sufficient records of the trip for any potential post-field trip projects.

After the warmup, students will complete the indirect measurement task. Each group will be assigned a waterfall (there are eight) and asked to complete a worksheet which guides them through the process of estimating the waterfall's height. In short, students will measure angles of elevation and depression twice from two different points and use the tangent ratio to calculate the waterfall's height. The accuracy of this final height is contingent on successfully measuring the angles and the distance between the measurements. If tape measures are not available or unwieldy, consider using the Measure app available on iPhones to measure horizontal distances. A variety of apps are available, but their accuracy should be a point of discussion amongst students. For the reflection part of the lesson, students will use the park brochure to determine their waterfall's true height, compare it to their calculated height, and note possible reasons for any discrepancy.

Ideally, the field trip lesson will have shown students how to effectively frame a mathematical question, answer it, and assess its sensibility. Ideally, students would then create their own projects where they choose one of their warmup questions or observations, and then refine, research, solve, and assess it. This would be an excellent illustration to students "that mathematics is an art. Math is something you *do*" (Lockhart, p. 92). Thereby, students are actively challenging the notion that "mathematics content becomes nothing more than a set of isolated skills" (NCTM, p. 71).

The standards for mathematical practice that are addressed by this trip are MP1, 3, 4, 5, and 6. MP1, make sense of problems and persevere in solving them, is relevant because students will be carrying out computations in a very unfamiliar setting with many more variables than they are accustomed to. MP3, construct viable arguments and critique the reasoning of others, is primarily used in the reflection section of the lesson, though students should productively argue throughout the solution process, attending to key elements such as how to properly measure angles and distances (attending to precision, MP6), how much to round, and which ratios to use. The entire lesson is centered around modeling with mathematics (MP4) along with using tools appropriately (MP5) to accomplish this (New Jersey Student Learning Standards: High School, p. 4). The relevant New Jersey learning standards are: 4.G.SRT.6 (understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles) and 4.G.SRT.8 (use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems) (New Jersey Student Learning Standards: High School, p. 24).

Interdisciplinary Connections

In terms of interdisciplinary connections, the site boasts a wide array of stunning scenes. Exposed shale is dotted with mosses, ferns, and rhododendrons, the water dripping from them conjuring images of jungles. The entrance has taxidermy of extirpated and extant native wildlife, and the water itself is dyed with tannins from fallen leaves. Thus, there are countless science related topics that can be explored. Some of these topics are discussed below and can be briefly mentioned to students before or after the trip to facilitate potential follow up projects.

In keeping with the overarching estimation theme, here are some ideas. Statistical sampling techniques can be used to calculate the number of plants, animals, or species in the entire site given a small sample space. Sampling can also be used to determine the number of visitors to the site during operating hours, with students identifying the advantages and disadvantages of conducting stratified sampling, cluster sampling, systematic sampling, and SRSs. The number of visitors observed during the duration of the trip can also be plotted and students can determine what type of distribution this follows (normal, uniform, or skewed). Finally, the Food and Agriculture Organization of the United Nations' website provides a detailed primer on calculating the rate of water flow in a river which would make an excellent extended project (WATER 3. ESTIMATES OF WATER FLOW).

Its adjacency to the Del Water Gap also opens avenues to investigations of the history of state parks, the push for nationalization of parks, the effect on local communities, history of local Native American peoples, and paleontology. As a specific example, consider exploring the economics of national parks. There was a recent push to make the Del Water Gap a national park site, and it proved to be divisive (Chergotis, 2023). Students could research the history of this issue, the economics, and, using data and a multidisciplinary approach, present whether or not the national park designation would be appropriate for the Gap (The Economic Impact of Local Parks). By incorporating math into a local issue, students are much more likely to be engaged and feel like their work is authentic and meaningful.

Conclusion

For a relatively low cost and short distance, students who attend this field trip can get a real glimpse of math in action. Hopefully, they are inspired enough by their visit to create their own projects and express ownership of their learning and doing. We leave you with this–stand at the base of a great cascade with Lockhart's susurrations emanating from the flow: "this kind of mathematical experience goes to the heart of what it means to be human…mathematics is nothing less than the distilled essence of *who we are*" (pp. 114-15).

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Proof of Why the Slide and Divide Method Works and its Modified Version Suitable for Educational Purposes

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Abstract: The Slide and Divide Method emerged as a fast alternative for factoring Quadratic Trinomials of the form ax^2+bx+c, where a, b, and c are integers, $a \neq 1$, and the Greatest Common Divisor (GCD) of a, b, and c is 1. Nevertheless, despite its seemingly magical effectiveness, this method lacks a formal mathematical proof explaining why it works and its algorithm exhibits steps that are not rationally justified, making it unsuitable for formal instruction in a college setting. This paper aims to achieve a dual objective: first, to formally prove the validity of the Slide and Divide Method, and second, to introduce a modified version that not only sustains its efficacy but also incorporates a logical pedagogical approach suitable for formal education.to promote evidence-based instructional change.

Introduction

Factoring is the process of rewriting an algebraic expression as a product of simpler expressions, typically polynomials. For instance, the quadratic expression $x^2 - 5x + 6$ can be factored into $(x - 2)(x - 3)$. This technique is fundamental in algebra as it simplifies complex expressions, facilitates the solving of equations, and provides insights into the properties of functions. Factoring is especially valuable in understanding the structure of equations and finding their solutions efficiently.

Why is Factoring Important for Teachers?

Factoring is a fundamental skill in algebra that appears in many areas of mathematics, including Calculus, Number Theory, and Linear Algebra. It plays a crucial role in determining the roots (or zeros) of polynomial functions, which is essential for graphing and analyzing their behavior. Beyond pure mathematics, factoring is widely used in fields such as physics, engineering, economics, and cryptography. For example, in calculus, factoring polynomials is key to solving optimization problems. Additionally, many standardized tests, such as the SAT, GRE, and other entrance exams, include factoring questions, highlighting its importance in mathematical proficiency.

Common Misunderstandings about Factoring

Some students believe that all polynomials can be factored over the integers. This misconception arises because many introductory algebra problems focus on polynomials that factor neatly over the integers, such as quadratics with integer roots or special cases like the difference of squares. As a result, students may think all polynomials can be factored over integers. Additionally, factoring techniques like grouping, the rational root theorem, and synthetic division often lead students to expect integer factorization to always be possible. However, polynomials like $x^2 + 1$ or $x^4 + 4$ demonstrate that not all polynomials can be factored over the integers. Research also indicates that students often struggle to identify polynomials that are not factorable over the integers. A study analyzing engineering students' misconceptions in algebra found that 88% of participants had difficulty recognizing such polynomials. Ancheta and Subia (2020, p. 68) attribute these errors to a lack of conceptual understanding and reliance on rote memorization.

Many students struggle with factoring out the greatest common factor first. For example, they may try to factor $2x^2 - 2x - 40$ directly without first extracting the common factor of 2, which can make the process more complicated than necessary. While factoring is a useful method, not all quadratic equations can be solved this way. The quadratic formula or completing the square may be necessary.

Method

Factoring quadratic trinomials

Traditionally, students are instructed in two primary approaches for factoring quadratic trinomials of the form $ax^2 + bx + c$, where a, b, and c are integers. Note that $a \neq 0$ (if $a = 0$, then $ax^2 + bx + c$ becomes $bx + c$, a linear binomial) and $a \neq 1$ (if a is 1, then $ax^2 + bx + c$ becomes $x^2 + bx + c$, which we already know how to factor). One of the most common methods of factoring is employing the *ac* Method as described by Pieronkiewicz and Tanton (2019) on page 109. Below is a brief summary of the *ac* method.

Step 1. Multiply a and c (the coefficient of x^2 and the constant term).

Step 2. Find two numbers that multiply to ac and add to b (the coefficient of x).

Step 3. Split the middle term bx using these two numbers.

Step 4. Group the terms into two binomials.

Step 5. Factor the common terms in each group.

Example: Factor $6x^2 + 11x + 4$.

Step 1. $ac = (6)(4) = 24$.

Step 2. Find two numbers that multiply to 24 and add to 11: 8 and 3.

Step 3. Rewrite: $6x^2 + 8x + 3x + 4$.

Step 4. Group: $(6x^2 + 8x) + (3x + 4)$.

Step 5. Factor: $2x(3x + 4) + 1(3x + 4)$.
Final answer: $(2x + 1)(3x + 4)$.

The *ac* Method works well, but it requires knowledge of factorization by grouping. On the other hand, students may have difficulty understanding why they need to multiply a by c.

Alternatively, students are also taught The Trial and Error Method, as outlined with examples in the OER textbook OpenStax Intermediate Algebra 2e. While this method works, it often involves a significant number of trials, posing a challenge for students to determine whether the polynomial is factorable or a prime polynomial (not factorable).

An additional method, widely known as the Slide and Divide Method (also identified as "bottoms-up factoring"), appeared as an "informal" technique to simplify and expedite the two methods mentioned above, as explained by Fosnaugh and Mitchell (2014). The Slide and Divide Method is commonly called "informal" because it seems to work almost magically. Most of the steps in this method lack a clear logical foundation. Additionally, there is no formal proof explaining why it works. Because of this, instructors do not teach it as an alternative for factoring Quadratic Trinomials.

Slide and Divide Method

Now, the Slide and Divide method will be explained in detail as to how it works, and the procedures involved with the method. Let's start by presenting the Slide and Divide Method and then formally proving why it works, as we know prerequisite knowledge is always important to understand mathematical

The Slide and Divide Method is commonly called "informal" because it seems to work almost magically. Most of the steps in this method lack a clear logical foundation. Additionally, there is no formal proof explaining why it works. Because of this, instructors do not teach it as an alternative for factoring Quadratic Trinomials.

concepts. The prerequisite knowledge needed for this method is shown below:

We assume that a quadratic trinomial is a polynomial of the form $ax^2 + bx + c$, where a, b, and c are integers and $a \neq 0$.

- A quadratic trinomial is factorable if it is factorable over the set of integers.
- We assume that students have mastered factoring monic quadratic trinomials of the form $x^2 + bx + c$.
- We assume that students are familiar with the use of the following arithmetic properties: Commutative property, Associative property, and Distributive property.
- We assume that students have mastered factoring and finding the Greatest Common Divisor (GCD) of a binomial and of a trinomial (if needed).
- When factoring a quadratic trinomial $ax^2 + bx + c$, we assume that $a > 0$; if not we can always factor out -1 and make the leading coefficient positive.

The Slide and Divide method is used to factor quadratic trinomials of the form $ax^2 + bx + c$ when $a \neq 1$, because if a is equal to 1 then $ax^2 + bx + c$ becomes $x^2 + bx + c$ and we already know how to factor it. Below is an explanation of the Slide and Divide method as it stands now.

The Slide and Divide Method for factoring quadratic trinomials

Step 1: We first verify that GCD(a, b, c) = 1: for if not, we factor out the GCD, multiply a and c (the leading and constant coefficients) to obtain a new trinomial $x^2 + bx + ac$.

Step 2: Factor the new trinomial as if the leading coefficient were 1.

Step 3: Divide the constants of the binomial factors by a. If the result of the division after simplifying is a fraction, place the denominator in front of x as needed.

Step 4: Write the final factor form.

Example: Factor $15x^2 - 7x - 2$

Step 1: After verifying that GCD(15,-7,-2) = 1, multiply the first and the last coefficients,

$(15)(-2) = -30$

Step 2: After replacing the leading coefficient 15 for 1 and the constant term -2 for -30, factor $x^2 - 7x - 30$, which leads to:

$(x - 10)(x + 3)$.

Step 3: Divide -10 and 3 by 15 and simplify:

$(x - 10/15)(x + 3/15)$.

Simplifying, we obtain $(x - 2/3)(x + 1/5)$.

Place the remaining denominators, 3 and 5, in front of x in each binomial factor (This is why this method is often called “bottoms-up factoring”).

Step 4: We obtain the final factorization

$(3x - 2)(5x + 1)$.

Discussion

Although the Slide and Divide Method successfully factored $15x^2 - 7x - 2$, it presents certain challenges. The issues are discussed below:

- In Step 1, no explanation is given for why we multiply (15)(-2) = -30
- In Step 2: The polynomials $15x^2 - 7x - 2$ and $x^2 - 7x - 30$ are not the same, so why should we factor $x^2 - 7x - 30$ in order to factor $15x^2 - 7x - 2$?
- In Step 3: Why should we divide only the constant parts of the binomial factors?
- In Step 3: Why after simplifying must we place the denominators in front of x on each binomial factor?

