Proof of Why the Slide and Divide Method Works and its Modified Version Suitable for Educational Purposes

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Abstract: The Slide and Divide Method emerged as a fast alternative for factoring Quadratic Trinomials of the form ax^2+bx+c, where a, b, and c are integers, $a \neq 1$, and the Greatest Common Divisor (GCD) of a, b, and c is 1. Nevertheless, despite its seemingly magical effectiveness, this method lacks a formal mathematical proof explaining why it works and its algorithm exhibits steps that are not rationally justified, making it unsuitable for formal instruction in a college setting. This paper aims to achieve a dual objective: first, to formally prove the validity of the Slide and Divide Method, and second, to introduce a modified version that not only sustains its efficacy but also incorporates a logical pedagogical approach suitable for formal education.to promote evidence-based instructional change.

Introduction

Factoring is the process of rewriting an algebraic expression as a product of simpler expressions, typically polynomials. For instance, the quadratic expression $x^2 - 5x + 6$ can be factored into $(x - 2)(x - 3)$. This technique is fundamental in algebra as it simplifies complex expressions, facilitates the solving of equations, and provides insights into the properties of functions. Factoring is especially valuable in understanding the structure of equations and finding their solutions efficiently.

Why is Factoring Important for Teachers?

Factoring is a fundamental skill in algebra that appears in many areas of mathematics, including Calculus, Number Theory, and Linear Algebra. It plays a crucial role in determining the roots (or zeros) of polynomial functions, which is essential for graphing and analyzing their behavior. Beyond pure mathematics, factoring is widely used in fields such as physics, engineering, economics, and cryptography. For example, in calculus, factoring polynomials is key to solving optimization problems. Additionally, many standardized tests, such as the SAT, GRE, and other entrance exams, include factoring questions, highlighting its importance in mathematical proficiency.

Common Misunderstandings about Factoring

Some students believe that all polynomials can be factored over the integers. This misconception arises because many introductory algebra problems focus on polynomials that factor neatly over the integers, such as quadratics with integer roots or special cases like the difference of squares. As a result, students may think all polynomials can be factored over integers. Additionally, factoring techniques like grouping, the rational root theorem, and synthetic division often lead students to expect integer factorization to always be possible. However, polynomials like $x^2 + 1$ or $x^4 + 4$ demonstrate that not all polynomials can be factored over the integers. Research also indicates that students often struggle to identify polynomials that are not factorable over the integers. A study analyzing engineering students' misconceptions in algebra found that 88% of participants had difficulty recognizing such polynomials. Ancheta and Subia (2020, p. 68) attribute these errors to a lack of conceptual understanding and reliance on rote memorization.

Many students struggle with factoring out the greatest common factor first. For example, they may try to factor $2x^2 - 2x - 40$ directly without first extracting the common factor of 2, which can make the process more complicated than necessary. While factoring is a useful method, not all quadratic equations can be solved this way. The quadratic formula or completing the square may be necessary.

Method

Factoring quadratic trinomials

Traditionally, students are instructed in two primary approaches for factoring quadratic trinomials of the form $ax^2 + bx + c$, where a, b, and c are integers. Note that $a \neq 0$ (if $a = 0$, then $ax^2 + bx + c$ becomes $bx + c$, a linear binomial) and $a \neq 1$ (if a is 1, then $ax^2 + bx + c$ becomes $x^2 + bx + c$, which we already know how to factor). One of the most common methods of factoring is employing the *ac* Method as described by Pieronkiewicz and Tanton (2019) on page 109. Below is a brief summary of the *ac* method.

Step 1. Multiply a and c (the coefficient of x^2 and the constant term).

Step 2. Find two numbers that multiply to ac and add to b (the coefficient of x).

Step 3. Split the middle term bx using these two numbers.

Step 4. Group the terms into two binomials.

Step 5. Factor the common terms in each group.

Example: Factor $6x^2 + 11x + 4$.

Step 1. $ac = (6)(4) = 24$.

Step 2. Find two numbers that multiply to 24 and add to 11: 8 and 3.

Step 3. Rewrite: $6x^2 + 8x + 3x + 4$.

Step 4. Group: $(6x^2 + 8x) + (3x + 4)$.

Step 5. Factor: $2x(3x + 4) + 1(3x + 4)$.
Final answer: $(2x + 1)(3x + 4)$.