The Slide and Divide Method proved to be highly effective, but its underlying logic often gets lost in the process. This is one of the primary reasons why many mathematics instructors don't prefer to teach

Based on my own teaching experiences over the years at various colleges, I've often observed instructors using this method in their own classrooms to help students quickly factor quadratic trinomials.

Proof of Slide and Divide Method...............(0)

Below, an algebraic proof of the Slide and Divide Method is explained.

Given a Quadratic Trinomial $ax^2 + bx + c$ such that GCD(a, b, c) = 1, and $a \neq 1$.

Factor $ax^2 + bx + c$.

Assume that GCD(a, b, c) = 1; for otherwise we can always factor out the GCD(a, b, c) and continue with our method.

Firstly, if $ax^2 + bx + c$ is factorable, then there exists d, e, f, and g integers such that $ax^2 + bx + c = (dx + e)(fx + g)$.

In order to factor $ax^2 + bx + c$ we must find d, e, f, and g.

Since $ax^2 + bx + c = (dx + e)(fx + g)$ after expanding the right side of the equation, then
$ax^2 + bx + c = dfx^2 + (dg + ef)x + eg$.

Where $a = df$, $b = dg + ef$, and $c = eg$............(1)

Now, we have $df = a$ and $eg = c$ (This is the case because both a and c are given).

We need to find ef and dg satisfying $ef + dg = b$ and $efdg = ac$.

From the above steps, this is where the Slide and Divide Method starts. We noticed that the above expression is equivalent to replacing c for ac, and factoring $x^2 + bx + ac$ (These are Steps 1 and 2 in the Slide and Divide Method).

Now,

$$ax^2 + bx + c = a(ax^2 + bx + c) \div a$$

$$ax^2 + bx + c = (a^2x^2 + abx + ac) \div a$$

$$ax^2 + bx + c = ((ax)^2 + b(ax) + ac) \div a$$

Now, suppose we find $m = ef$ and $n = dg$ satisfying $ef + dg = b$ and $efdg = ac$

Then using a change of variable $z = ax$, we can factor:

$$(ax)^2 + b(ax) + ac = (ax + ef)(ax + dg)$$

Then

$$ax^2 + bx + c = (ax + ef)(ax + dg) \div a$$

$$ax^2 + bx + c = a(x + ef/a)a(x + dg/a) \div a$$

$$ax^2 + bx + c = a(x + ef/a)(x + dg/a)$$

From (1), substituting $a = df$, we obtain:
$ax^2 + bx + c = a(x + ef/df)(x + dg/df)$

Clarification: This is where the Slide and Divide Method instructs us to divide the constant terms of the binomial factors by $a = df$ and simplify (Step 3 in the Slide and Divide Method).

After simplifying, we obtain:

$$ax^2 + bx + c = a(x + e/d)(x + g/f)$$

$$ax^2 + bx + c = a/df\,(dx + e)(fx + g)$$

From (1), substituting $a = df$, we obtain:

$ax^2 + bx + c = (dx + e)(fx + g)$……………..(2)

Clarification: the last three equations above are equivalent to the last step in The Slide and Divide Method where we are asked to place the denominators left, after simplifying, in front of x on both binomial factors (Step 3 of the Slide and Divide Method).

In equation (2) d and e are relatively prime, because otherwise they would generate a common divisor different than 1 for a, b, and c, which would contradict the fact that GCD(a, b, c) = 1.

For the same reason explained above f and g also are relatively prime……......……………………(3)

Then the factorization $ax^2 + bx + c = (dx + e)(fx + g)$ is a full factorization.

As we have seen, the Slide and Divide Method works by extracting 'valid partial results' that naturally arise from a formal sequence of logical steps, but it does not explain why these results are true. Since the logical foundation is not explicitly shown in the Slide and Divide Method, its 'simplified form' becomes an attractive and efficient algorithm for students, especially when the final factorization is the primary focus.

Now that we have formally proven why the Slide and Divide Method works, let's discuss the relationship between the polynomials $ax^2 + bx + c$ and $x^2 + bx + ac$.

Properties between the polynomials $ax^2 + bx + c$ and $x^2 + bx + ac$

- The Discriminants of $ax^2 + bx + c$ and $x^2 + bx + ac$ are the same and equal to $b^2 - 4ac$. This can be proven by applying the discriminant formula for both quadratic trinomials.
- Since $ax^2 + bx + c$ and $x^2 + bx + ac$ have the same Discriminant then they have the same type of roots, they could be both real or complex numbers.
- If the roots of $ax^2 + bx + c$ are x_1 and x_2, then the roots of $x^2 + bx + ac$ are ax_1 and ax_2. This can be proven by applying the quadratic formula for both quadratic trinomials.
- Regarding transformations, converting from $ax^2 + bx + c$ to $x^2 + bx + ac$ involves a horizontal stretch by a factor of a, followed by a vertical stretch by a factor of a. This is achieved through a change of variable (substituting x/a for x) and then multiplying the resulting polynomial by a.

A visualization of the last property via an example is shown below.

Figure 1 illustrates how the polynomial $6x^2 - 11x + 3$, after the two transformations mentioned above, becomes $x^2 - 11x + 18$. The dynamic geometry software Desmos is used to provide visual representation of the procedure.

Notice that the roots of $6x^2 - 11x + 3$ are 1/3 and 3/2. After the first transformation, these roots are multiplied by 6, resulting in the roots of $6(x/6)^2 - 11(x/6) + 18$, which are $6(1/3) = 2$ and $6(3/2) = 9$. These roots remain unchanged in the last transformation, as it only affects the y-values.

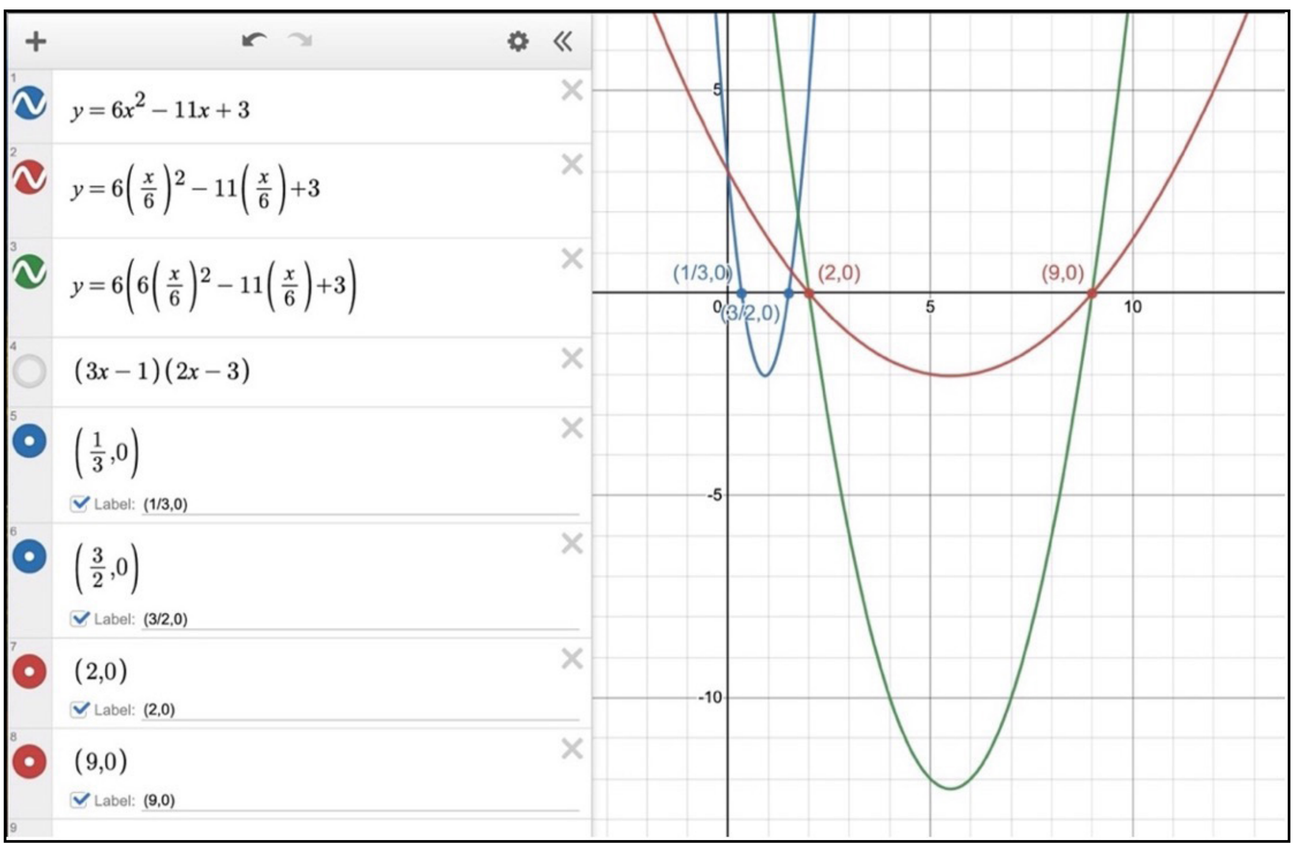

Figure 1: transformation of $6x^2 - 11x + 3$ into $x^2 - 11x + 18$ (Graphed via DESMOS)

The properties mentioned above suggest that the factorization of a quadratic trinomial $ax^2 + bx + c$ and the factorization of its associated monic quadratic trinomial $x^2 + bx + ac$ must be related; in other words, if we want to factor $ax^2 + bx + c$, we can factor $x^2 + bx + ac$ and then transform it into the given quadratic trinomial in factored form, all these remarks together with the proof of why the Slide and Divide Method works in (0), served as a motivation for developing a modified version of the Slide and Divide Method.

There are many algorithms for factoring quadratic trinomials, some of which use formulas involving radicals and derivatives from calculus, as seen in Joarder (2021). However, we will develop a method that is a variation of the Slide and Divide Method, suitable for college algebra students. For this purpose, we assume that students have mastered factoring quadratic trinomials of the form $x^2 + bx + c$ using the reverse FOIL method.

Because our proof in (0) was constructive, we can now introduce a logical algorithm for factoring a quadratic trinomial, called the Modified Slide and Divide Method.

The Modified Slide and Divide Method

Recall that our goal is to present an algorithm with a clear logical flow, enabling students to follow each step with full understanding.

Algorithm

Factor $ax^2 + bx + c$ where a, b, and c are integers, GCD(a, b, c) = 1, and $a \neq 1$.

Step 1: Multiply $ax^2 + bx + c$ by a on the left and divide it by a on the right.

$$ax^2 + bx + c = a(ax^2 + bx + c) \div a$$

Clarification: Multiplying a polynomial on the left and right by the same number does not change its value.

Step 2: Distribute the multiplication on the left.
$ax^2 + bx + c = (a^2x^2 + abx + ac) \div a$

Step 3: Write a^2x^2 as $(ax)^2$, apply the commutative and associative properties to obtain:

$$ax^2 + bx + c = ((ax)^2 + b(ax) + ac) \div a$$

Step 4: Letting $z = ax$ and factoring $(ax)^2 + b(ax) + ac$ by finding two numbers $m = dg$ and $n = ef$ such that $m + n = b$ and $mn = ac$, to obtain:

$$ax^2 + bx + c = (ax + m)(ax + n) \div a$$

$$ax^2 + bx + c = (ax + dg)(ax + ef) \div a$$

Step 5: Factor out the Greatest Common Divisor (GCD) of a and dg, which will be **d**, since $a = df$ and f is relatively prime to g, as explained in (1) and (3).

Factor out the Greatest Common Divisor (GCD) of a and ef, which will be f, since $a = df$ and d is relatively prime to e as it was explained in (1) and (3).

So, we obtain:

$$ax^2 + bx + c = d(fx + g)f(dx + e) \div a$$

Step 6: Since $a = df$ as shown in (1), then after simplifying, the final full factorization will be:

$$ax^2 + bx + c = (fx + g)f(dx + e)$$

Examples of the application of The Modified Slide and Divide Method.

Example 1: Factor $8x^2 + 10x - 3$

Step 1: Multiply $8x^2 + 10x - 3$ by 8 on the left and divide by 8 on the right.

$$8x^2 + 10x - 3 = 8(8x^2 + 10x - 3) \div 8$$

Step 2: Distribute the multiplication on the left.

$$8x^2 + 10x - 3 = (8^2x^2 + 8(10x) - 8(3)) \div 8$$

Step 3: Write 8^2x^2 as $(8x)^2$, apply the commutative, and the associative properties to obtain:

$$8x^2 + 10x - 3 = ((8x)^2 + 10(8x) - 24) \div 8$$

Step 4: Letting $z = 8x$ and factoring $(8x)^2 + 10(8x) - 24$ by finding two numbers m and n such that $m + n = 10$ and $mn = -24$ we obtain:

$8x^2 + 10x - 3 = (8x - 2)(8x + 12) \div 8$

Step 5: Factoring out the Greatest Common Divisor (GCD) from each binomials gives:

$8x^2 + 10x - 3 = 2(4x - 1)4(2x + 3) \div 8$

Step 6: Simplifying, we obtain the factorization:

$8x^2 + 10x - 3 = (4x - 1)(2x + 3)$

Example 2: Factor $3x^2 + 19x + 20$

Step 1: Multiply $3x^2 + 19x + 20$ by 3 on the left and divide by 3 on the right.

$3x^2 + 19x + 20 = 3(3x^2 + 19x + 20) \div 3$

Step 2: Distribute the multiplication on the left.

$3x^2 + 19x + 20 = (3^2x^2 + 3(19x) + 3(20)) \div 3$

Step 3: Write 3^2x^2 as $(3x)^2$, apply the commutative and the associative properties to obtain:

$3x^2 + 19x + 20 = ((3x)^2 + 19(3x) + 60) \div 3$

Step 4: Letting $z = 3x$ and factoring $(3x)^2 + 19(3x) + 60$ by finding two numbers m and n such that $m + n = 19$ and $mn = 60$ we obtain:

$3x^2 + 19x + 20 = (3x + 4)(3x + 15) \div 3$

Step 5: Factoring out the Greatest Common Divisor (GCD) from each binomial gives:

$3x^2 + 19x + 20 = (3x + 4)3(x + 5) \div 3$

Step 6: Simplifying, we obtain the factorization:

$3x^2 + 19x + 20 = (3x + 4)(x + 5)$

CONCLUSION

The Modified Slide and Divide Method to factor quadratic trinomials of the form $ax^2 + bx + c$ where a, b, and c are integers and $a \neq 1$, fundamentally addresses the lack of logical foundation in the Slide and Divide Method, providing mathematics Instructors an alternative method for factoring quadratic trinomials. This method can be taught as a sequence of logical steps that students can follow, based on formal mathematical reasoning and fundamental properties such as the commutative, associative, and distributive properties. It also uses the principle that multiplying and dividing a polynomial on the left and right by the same nonzero number does not change the polynomial's value. Additionally, our method does not require knowledge of factoring by grouping; instead, it relies only on the factorization of monic polynomials of the form $x^2 + bx + c$, a skill commonly mastered by undergraduate students in college algebra courses.

From an educational perspective, our method introduces the basic concept of a change of variable ($z = ax$, where a is an integer) early on. This illustrates how a one-to-one linear function can treat ax as a new variable in one-to-one correspondence with x, rather than simply as a constant times a variable.

While the Modified Slide and Divide Method may not be as appealing as the original Slide and Divide Method (it has two additional steps), its logical structure compensates for this. Additionally, it highlights the benefits of the Slide and Divide Method, even in its original form. For example, we proved in (0) that it can be used to quickly determine whether a quadratic trinomial (with $a \neq 1$) is factorable by checking if two integers m and n can be found such that $m + n = b$ and $mn = ac$. Most importantly, our Modified Slide and Divide Method provides a formal justification for instructors to teach the Slide and Divide Method openly, rather than implicitly. Using simple mathematics based on our Modified Slide and Divide method, instructors can explain to students the benefits and limitations of the original Slide and Divide Method and present it as an alternative to quickly factor quadratic trinomials.

The Modified Slide and Divide Method for factoring quadratic trinomials serves as both an alternative and a validation tool for the widely used and effective original Slide and Divide Method. It enhances understanding by emphasizing the logical foundations of mathematics and will be particularly appreciated by students who are critical thinkers, not just those who follow procedures mechanically. Lastly, hopefully it will make instructors who teach the original Slide and Divide Method feel that they are not magicians but mathematicians, who truly care about the reasons behind the algorithms that they present in class.

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Using Excel to Learn Statistics through Project-based Learning

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Abstract: In higher education, technology has been useful for teaching statistics. Many college statistics textbooks include technology features like "StatCrunch", "R", "Excel", "TI 84", " and "SPSS". Thus, statistics tools have been incorporated into teaching statistics. As a result of this trend of incorporating technology into statistics courses, many community colleges are also encouraging students to incorporate technology into their courses. It has been suggested that community colleges can improve their teaching of statistics by using more than just a scientific calculator.

Microsoft Excel is a widely used and free software program. Despite the availability of other statistics tools, such as "R," "SPSS," "StatCrunch," etc., many community students cannot afford them, so Excel has been used for public education. Many community colleges also offer free Excel classes as part of their STEM programs. As a result of this project, many teachers have understood the importance of using Excel in statistics courses. In this study, researchers have developed a statistics curriculum that utilizes Excel and teaches statistics at an urban community college. The researchers explain how the Excel curriculum and statistics classes were designed. In addition, feedback from students on how they learned statistics using Excel was also included in the study. Moreover, this study found that Excel's graphical capabilities can help students interpret statistical concepts and visualize data, improving their understanding of statistical concepts.

Introduction

In statistics, data are collected, summarized, analyzed, interpreted, and conclusions are drawn (Lee, & Ban, 2020). To generate a statistical result, a complex computational formula is typically required (Mairing, 2020). Thus, most statistics classes allow students to analyze data using a scientific calculator or statistical software (Lee & Ban, 2020: Mairing, 2020). In spite of the fact that calculators are excellent tools for analyzing data, students are difficult to learn statistics from because of their multiple algebraic operations. Each time a student clicks the calculator button, the entire procedure must be repeated. It is common for students to input numbers incorrectly into a calculator and then struggle to find the correct answer. Some students appear to intuitively understand or think about these working processes. Others find it more challenging. Moreover, unless presented with relevant and relatable examples, statistics may seem disconnected from real-life situations. Without understanding the practical applications of statistical methods, students find it challenging to see the significance and value of learning statistics. Consequently, students face many challenges when learning statistics and analyzing data. Students have difficulty grasping many statistical concepts without visual representations or hands-on exercises (Alias, 2009; Ridgway, Nicholson, & McCusker, 2007).

To address these challenges, educators have implemented active learning strategies, provided relevant and practical examples, and created a supportive learning environment in a statistics course. This encourages students to explore and practice statistical and analytical thinking. In particular, using technology and data sets with software helps students gain confidence when calculating statistical formulas. Statistics software such as Excel, Minitab, SPSS, and R were introduced, and many researchers and instructors utilized these programs to teach statistics courses (Basturk, 2005: Jatnika, 2015). By using this software, the results of data analysis were displayed immediately without showing the processes involved. When it comes to statistical analysis, Excel can be a viable option, especially for beginners or people unfamiliar with specialized software. Many basic statistical calculations and analyses can be performed using Excel's statistical functions and tools.

Although some researchers claim Excel is not very convenient for data analysis, it is easier to use than some other software, and it uses a similar algorithm (Mairing, 2020). Unlike other software, it requires inputting formulas into Excel spreadsheets but is less work than using a scientific calculator. Hence, replacing Excel with a scientific calculator has been an alternative way to collect, orgnize, and analyze the data, and this study examines how Excel can increase student satisfaction and statistical thinking skills. The purpose of this study is to demonstrate how the statistics curriculum is designed using Excel, as well as to report students' satisfaction with learning the subject using Excel. In addition, feedback from students will be reported.

Literature Review

Statistical Software in Statistics

The study of statistics builds critical thinking skills as well as the ability to analyze data based on students' research (Lee & Ban, 2020). Research has shown that statistical software is a helpful tool for analyzing data and makes learning statistics easier compared to the old-fashioned method of solving problems using a formula and a calculator. (Alias, M. 2009). Many statistics software programs, such as SPSS, R, Minitab, etc., have recently been made available and used by institutions. Students can interact with the software and visualize statistical concepts dynamically. This makes it an attractive factor in statistics classes (Oldknow, Tylor, & Tetlow, 2010). That is, statistical software can enhance the learning experience and help students better grasp statistical concepts when used in statistics courses. (Pratt, Davies, & Connor, 2011; Tishkovskaya & Lancester, 2012). Especially with Excel, students can perform various types of data analysis, such as calculating means, medians, standard deviations, correlations, and regressions. Students can apply statistical concepts directly to real-world data in this way. A lack of research has been conducted on which statistical software is beneficial to increasing students' statistical thinking and analytics skills. In order to determine which software is appropriate, students' characteristics and an institution's population should be taken into account (Mairing, 2020: Oldknow, Tylor, & Tetlow, 2010). It implies that learning statistics using software is difficult for students who have never used it before.

Project-Based Learning

Students often express negative attitudes about elementary statistics courses and end up with few useful skills (Delia, 2017). Many students find statistics courses rigid, abstract, and demanding (Wen Huang, Jeremi S. London & Logan A. Perry, 2022). Project-based learning (PBL) is one of the student center learning strategies (Lee, 2011; Lee & Ban, 2021; Viro et al., 2020). Through PBL, which is defined as a teaching method based on real-life and authentic situations that engage students' interest and enthusiasm (Krajcik and Blumenfeld, 2006), statistics can be made more relevant to students and emphasize analytical statistical thinking. The purpose of project-based learning is to improve learners' thinking and problem-solving skills (Siswone et al., 2017a) and to engage them in the classroom (Berends, Boersma, and Weggemann, 2003). Through Project-Based Learning, students gain experience in problem-solving (Kaldi et al., 2011), critical thinking and decision-making skills (Saracaloglu et al., 2006), and the ability to think critically.

According to Demirham (2002), Project Based Learning promotes individual and collaborative learning in everyday life situations based on personal interests. In order to determine and evaluate the effectiveness of Project Based Learning in education,

Software has strengths and weaknesses, and what might be considered "better" depends on the educational setting, goals, and needs of the students.

numerous research studies have been conducted. For example, Siswone et al., (2018) in their study of the effectiveness of Project Based Learning on Statistical Learning for Secondary School, showed that Project Based Learning is effective in statistical learning. As a result of active discussions with peers in the classroom, the students were enthusiastic about working on the given project. A similar finding was found by Miswanto (2022), Dede, and Yaman (2013) that Project Based Learning is effective in science and mathematics classrooms. It is consistent with the results of a study by Dierker et al. (2017) that shows an increased interest in working with advanced statistics and an expectation of using statistics in the future.

In Dierker's study (2017), he tested whether guided project-based learning instructional methods influenced students' attitudes and academic performance in a college-level elementary statistics course. His study shows that guided project-based learning enhances students' attitudes towards statistics and academic performance (P 3). As part of student-centered learning, PBL provides students with opportunities to explore solving problems, so that they can develop positive learning attitudes and enhance their critical thinking skills (Lee & Ban, 2021).

Project-Based Learning with Technology in a Statistics Classroom

Student engagement is enhanced by PBL with technology (Crocco, F., Offenholley, K., & Hernandez, C., 2016; Poon, 2018). Using computing software helps students learn mathematics, according to Poon (2018). A mathematical computing game can be used to teach Algebra topics according to Crocco, F., Offenholley, K., & Hernandez, C. (2016). According to them, students enjoyed playing games, and they improved dramatically in learning mathematics since they felt like playing a game instead of studying. Thus, PBL can be a rewarding experience when technologies are implemented to support it. According to Donnelly (2005), one of the most important uses of technology in curriculum and instruction is allowing students to explore and solve real-world problems.