The *ac* Method works well, but it requires knowledge of factorization by grouping. On the other hand, students may have difficulty understanding why they need to multiply a by c.

Alternatively, students are also taught The Trial and Error Method, as outlined with examples in the OER textbook OpenStax Intermediate Algebra 2e. While this method works, it often involves a significant number of trials, posing a challenge for students to determine whether the polynomial is factorable or a prime polynomial (not factorable).

An additional method, widely known as the Slide and Divide Method (also identified as “bottoms-up factoring”), appeared as an "informal" technique to simplify and expedite the two methods mentioned above, as explained by Fosnaugh and Mitchell (2014). The Slide and Divide Method is commonly called "informal" because it seems to work almost magically. Most of the steps in this method lack a clear logical foundation. Additionally, there is no formal proof explaining why it works. Because of this, instructors do not teach it as an alternative for factoring Quadratic Trinomials.

Slide and Divide Method

Now, the Slide and Divide method will be explained in detail as to how it works, and the procedures involved with the method. Let’s start by presenting the Slide and Divide Method and then formally proving why it works, as we know prerequisite knowledge is always important to understand mathematical

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concepts. The prerequisite knowledge needed for this method is shown below:

We assume that a quadratic trinomial is a polynomial of the form $ax^2 + bx + c$, where a, b, and c are integers and $a \neq 0$.

- A quadratic trinomial is factorable if it is factorable over the set of integers.
- We assume that students have mastered factoring monic quadratic trinomials of the form $x^2 + bx + c$.
- We assume that students are familiar with the use of the following arithmetic properties: Commutative property, Associative property, and Distributive property.
- We assume that students have mastered factoring and finding the Greatest Common Divisor (GCD) of a binomial and of a trinomial (if needed).
- When factoring a quadratic trinomial $ax^2 + bx + c$, we assume that $a > 0$; if not we can always factor out -1 and make the leading coefficient positive.

The Slide and Divide method is used to factor quadratic trinomials of the form $ax^2 + bx + c$ when $a \neq 1$, because if a is equal to 1 then $ax^2 + bx + c$ becomes $x^2 + bx + c$ and we already know how to factor it. Below is an explanation of the Slide and Divide method as it stands now.

The Slide and Divide Method for factoring quadratic trinomials

Step 1: We first verify that GCD(a, b, c) = 1: for if not, we factor out the GCD, multiply a and c (the leading and constant coefficients) to obtain a new trinomial $x^2 + bx + ac$.

Step 2: Factor the new trinomial as if the leading coefficient were 1.

Step 3: Divide the constants of the binomial factors by a. If the result of the division after simplifying is a fraction, place the denominator in front of x as needed.

Step 4: Write the final factor form.

Example: Factor $15x^2 - 7x - 2$

Step 1: After verifying that GCD(15,-7,-2) = 1, multiply the first and the last coefficients,

(15)(-2) = -30

Step 2: After replacing the leading coefficient 15 for 1 and the constant term -2 for -30, factor $x^2 - 7x - 30$, which leads to:

$(x - 10)(x + 3)$.

Step 3: Divide -10 and 3 by 15 and simplify:

$(x - 10/15)(x + 3/15)$.

Simplifying, we obtain $(x - 2/3)(x + 1/5)$.

Place the remaining denominators, 3 and 5, in front of x in each binomial factor (This is why this method is often called "bottoms-up factoring").

Step 4: We obtain the final factorization

$(3x - 2)(5x + 1)$.

Discussion

Although the Slide and Divide Method successfully factored $15x^2 - 7x - 2$, it presents certain challenges. The issues are discussed below:

- In Step 1, no explanation is given for why we multiply (15)(-2) = -30
- In Step 2: The polynomials $15x^2 - 7x - 2$ and $x^2 - 7x - 30$ are not the same, so why should we factor $x^2 - 7x - 30$ in order to factor $15x^2 - 7x - 2$?
- In Step 3: Why should we divide only the constant parts of the binomial factors?
- In Step 3: Why after simplifying must we place the denominators in front of x on each binomial factor?

The Slide and Divide Method proved to be highly effective, but its underlying logic often gets lost in the process. This is one of the primary reasons why many mathematics instructors don't prefer to teach

Based on my own teaching experiences over the years at various colleges, I've often observed instructors using this method in their own classrooms to help students quickly factor quadratic trinomials.