Similarly, a lot of statistics learning relies on dynamic software, such as SPSS, StatCrunch, R, Desmo, Google sheets, etc. Desmos and Google sheets provide several statistical tools to calculate normal distribution curves, pie charts, scatter plots, etc. Thus, technology and project-based learning (PBL) can be a powerful combination to engage students, foster critical thinking, and develop practical skills. Students are placed at the center of the learning process in PBL, allowing them to actively participate in their education (Lee, 2011).

Various factors influence the choice of statistical software, including the specific context, course objectives, and the level of complexity required for statistical analysis. Software has strengths and weaknesses, and what might be considered "better" depends on the educational setting, goals, and needs of the students. Statistical software such as SPSS, R, Excel, Google Sheets, etc., are widely known. Many senior colleges and universities use SPSS and R among a variety of dynamic software. A two-year college, on the other hand, often has students who do not understand computing skills or do not have the budget to purchase the needed software. Educators in a community college setting should carefully evaluate the course objectives, student backgrounds, and the level of statistical analysis required before choosing the best statistical software. The simplicity and accessibility of Excel may make it an appropriate choice for introductory statistics courses, while more advanced courses or research projects may benefit from specialized statistical software such as SPSS or R, which provide more advanced statistical methods and robust analysis capabilities.

In statistics courses, technology, including statistical software such as Excel, can play an essential role in fostering PBL. Excel is most commonly used in business and education settings. Students can actively engage with data, perform analyses, and interpret results in statistics courses using Excel. The familiarity of Excel makes it a suitable tool for introductory statistics courses. If students are already familiar with

Excel, they will have a shorter learning curve for your basic statistical tasks. As a result, students gain practical experience and reinforce their understanding of statistical concepts by working on real-world datasets. In addition, statistical software like Excel can be customized to meet each student's needs and skill levels. Data and statistical results can be visualized in various formats using Excel's graphing and charting capabilities. As a result, students are able to interpret their analyses more effectively. Furthermore, Excel is widely used for data analysis in many industries, so it is a valuable tool for students preparing for future careers. By understanding statistical analysis in Excel, students can become more employable and adaptable. Students gain practical experience, develop essential skills, and learn how statistical analysis can be used to solve problems in various fields by integrating project-based learning with Excel.

This study examines whether Excel could be useful under project-based learning for Statistics classes. In this study, qualitative data was collected to determine students' satisfaction with Excel's statistical data analysis abilities. In our literature review, the researchers investigated three affective elements - Effective Excel topics, Student Satisfaction, and Project-based Learning - and how students perceived these elements in Excel statistics. This study aims to answer the following research questions.

1. When using Excel to understand Statistics topics, which topics were most effective?

2. For marginalized students in Statistics courses, how does using Excel implement students' satisfaction and mathematical thinking in learning Statistics concepts?

3. Which aspects of the project-based learning approach and instruction are affected by using an Excel in Statistics course? OR What is the relationship between a) the level of implementation of the curriculum guide with Excel, pedagogy, and assessment, and b) effective topics in Statistics?

Methodology

The purpose of this study was to begin learning Excel in a statistics class at an urban community college in New York City. Students taking introductory statistics classes participated in this study. Analyzing qualitative data was the focus of the study. A total of forty-two students were involved in the study at the beginning, and 35 students completed the course at the end of the semester. (in-person course?yes) One semester was the duration of the study. To investigate the research questions, researchers conducted semi-structured interviews during the class session with a sample of instructors (n=35) who took the Introductory Statistics course in the Spring of 2023.

At the beginning of the semester, students had confusion and learning difficulties and struggled with Excel instruction. Most students were unfamiliar with Excel commenders and had never learned statistics using Excel, so they struggled to understand how to use Excel on statistics problems. During classes, the instructor reviewed how to use Excel to solve problems multiple times during each class. The instructor recorded YouTube mini-lectures that students watched as part of the project. Most students understood the instructions after a few weeks and began completing Excel-based projects. Some students still struggled with Excel; they mostly didn't understand it, so they preferred to use a scientific calculator. Instructors observed that some students sat quietly and did nothing because they did not understand Excel instructions. The instructor re-explained detailed Excel instructions and encouraged them to watch the related videos repeatedly.

In particular, the students at this urban community college were not familiar with learning statistics in Excel. Students were historically taught how to analyze data using a scientific calculator rather than Excel. Students in introductory Statistics courses at the 2-year college level lacked sufficient experience with advanced software and Excel to analyze the data. Because most students had never used Excel in Statistics classes, the class created a recorded video that explained how to use Excel in the class. An instructor recorded videos explaining how to use Excel for each topic in Statistics. To make sure that students could follow the instructions effectively, each video clip was no longer than 15 minutes. Additionally, all videos were organized by category (Figure 1).

Videos were recorded and published on YouTube, and students were able to learn how to use Excel for each topic of Statistics anytime and anywhere. The

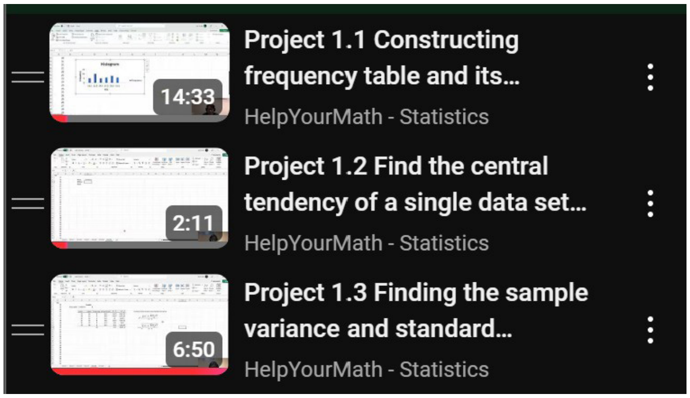

Figure 1: Example of recorded videos explaining how to use Excel for each topic in Statistics

Statistics projects required students to watch the videos to learn how to use Excel before they began working on them. Whenever related topics were discussed in class, each project was assigned. The instructor assigned students a week to complete and submit their projects. Excel statistics projects covered the following topics: 1) Construct a frequency table and its histogram, 2) Linear regression line and its correlation, 3) Boxplot and Outliers, 4) Central Tendencies, Variance, and Standard deviation, 5) Conditional probability, 6) Sampling distribution, 7) Confidence intervals, and 8) (only group) Hypothesis Testing. Students were required to complete five Excel projects during the semester. (during the in-person lecture + PBL over the semester)

Students with experience in Excel were able to follow the Excel project progress with the instructor's guidelines. Students' project guidelines were created by the instructor and video instructions were posted on how to complete each project as part of a Project Based Learning (PBL) approach. An instructor guided step-by-step instructions on how to create a bar chart or histogram using an Excel spreadsheet in one of the projects (Figure 2). Students followed the instructions and were able to create the visual representations of data. They were then able to interpret the data and draw meaningful conclusions. For projects, Wikipedia was used as a source of data. The students generated the outcomes using Excel based on the data. The results were then displayed in a presentation. The students discussed the findings and drew conclusions from the data. The project consisted of five different projects: students completed projects #1 and #4 independently, and only project #5 was completed in groups. Due to the difficulty, students had understanding Excel instruction, an instructor decided to have

Final Project Assignment #1:

Throughout Final Project Assignment #1, you would choose one quantitative topic. You would write a short paragraph based on the following steps. This project is based on surveying people

Step 1: Determine your quantitative topic

Step 2: Collect 50 samples with both genders

Step 3: Create a Frequency table with 6-class.

Step 4: Create a histogram

Step 5: Find the central Tendency

Step 6: Create Boxplot

Step 7: Find the variance and standard deviation.

Figure 2: Example of project guideline

Student Survey Methods

all projects due at the end of the semester. As a result, students worked at their own pace, meeting deadlines for each project.

The instructor created a survey that reflects research questions thoroughly at the end of the semester to see students' satisfaction using an Excel in Statistics course. To develop the survey items, the instructor reviewed the literature of the Statistics classroom that impacts learning with Excel. In our literature review, the researchers focused on three affective elements — Effective Excel topics, Student Satisfaction, and Project-based Learning — and developed survey item statements that elicited students' perceptions of how these elements were experienced in Excel statistics runs.

Additionally, we created a survey form with informational items, one to gather information on the effective topics in Statistics using Excel, and one to gather students' satisfaction information. A recording of the discussion between the instructor and students was made, and the researchers reviewed it. Students freely shared their opinions about using Excel for data analysis of projects during this discussion. The authors reviewed and summarized survey results and discussions to determine whether students were satisfied with Excel and in which topic Excel was most effective. Researchers received feedback on all the items from the participants and made final edits to the survey items and structure based on this feedback.

The Outcome of the Study

Based on the results of the students' survey and discussion sessions, the sample of students who participated in the project benefited from the mini-lectures regarding the use of the Excel guide, pedagogical strategies, and project-based learning assessment. All participants reported using multiple project-based learning strategies aligned with the Excel guide, despite varying reports of Excel use. Over the course of the semester, students shared "*I watched the project video over and over again.. I might have watched the same video clip more than 100 times. After understanding one video, others are easier to understand.*" An additional participant discussed how mini-lecture videos assist students in understanding projects through visualization. As the project progressed, students preferred to use Excel instead of a scientific calculator to solve problems. Students began using Excel to solve problems for classwork, exams, and projects.

A similar trend occurred in students reading Excel instructions and incorporating Excel skills into assignments. The students learned how to use a calculator and Excel to solve problems, and they selected the tool that was most efficient for each problem. Most participants mentioned Excel as being particularly useful. One student shared, "*Professor, when solving binomial probabilities, Excel is the greatest and easiest tool to use. It took only a second to set up and solve binomial probabilities compared to using a scientific calculator. It takes forever to solve binomial probabilities using a calculator.*" Another student discussed how Excel is difficult to use to analyze conditional probabilities: "*I don't want to use Excel for conditional probabilities. Excel required extra steps to set up a problem.*" Overall, most students described the benefits of using Excel which aligned with the project guidelines, although several noted that the use of Excel was complicated and required additional steps to produce the results. As students analyzed data based on the topics covered in Statistics, they favored either a calculator or Excel.

In answering the first research question, the instructor conducted a survey to determine which Statistics topics helped students use Excel most effectively. Student responses indicate that Excel can be useful for most topics, according to the outcomes. According to students, the following topics are very effective when used with Excel (Figure 3).

1. Frequency table and histogram
2. Central Tendency, Variance, and standard deviation
3. Normal Distribution

Based on the responses, Excel can be used for a variety of purposes. When it comes to creating a frequency distribution table, students found Excel to be the most helpful. Frequency distribution tables were especially useful for creating histograms with big data. Frequency distribution tables in Excel can also be used to find the mean, median, mode, and confidence interval of a dataset (Figure 4).

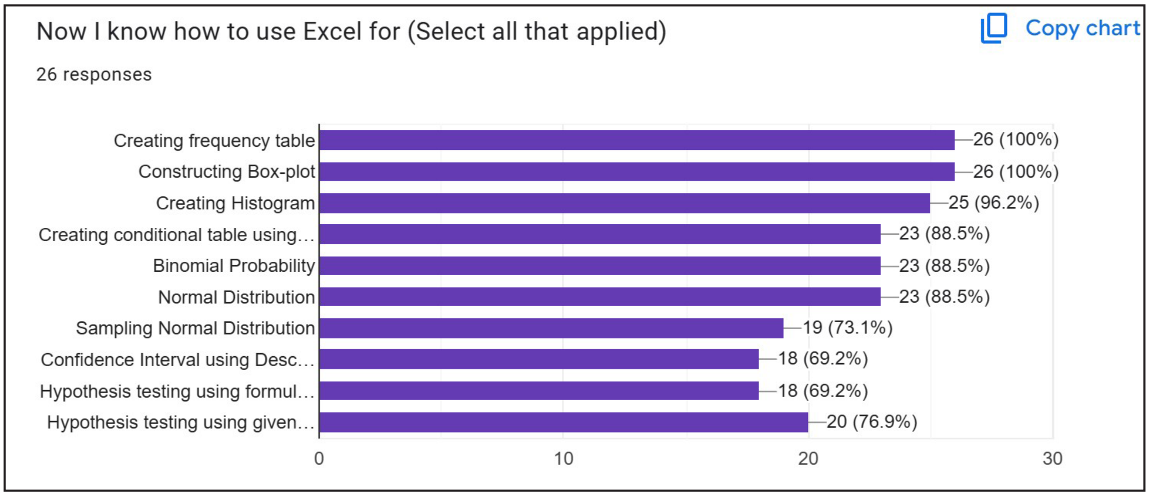

Figure 3: Students' opinions of most suitable topics using Excel

This was useful for students' visual learning in Statistics to identify outliers and analyze how the data is distributed. Thus, the results indicate that Excel allows students to create various types of charts and graphs, such as bar charts, pie charts, line graphs, scatter plots, and more. It is especially beneficial to use Excel for visual learning since it is a versatile tool for creating and presenting data visually. In visual learning, concepts are represented visually through charts, graphs, and diagrams, which make them easier to understand and remember. By using these visuals, students were able to easily interpret and analyze data, making it easier for them to identify trends and patterns. The topic of "Conditional Probability," however, was considered the least effective to use Excel for. Students felt that they needed to calculate the event in several steps as part of the conditional probability formula. A calculator was found to be more convenient than following several steps in Excel to calculate conditional probability. Even though Excel is a valuable tool for many mathematical and statistical tasks, it has limitations when it comes to advanced symbolic math and complex statistical analyses. Students may feel the need to supplement Excel with another software or programming language to overcome these limitations, depending on their specific prior knowledge of Excel. Most students in an introductory Statistics course found Excel challenging to reproduce complex mathematical or statistical analyses, especially when dealing with multiple formulas and dependencies.

To answer the second research question, the instructor conducted both a student survey and a discussion to determine students' satisfaction with Excel. Students built Excel skills according to each topic, even though Excel function values can differ. By reading Excel results, students interpreted and analyzed data; they used a calculator to find the solutions only by applying the statistics formula. Students were able to improve their statistics skills over the semester.

Commuting time between home and school							
Female	25						
Men	35	*Column1*		*Column1*		*Column1*	
Female	40						
Female	28	Mean	52.6078431	Mean	52.6078431	Mean	52.6078431
Men	40	Standard Error	3.01708578	Standard Error	3.01708578	Standard Error	3.01708578
Female	70	Median	50	Median	50	Median	50
Men	50	Mode	50	Mode	50	Mode	50
Female	20	Standard Deviation	21.5463022	Standard Deviation	21.5463022	Standard Deviation	21.5463022
Men	25	Sample Variance	464.243137	Sample Variance	464.243137	Sample Variance	464.243137
Female	80	Kurtosis	0.93928964	Kurtosis	0.93928964	Kurtosis	0.93928964
Men	20	Skewness	0.85473854	Skewness	0.85473854	Skewness	0.85473854
Female	30	Range	100	Range	100	Range	100
Men	45	Minimum	20	Minimum	20	Minimum	20
Female	45	Maximum	120	Maximum	120	Maximum	120
Men	25	Sum	2683	Sum	2683	Sum	2683
Men	50	Count	51	Count	51	Count	51
Female	50	Confidence Level(90.0%)	5.05634922	Confidence Level(95.0%)	6.05999514	Confidence Level(99.0%)	8.079132
Female	25						
Female	50	Lower Bound	47.5515	Lower Bound	46.5478	Lower Bound	44.5287
Female	45	Upper Bound	57.6642	Upper Bound	58.6678	Upper Bound	60.6870
Men	45						
Female	70	90% confidence interval is [47.5515, 57.6642]		95% confidence interval is [46.5478, 58.6678]		99% confidence interval is [44.5287, 60.6870]	
Men	100						
Female	50						

Figure 4: An example of students analyzing mean and median with Excel

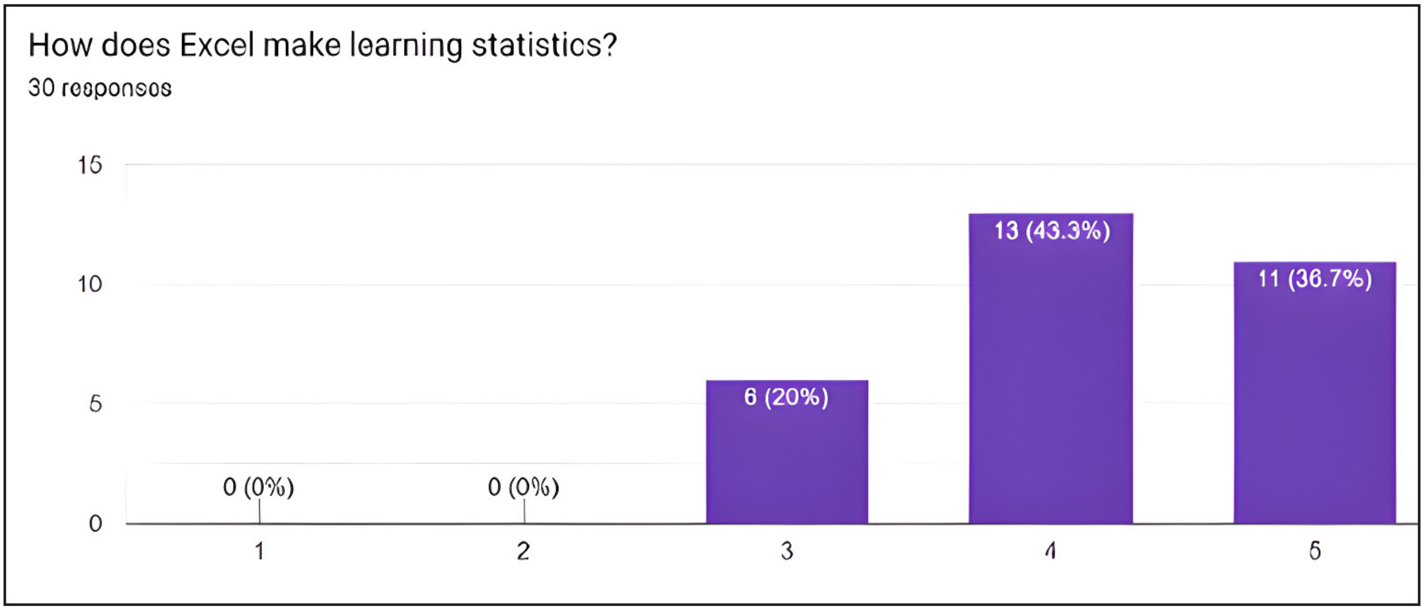

Figure 5: Students' learning experience with Excel in Statistics classes

Overall, students were satisfied with the way they learned statistics using Excel (Figure 5.) According to one of the survey questions, "What do you think about using Excel in Statistics classes? ", students felt Excel was comfortable to work with and saved their time analyzing data. In the survey, 1 indicates Strongly Disagree; 2 indicates Disagree; 3 indicates Neither Agree nor Disagree; 4 indicates Agree; 5 indicates Strongly Agree.

About 80% of students responded that learning Excel in statistics classes was effective. The majority of students felt that using Excel was beneficial for their learning. They found it easy to use and were able to understand the concepts better. The 20% who responded "Neutral" had a mixed opinion, and did not feel strongly one way or the other.

According to the discussion results of the same survey question, "What do you think about using Excel in introductory statistics courses?" students responded as follows:

- *"I think that Excel is the most convenient resource for the Stats course. Most of the work that would usually be done would take a lot longer using a calculator in comparison to using Excel. Excel is much quicker and more efficient."*

- *"When using Excel in Statistics I find it a lot easier than the calculator. it is less time-consuming and a lot more interesting to me. Though it is fewer steps it is still difficult if you do not understand how to use Excel formulas or commands"*

- *"Using Excel in statistics was very helpful for questions that were hard to do algebraically. When I first used Excel, it was hard to understand the function of it. After a while of using it in class, it became easier to use"*

- *"After completing this intro course into statistics, I feel more confident in using Excel as a tool. excel was very intimidating at first because there were so many functions and tools in the software. After this class, I realized how important it is to know how to navigate Excel, how commonly Excel is used as a tool, and how desirable knowing how to utilize the software, as many don't know how to operate Excel on a basic skill level."*

With Excel, students felt that graphic representations of data simplify complex information and make it more accessible. Visual learners grasp concepts more easily when they can see them graphically. Students also felt that they could retain information better when it is presented visually and organized. Overall, Excel for visual learning helped students better understand and remember complex information, analyze data effectively, and communicate their findings clearly. To enhance learning and decision-making processes, students learned that it can be used in various educational, professional, and personal contexts.

Although students found it useful to use Excel for statistical analysis, less than 20% of the students felt overwhelmed by its complexity. They also felt that it was difficult to learn the different functions and features of Excel. As a result, they suggested that more instruction and guidance should be provided in order to help them better understand the software. Other students' responses to the same survey question, "Do you think Excel should be used in introductory statistics courses?", were as follows:

- *"I am concerned about the Excel formula. It is difficult for me to memorize the formula. If I miss something from the start of class, I can lose the theory."*
- *"Although Excel can be very helpful when it comes to data, it can be complicated to understand its functions. I wish I could understand more about the tools that Excel provides. Sometimes it makes me confused about how I can organize my work better."*
- *"I would say my only concern would be knowing each function and how to find certain concepts."*
- *"The most frequent problem that I faced while working through these assignments was knowing which formula to use for certain computations and also when we use Excel in a statistics course we become very dependent on it, and then we do not want to do it manually because it's more complicated and time-consuming to do manually, and sometimes we forget how to do it manually."*
- *"My concerns are that we won't know the step-by-step process of solving problems and therefore we skip the analytical part of it making us not know how exactly we got to the answer."*

These responses indicate that Excel has limitations when it comes to handling complex mathematical problems, advanced mathematical concepts, and high-precision calculations. In order to deal with complex mathematical operations, students should be advised to utilize specialized software such as Matlab or Mathematica or more guidance with Excel. Students can better understand complex mathematical concepts and problems by utilizing specialized software or other teaching assessments such as teacher guidance, step-by-step work, and procedural understanding. Therefore, they become more confident in using Excel after making more accurate calculations.

Another concern was the dependence on data analysis by Excel, which prevented students from understanding each concept and calculation manually. Therefore, instructors must find a balance between teaching students how to use Excel and teaching them the underlying concepts. It can also be beneficial to assign exercises that require students to perform calculations by hand and then compare the results to those generated by Excel.

As part of the survey, students were also asked how using Excel improved their math skills (Figure 6). In the survey, 1 indicates Strongly Disagree; 2 indicates Disagree; 3 indicates Neither Agree nor Disagree; 4 indicates Agree; 5 indicates Strongly Agree.

In this study, 53.3% of the students agreed that Excel helped them develop mathematical thinking skills as they understood statistics conceptually and

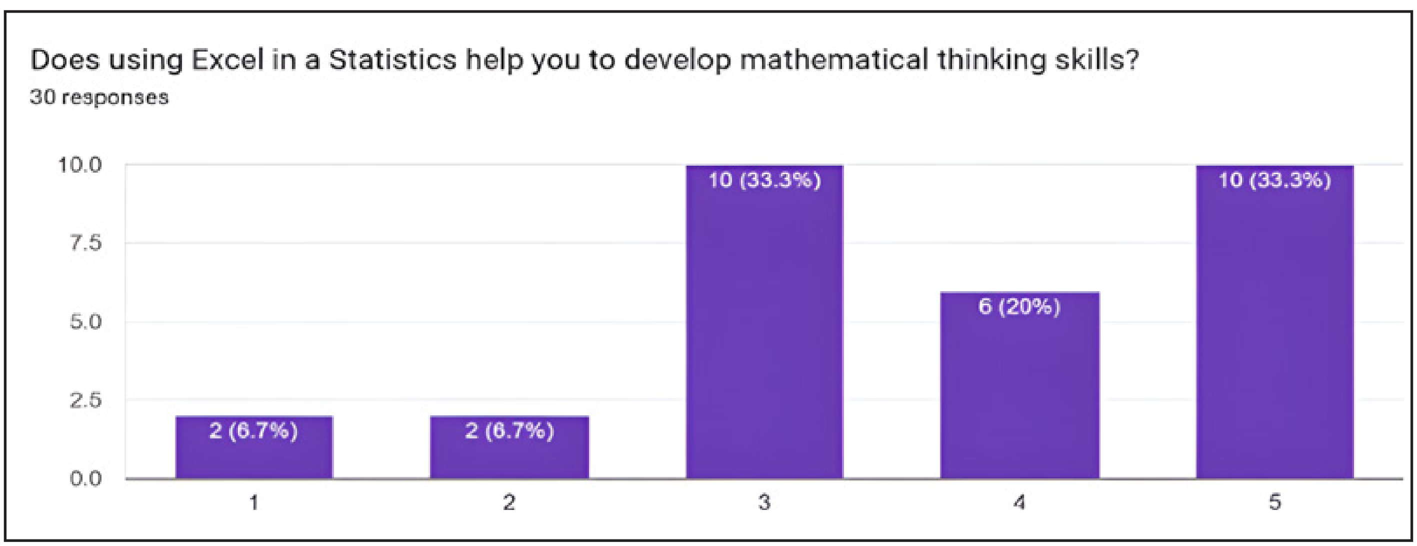

Figure 6. Mathematical thinking skills of students using Excel

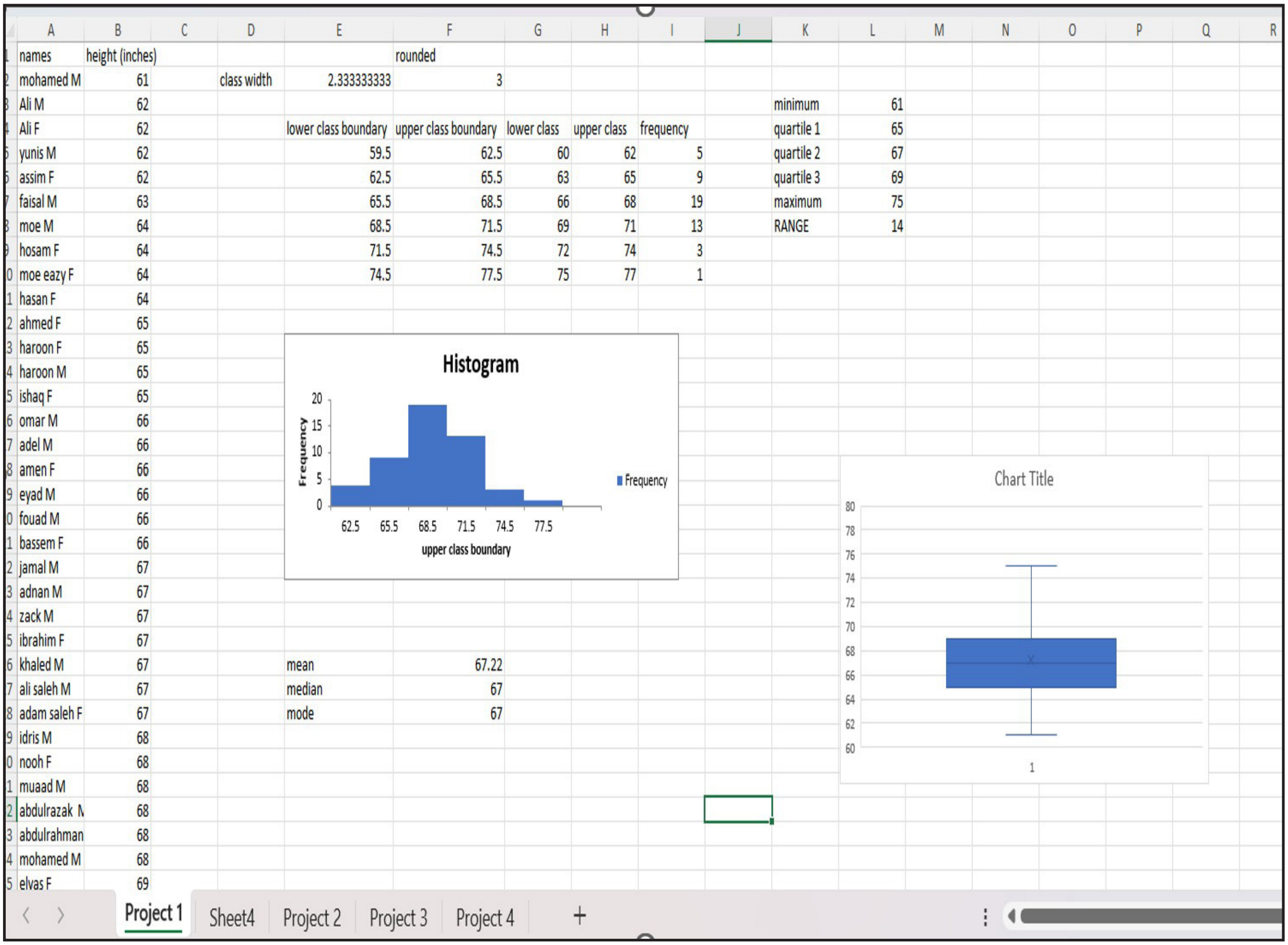

Figure 7: Example of Activity 1, using Excel to create visual representations

interpreted data. However, 46.7% of the students were neutral or disagreed that Excel was not helpful to their mathematical thinking skills in Statistics. According to these results, Excel should be used with other methods, such as hands-on activities and discussions. Assessments and guidance will also be important to improve students' mathematical thinking skills with the diverse practices of the teachers. Teachers should also provide feedback to students to help them identify their strengths and weaknesses in their mathematical thinking skills. It is also important for teachers to find ways to make their lessons interesting and engaging, in order to keep students motivated and focused. Diverse teaching practices can help foster a deeper understanding of the concepts and help students develop critical thinking skills.

To answer the third research question, the instructor created a project involving real-world examples and data for students to apply mathematical and statistical concepts and skills at the end of the semester. With a Project-based Learning approach, students were encouraged to use Excel for hands-on, group-based, and inquiry-based learning activities. Students selected their own topic and gathered their own data for this final project, ensuring that each student had a unique project. In cases where students selected the same topic, the instructor encouraged them to focus on different population groups, ensuring that the data was diverse. Under the project-based learning approach, the final project is constructed with four discrete parts.

- Activiy 1. Using Excel, show a frequency t ble, calculated variance and standard deviation, and create both histograms and box plots based on your own data.

- Activity 2. By using Excel's "Contif" function, show how to analyze conditional probability based on your chosen data.

- Activity 3. shows binomial proabilty calculuation using an Excel functions.

- Activity 4. Create a confidence interval based on your own data.

Throughout PBL in Statistics, all projects are designed using the same data that was collected by each student at the beginning of the course. In activity 1, by using Excel functions such as mean, median, mode, and standard deviation, students were able to summarize the data using descriptive outcomes. Students completed descriptive statistics, constructed a frequency table, histogram, and calculated the central tendency, variance, and sample standard deviation, and created a box plot using Excel (Figure 7). Students also interpreted their data and identified outliers. They used the data from Excel to create a scatter plot and draw conclusions from their observations. Finally, they discussed their findings and wrote a report on the data analysis.

In the second activity, students imported the central tendency values from activity 1 and completed the conditional probability project using the Excel function (Figure 8). The students were asked to predict the chances of an event occurring given the probability of each event. The conditional probability function allowed them to calculate the probability of an event occurring given the probability of another event occurring. They used these values to draw conclusions about the relationship between the two events and to determine if the events were independent or dependent.

As part of the third activity, students imported the basic probabilities of male, and female, greater than or equal to the mean, or less than the mean, and then used these probabilities to complete the binomial probability table in the third activity. Students created conditional tables using the pivot table tool (Figure 9). The students were asked to collect fifty samples of both genders (males and females). The students then used the table to calculate the expected number of successes of a trial. They also used the table to calculate the probability and expected value of a given number of successes (Figure 10). Finally, the students used the table to calculate the probability and expected value of a given number of successes given the number of trials using Excel. For example, the number indicates the target probability, and the number of trials indicates the total number of selections. As an example, if p(male) is 0.67, and there are 10 selections. P(6) = binom.dist(6,10,0.678,false) can be used to determine the probability of choosing exactly 6 males (Figure 11). The advantage of using Excel for finding the binomial probability is that students only have to work on the first row's binomial probability, and they can copy the procedure for the other rows as well. This saves time and effort, as students do not have to manually calculate the binomial probability for each row. Furthermore, Excel's formulas are also convenient to use, allowing students to quickly calculate the binomial

A	B	C	D	E	F	G	H	I
Height								
Male	Female							
61	62							
62	62			Mean Age	67.22			
62	64							
63	64							
64	64			Column1	Male	Female	total	
65	65				Male	Female	Total	
66	65			Greater than or equal to mean	16	7	23	
66	65			Less than mean	15	12	27	
66	66			Total	31	19	50	
66	66							
67	67							
67	67							
67	68			P(male)	0.62			
67	68			P(female)	0.38			
67	69			P(Greater than or equal to mean)	0.46			
68	69			P(less than mean)	0.54			
68	69			P(Male/Greater than or equal to mean)	0.696			
68	72			P(less than mean/Female)	0.632			
68	72			P(Female/ less than mean	0.444			
69								
69								
69								
70								
70								
70								
71								
71								
71								
71								
73								
75								

Figure 8. Example of Activity 2, using an Excel to create conditional probability

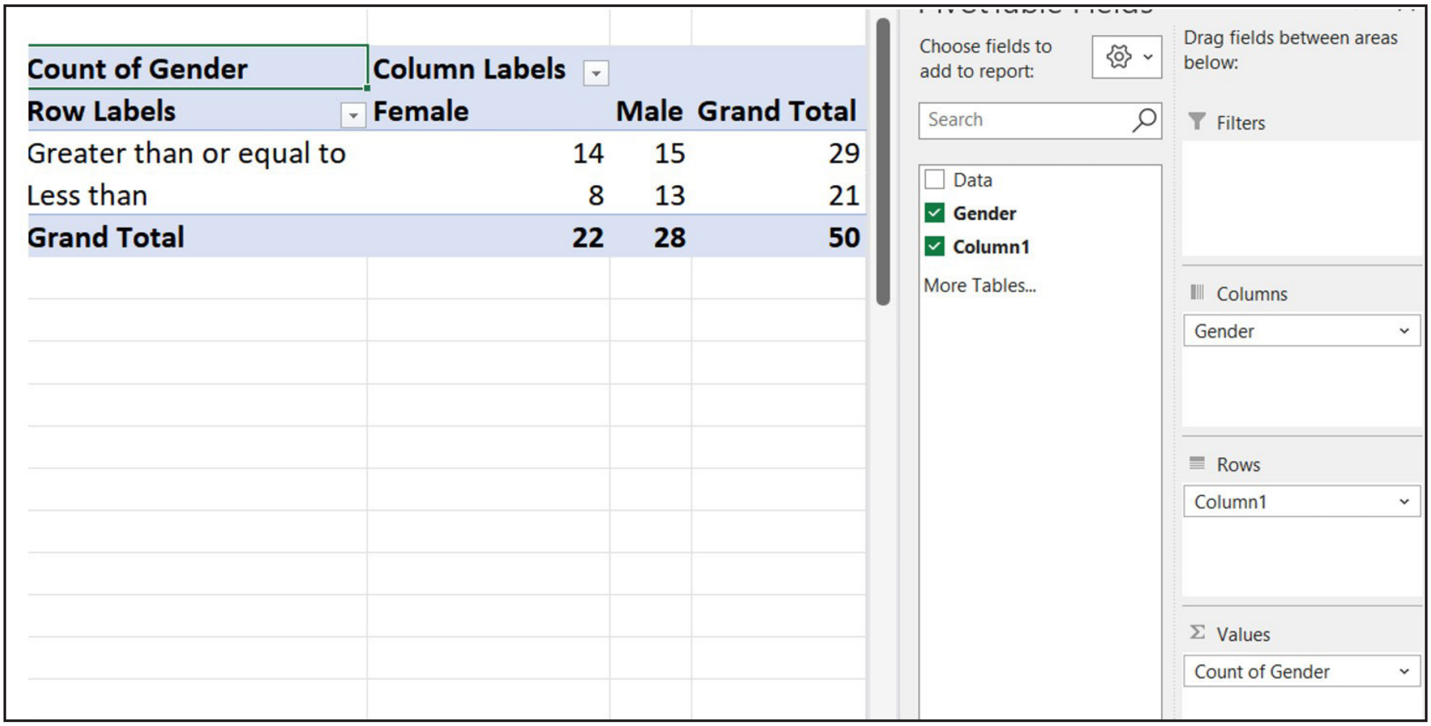

Figure 9. Example of Activity 3, using anPivot table to create binomial Probability probability for each row.

Lastly, the students learned how to use the "Descriptive Statistics" tool to determine a 90%, 95%, and 99% confidence interval based on their sample mean using an Excel function in activity 4 (Figure 12). The students also learned how to calculate the sample size needed to reach a certain confidence level. They could also use the Excel function to calculate the margin of error for a confidence interval. Finally, they learned how to interpret their results.

Thus, the purpose of these activities was to help students realize that all statistics topics are interconnected and that previous knowledge is essential for moving on to the next topic. Additionally, these activi

P(male)	0.62
P(female)	0.38
P(greater than or equal to the mean	0.46
P(less than mean)	0.54

n	10				n	10		
p	0.38				p	0.46		
x	P(x)				x	P(x)		
0	0.00839299		P(3)	0.232	0	0.00210833	P(6)	0.169
1	0.05144093		P(at least 4)	0.566	1	0.01795981	P(more than 5)	0.283
2	0.1418774		P(6 and 7)	0.126	2	0.06884593		
3	0.23188564				3	0.15639075		
4	0.24871605				4	0.23313806		
5	0.18292665				5	0.23831891		
6	0.09343028				6	0.169177		
7	0.03272212				7	0.08235071		
8	0.00752081				8	0.02630648		
9	0.00102434				9	0.00497983		
10	6.2782E-05				10	0.00042421		
	1					1		

Figure 10. Example of Activity 3, finding the probability of success

P(Male)	0.67
n	10
x	p(x)
0	2.25393E-05
1	0.00043173
2	0.003721313
3	0.01900799
4	0.063715558
5	0.146452904
6	0.233769578
7	0.255871075
8	0.183790776
9	0.078231642
10	0.014984895

Figure 11. Example of Activity 3, finding the probability of male

ties allowed students to practice their ability to analyze and interpret data, as well as their ability to draw conclusions. This experience provided students with a strong foundation of statistical knowledge.

Through the use of individual data, students were able to explore their chosen topics in greater depth, recognizing the connections between various statistical concepts. Before working on these projects, students faced challenges with calculating conditional probabilities, binomial probabilities, and confidence intervals. However, the hands-on nature of this project-based learning approach helped them become more skilled with Excel. Additionally, the experience emphasized the significance of data analysis in the decision-making process and provided them with practice in interpreting data-driven results. In the end, the project reinforced the importance of using data analysis, particularly through Excel, for making informed decisions.

Further Study

Comparing two groups is one way to determine the effectiveness of learning statistics using Excel. Compared to a normal class setting where students use scientific calculators and formulas to calculate statistics outcomes, the Excel setting did not provide a significant difference. The following study needs to set up two different groups and compare students' learnings, and both groups require students' projects. Excel will be compared to a normal classroom setting in providing statistics learning. The next study could provide a more detailed comparison between the two methods by measuring students' understanding of the material, their ability to recall the information and their overall performance on exams. This would allow the researchers to determine if there is a significant difference in learning outcomes between the two groups.

Height (In Inches)		*Height (In Inches)*		*Height (In Inches)*	
Mean	67.3469388	Mean	67.3469388	Mean	67.3469388
Standard Error	0.43207991	Standard Error	0.43207991	Standard Error	0.43207991
Median	67	Median	67	Median	67
Mode	67	Mode	67	Mode	67
Standard Deviation	3.02455934	Standard Deviation	3.02455934	Standard Deviation	3.02455934
Sample Variance	9.14795918	Sample Variance	9.14795918	Sample Variance	9.14795918
Kurtosis	-0.2163296	Kurtosis	-0.2163296	Kurtosis	-0.2163296
Skewness	0.15694478	Skewness	0.15694478	Skewness	0.15694478
Range	13	Range	13	Range	13
Minimum	62	Minimum	62	Minimum	62
Maximum	75	Maximum	75	Maximum	75
Sum	3300	Sum	3300	Sum	3300
Count	49	Count	49	Count	49
Confidence Level(95.0%)	0.86875488	Confidence Level(90.0%)	0.72469487	Confidence Level(99.0%)	1.15892646
lower limit	66.478	lower limit	66.622	lower limit	66.188
upper limit	68.216	upper limit	68.072	upper limit	68.506
95% confindence interval is (66.478,68.216)		90% confidence interval is (66.622,68.072)		99% confidence interval is (66.188, 68.506)	

Figure 12. Example of Activity 4, using an Excel to create a confidence interval

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Exploring Quantitative Reasoning Skills in School Mathematics

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Abstract: Quantitative Reasoning (QR) has been a meaningful way to explore mathematical ideas because of its focus on thinking and reasoning skills. It refers to a measurable attribute of an object, where students should be able to understand various aspects of the quantifiable attributes, rather than just knowing quantitative information. One of the main elements of QR is its connection to reasoning with real-world problems, so it requires students to think creatively in conjunction with real-world math problems. Students usually find math learning enjoyable and engaging when math relates to real-world issues. Therefore, it is essential to incorporate the QR approach in teaching mathematics at the school level. It has been highly recommended to mathematics teachers to integrate the QR approach for effective mathematics lessons.

Introduction

Numbers and quantities are integral to human life, prevalent in an individual's everyday life. However, it is not that numbers and quantities are essential because they are prevalent; instead, they are prevalent because they are important (Lutsky, 2008). Individuals use numbers and quantities to make personal and professional decisions throughout their lives (Madison & Steen, 2008; Mayes & Shader, 2011). Lutsky asserts that numbers and quantities have the power to inform individuals and help them think thoughtfully and critically. A domain of mathematics that explicitly highlights individuals' reasoning and critical thinking is quantitative reasoning. Smith and Thompson (2007) state that "quantitative reasoning is a central dimension of students' mathematical development" (p. 41).

Quantitative reasoning (QR) is an individual's capacity to understand and use numbers and quantities in relation to one's context. QR capacities include (a) reading, understanding, and using numbers and quantities, (b) to solve everyday problems using mathematics and statistics, and (c) reasoning using quantitative evidence (Budhathoki, 2022; Madison & Steen, 2003; Thompson, 2011). QR is directly related to individuals' personal and professional lives, by providing them with a lens to understand and articulate mathematics in real-world contexts and solve problems from the real world (Madison & Steen, 2008) and model real-world phenomena in professional lives (Mayes & Shader, 2011). Thompson (1990) further states that "quantitative reasoning is an individual's analysis of a situation into a quantitative structure" (p. 13). Powell and Leveson (2004) state that quantitative reasoning refers to applying logical arguments and developing an understanding of how numbers and mathematics are applied. The Dana Center at the University of Texas (2014) and the National Numeracy Network (NNN) (2021) define QR as a habit of mind to work with numbers. Furthermore, the NNN explains it as the ability to use "higher-order reasoning and critical thinking skills needed to understand and to create sophisticated arguments supported by quantitative data" (p. 1).

Thompson (2011) further suggests that quantitative reasoning involves quantifying and reasoning about the relationship between quantities. Foley and Wachira (2021) state that QR refers to students' ability to think critically about how the world around us connects to mathematics and vice versa. Quantitative reasoning involves developing students' ability to interpret quantitative information and identify the appropriate skills and procedures to solve a specific problem (NCTM, 2000). In fact, critical thinking lies at the heart of QR (Budhathoki et al., 2025; Elrod, 2014).

Budhathoki (2024) stated that critical thinking is the sine qua non for a QR course. Thus, in general, quantitative reasoning refers to a measurable attribute of an object, where students should be able to understand various aspects of the measurable attributes, rather than just understanding quantitative aspects. Therefore, the QR advocates the constructivist learning approach, where students develop conceptual understanding of mathematics. To enhance critical thinking in learners, QR should be an essential aspect of teaching and learning mathematics at the school level. Infusing QR into mathematics lessons will promote conceptual understanding in learners (Johnson et al., 2022) and encourage students to explore and analyze new applications, emphasizing deep understanding and insight over rote procedures (Augistin et al., 2021). This paper aims to provide a brief overview of QR in the context of school mathematics, why it is important and how it can be embedded in instructional approaches to teaching and learning.

Historical ackground

It is worth mentioning that quantitative reasoning is referred to by other terms, like quantitative literacy, numeracy, or mathematical literacy, in different countries and contexts. Still, educators have discrepancies regarding whether these terms refer to the same or different attributes. For example, Budhathoki (2022), Madison and Steen (2008), Steen (1997), and Vacher (2014) argue that the terms are synonymous, referring to cultural contexts and may be used interchangeably, while others like Mayes et al. (2012), Powell and Leveson (2002), and Foley and Wachira (2021) consider the terms differ as they have different meanings and represent different aspects of a quantitative phenomenon (Budhathoki, 2022).

The Crowther Report (1959), a public report in the United Kingdom, is credited with using the term numeracy for the first time. This report defined numeracy as the "mirror image of literacy" (p. 269), a high school student's ability to solve sophisticated quantitative problems and understand and reason with scientific methods. However, numeracy did not receive importance until the publication of the Cockcroft Report (Cockcroft, 1982), another public report in the United Kingdom, which explained two prominent features of numeracy: (a) the capacity to use mathematics in everyday life, and (b) to figure out and appreciate quantitative information in mathematical terms (Budhathoki, 2022). This report even fostered discussion about numeracy (or quantitative literacy or quantitative reasoning) in the United States.

Several educators and professional organizations have identified quantitative reasoning as the most important 21st-century skill for adults in the United States. "There is a growing consensus in the United States that high school and college-level education should focus on developing students' QR skills" (Budhathoki, 2022). Several professional organizations, such as the National Council of Education and Disciplines (NCED), the Mathematical Association of America (MAA), and the National Council of Teachers of Mathematics (NCTM), have emphasized the importance of QR for high school graduates. The NCED, through its publication, Mathematics and Democracy: The Case for Quantitative Literacy (Steen, 2001), stated quantitative literacy as the extension of mathematics into other subjects and recommended emphasizing QL across the curriculum. Similarly, the NCTM recommended QR as the necessary goal for high school graduates. In Catalyzing Change in High School Mathematics: Initiating Critical Conversations (NCTM, 2018), NCTM stated, "students should leave high school with the quantitative literacy and critical thinking processes needed to make wise decisions in their personal lives" (p. xi). Thus, the importance of QR has been growing and getting significant attention in the last couple of decades in the domain of mathematics.

QR in School Mathematics

Critical thinking is central to quantitative reasoning since QR fosters students' critical thinking and mathematical reasoning (Budhathoki et al., 2024; NCTM, 2018). Thus, developing students' QR skills in school mathematics is important. Nguyen and Tran (2025) explain the various phases of quantitative reasoning. They suggested that quantitative reasoning includes conceptualizing a situation (e.g., aircraft landing), quantities (e.g., distance from a stacking point), and relationships between quantities (e.g., distance and time). One of the main components of QR is the connection of reasoning with real-world problems.

As a result, quantitative reasoning requires students to think creatively in conjunction with real-world problems, as shown in Figure 1 (Foley & Wachira, 2021). When learning mathematics, connecting with the real world and real-world situation models with mathematics, students develop QR and enhance their conceptual understanding of mathematics.

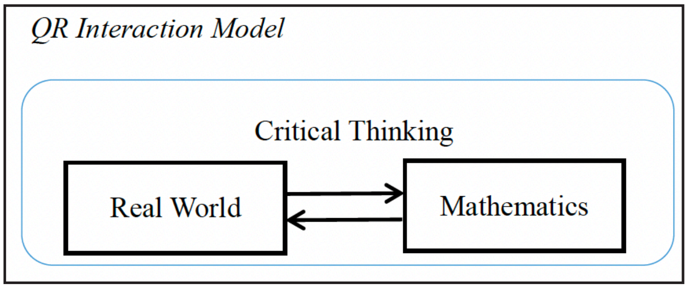


Figure 1: A model of student engagement in Quantitative Reasoning

The Association of American Colleges and Universities [AAC&U] (2009) listed six core competencies of QR: interpretation, representation, calculation, assumptions, analysis/synthesis, and communication. The latter three competencies refer to critical thinking, which refers to an individual's ability to critically understand quantitative information and data, infer conclusions, and communicate the conclusions. The six core competencies for quantitative reasoning) (omit)) are as follows:

- Interpretation: The ability to glean and explain mathematical information presented in various forms (e.g., equations, graphs, diagrams, tables, words, i.e. verbally).
- Representation: The ability to convert information from one mathematical form (e.g., equations, graphs, diagrams, tables, words) into another.
- Calculation: The ability to perform arithmetical and mathematical calculations.
- Analysis/Synthesis: The ability to make and draw conclusions based on quantitative analysis.
- Assumptions: The ability to make and evaluate important assumptions in estimation, modeling, and data analysis.
- Communication: The ability to explain thoughts and processes regarding what evidence is used, how it is organized, presented, and contextualized.

In instructional strategy, we can carefully design mathematical tasks that cannot be done traditionally, but rather the tasks would require students to go through the six core principles. For instance, Taylor (2008) suggested that teachers must provide students with more opportunities to make decisions that require gathering and evaluating information, conducting quantitative analyses, and communicating about quantitative concepts, rather than simply performing textbook-based mathematical calculations. We need to focus on learning mathematics as interconnected relations across the disciplines, rather than teaching it as a set of disconnected rules, formulas, and standard algorithms. Thus, the QR approach differs from the traditional teaching and learning mathematics approach. For example, Madison described the differences in just focusing on mathematics versus infusing the QR approach in Table 1.

Table 1 highlights the distinction between traditional mathematics and quantitative reasoning (QR).

Typical Mathematics	Quantitative Reasoning
Power in abstraction	Real, authentic contexts
Power in generality	Specific applications
Some context dependency	Heavy context dependency
Society independent	Society dependent
Apolitical	Political
Methods and algorithms	Ad hoc methods
Well-defined problems	Ill-defined problems
Approximation	Estimation is critical
Heavily disciplinary	Interdisciplinary
Problem solutions	Problem descriptions
Few opportunities to practice outside the classroom	Many practice opportunities outside the classroom
Predictable	Unpredictable

Table 1: Contrast between typical mathematics and quantitative reasoning

While mathematics emphasizes abstraction, generality, and structured, well-defined problems solved through established methods, QR focuses on applying mathematical thinking in real, authentic, and often unpredictable contexts. QR is highly context-dependent, interdisciplinary, and socially relevant, dealing with ill-defined problems that require estimation and flexible, ad hoc approaches. Unlike typical mathematics teaching, which is primarily confined to classroom settings, QR offers numerous opportunities for real-world practice, making it essential for preparing students to navigate complex societal and practical challenges. This distinction is important in education because it suggests that to prepare students for the real world, we must go beyond traditional math and include QR to build adaptable, critical thinkers.

Promoting QR

There are multiple pedagogical approaches through which mathematics teachers can integrate quantitative reasoning (QR) into mathematics instruction. A key strategy involves embedding real-life contexts into mathematical lessons and emphasizing conceptual understanding by encouraging exploration of the "why" and "how" behind mathematical reasoning, rather than focusing solely on procedural fluency. Weber et al. (2014) propose six practical strategies to foster QR within mathematics classrooms, aiming to enhance students' quantitative reasoning skills in meaningful and authentic ways. The six tips are as follows:

1. Rewrite a problem situation or prompt so the students can identify the quantities they believe are relevant to solving the problem.

2. Ask questions about a problem that focuses on why students chose to identify quantities and how they intend to or imagine measuring them.

3. Have students identify and test relationships between the quantities that they measure. Push them to justify why those relationships always or do not always hold.

4. Once students have created a model, ask them to determine how varying individual quantities affect the rest of the quantities in the model.

5. Have students develop a representation (physical, visual, etc.) of the situation they are modeling that consists of all the quantities and their interrelationships.

6. Have students revise and retest aspects of their model that may not have been accurate.

To incorporate QR in mathematics lessons, teachers can be creative in designing rich mathematical tasks that will likely enhance students' creative and critical thinking, emphasizing QR. One of the essential components of QR is the infusion of real-life context when designing mathematical tasks. In traditional pedagogy, mathematical concepts are often introduced in more abstract ways, without making the learning of mathematics more meaningful. The following is an example of a math task that compares traditional versus quantitative reasoning, as explained by Foley and Wachira (2021).

- Both tasks are designed to teach the concept of arithmetic progression; however, Task B is introduced within a real-life context. Task A has only one correct answer, whereas Task B presents an open-ended context, with a low floor and a high ceil-

Task A: Traditional	**Task B: Quantitative Reasoning**
If the sequence 3, 7, 11, 15, ... continues an arithmetic (additive) pattern, what is the value of the 100th term?	On the first day of April, Brianna has \$1.50 in her piggy bank. On April 2nd, she adds 25¢. Briana keeps adding a quarter dollar each day. Model this situation in at least two ways. Write and answer two questions about the situation.

Table 2: Example of traditional and QR tasks

To enhance critical thinking in learners, QR should be an essential aspect of teaching and learning mathematics at the school level. Infusing QR into mathematics lessons will promote conceptual understanding in learners (Johnson et al., 2022) and encourage students to explore and analyze new applications, emphasizing deep understanding and insight over rote procedures (Augistin et al., 2021).

ing. The real-life context in Task B allows students to approach the problem in various ways, whereas Task A can be solved simply by using a formula. Thus, Task B is cognitively more demanding than Task A. Furthermore, Task A involves only numbers, whereas Task B engages with quantities—numbers in context, with associated attributes and units of measure. The following are two additional examples that elaborate on the concept of task design based on QR.

- I walk home from school in 30 minutes, and my brother takes 40 minutes. My brother left 6 minutes before I did. In how many minutes will I overtake him? (Krutetski, 1976, p. 160)
- John is taller than Mary. John is shorter than Jane. Who is the tallest? (Krutetski, 1976)

Neither task requires a specific method to complete; instead, they demand more quantitative reasoning and mathematical thinking. Each task is based on a particular situation, using real-life examples that encourage students to engage with the problems—regardless of their weaknesses in mathematics—since the tasks do not rely on a specific procedure for completion.

QR is gaining widespread attention, particularly at the intermediate college level; however, it is equally important in school mathematics, especially at the high school level. Given that QR incorporates real-life components, it also appears equally relevant in lower-grade mathematics. One important pedagogical approach in elementary school mathematics is integrating real-life contexts to make learning more meaningful for students. Likewise, emphasizing conceptual understanding—rather than relying solely on standard algorithms—has become a key focus in the teaching and learning of mathematics at the elementary level. Therefore, emphasizing QR in mathematics lessons will likely enhance conceptual understanding and make learning mathematics more meaningful. At the elementary level, QR might involve, for example, completing an addition problem using multiple strategies, such as using a number line, breaking numbers into ten frames, or composing/decomposing numbers. Consider the following math tasks:

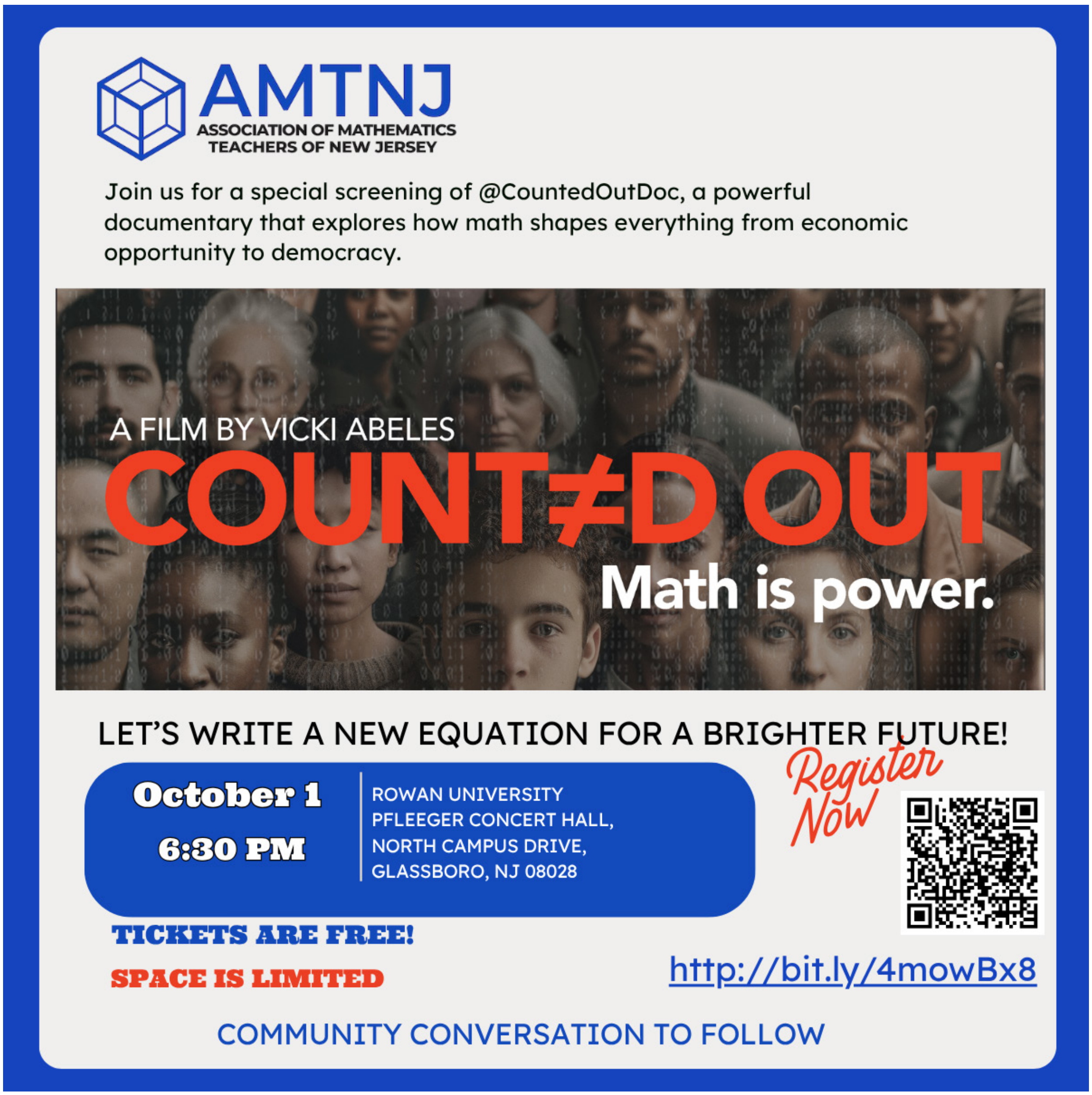
1. 6 +5 = ____
2. Bob has 11 fruits in his basket, including mangoes and apples. If the basket has 6 apples, how many mangoes are there?
3. Bob has 11 fruits in his basket, some are mangoes and some are apples. How many mangoes and apples can Bob have in his basket?

The three math tasks may appear similar; however, the first one is more abstract and lacks a real-life context, representing a more traditional approach to teaching the concept of addition at the elementary level. Although the second and third examples are based on real-life contexts, the third example requires significantly more mathematical reasoning and is cognitively more demanding. Therefore, the third example better fosters students' quantitative reasoning (QR) skills. A common assumption in teaching and learning mathematics is that there is only one correct answer to any given problem. However, the third example allows for multiple correct solutions. In this regard, students cannot immediately predict a single solution, which aligns with one of the key characteristics of QR as outlined in Table 1. Thus, we suggest that QR skills are equally important even at the elementary level. Introducing QR early in schooling is essential for developing a strong mathematical foundation that supports learning in later grades, including collegiate mathematics courses.

Conclusion

The 21st century is the age of information and technology. The sources of information are expanding rapidly, but at the same time, the information is becoming increasingly complex. One of the main elements of information is quantity and numbers. To prepare our students to be more competent in the 21st-century workforce, we must teach Quantitative Reasoning (QR) skills. As a result, the importance of QR skills has increased significantly over the past couple of decades.

Furthermore, QR is a practical pedagogical approach, as reasoning and critical thinking are essential components of teaching and learning mathematics. Emphasizing QR in mathematics instruction enhances effectiveness and makes learning mathematics more enjoyable and meaningful for students. Therefore, incorporating a QR approach into teaching and learning mathematics is important. While it may seem that employing a QR approach in math lessons is time-consuming, it is not as difficult as it appears. Appropriate planning and carefully selecting mathematical tasks are essential, but these efforts greatly enhance students' understanding of mathematical concepts. QR-oriented professional development would help teachers feel more confident and comfortable integrating QR approaches into their mathematics lessons.



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