Proof of Slide and Divide Method...............(0)

Below, an algebraic proof of the Slide and Divide Method is explained.

Given a Quadratic Trinomial $ax^2 + bx + c$ such that GCD(a, b, c) = 1, and $a \neq 1$.

Factor $ax^2 + bx + c$.

Assume that GCD(a, b, c) = 1; for otherwise we can always factor out the GCD(a, b, c) and continue with our method.

Firstly, if $ax^2 + bx + c$ is factorable, then there exists d, e, f, and g integers such that $ax^2 + bx + c = (dx + e)(fx + g)$.

In order to factor $ax^2 + bx + c$ we must find d, e, f, and g.

Since $ax^2 + bx + c = (dx + e)(fx + g)$ after expanding the right side of the equation, then
$ax^2 + bx + c = dfx^2 + (dg + ef)x + eg$.

Where $a = df$, $b = dg + ef$, and $c = eg$............(1)

Now, we have $df = a$ and $eg = c$ (This is the case because both a and c are given).

We need to find ef and dg satisfying $ef + dg = b$ and $efdg = ac$.

From the above steps, this is where the Slide and Divide Method starts. We noticed that the above expression is equivalent to replacing c for ac, and factoring $x^2 + bx + ac$ (These are Steps 1 and 2 in the Slide and Divide Method).

Now,

$$ax^2 + bx + c = a(ax^2 + bx + c) \div a$$

$$ax^2 + bx + c = (a^2x^2 + abx + ac) \div a$$

$$ax^2 + bx + c = ((ax)^2 + b(ax) + ac) \div a$$

Now, suppose we find $m = ef$ and $n = dg$ satisfying $ef + dg = b$ and $efdg = ac$

Then using a change of variable $z = ax$, we can factor:

$$(ax)^2 + b(ax) + ac = (ax + ef)(ax + dg)$$

Then

$$ax^2 + bx + c = (ax + ef)(ax + dg) \div a$$

$$ax^2 + bx + c = a(x + ef/a)a(x + dg/a) \div a$$

$$ax^2 + bx + c = a(x + ef/a)(x + dg/a)$$

From (1), substituting $a = df$, we obtain:
$ax^2 + bx + c = a(x + ef/df)(x + dg/df)$

Clarification: This is where the Slide and Divide Method instructs us to divide the constant terms of the binomial factors by $a = df$ and simplify (Step 3 in the Slide and Divide Method).

After simplifying, we obtain:

$$ax^2 + bx + c = a(x + e/d)(x + g/f)$$

$$ax^2 + bx + c = a/df\,(dx + e)(fx + g)$$

From (1), substituting $a = df$, we obtain:

$$ax^2 + bx + c = (dx + e)(fx + g)\text{……………..(2)}$$

Clarification: the last three equations above are equivalent to the last step in The Slide and Divide Method where we are asked to place the denominators left, after simplifying, in front of x on both binomial factors (Step 3 of the Slide and Divide Method).

In equation (2) d and e are relatively prime, because otherwise they would generate a common divisor different than 1 for a, b, and c, which would contradict the fact that GCD(a, b, c) = 1.

For the same reason explained above f and g also are relatively prime…….....…………………..(3)

Then the factorization $ax^2 + bx + c = (dx + e)(fx + g)$ is a full factorization.

As we have seen, the Slide and Divide Method works by extracting 'valid partial results' that naturally arise from a formal sequence of logical steps, but it does not explain why these results are true. Since the logical foundation is not explicitly shown in the Slide and Divide Method, its 'simplified form' becomes an attractive and efficient algorithm for students, especially when the final factorization is the primary focus.

Now that we have formally proven why the Slide and Divide Method works, let's discuss the relationship between the polynomials $ax^2 + bx + c$ and $x^2 + bx + ac$.

Properties between the polynomials $ax^2 + bx + c$ and $x^2 + bx + ac$

- The Discriminants of $ax^2 + bx + c$ and $x^2 + bx + ac$ are the same and equal to $b^2 - 4ac$. This can be proven by applying the discriminant formula for both quadratic trinomials.
- Since $ax^2 + bx + c$ and $x^2 + bx + ac$ have the same Discriminant then they have the same type of roots, they could be both real or complex numbers.
- If the roots of $ax^2 + bx + c$ are x_1 and x_2, then the roots of $x^2 + bx + ac$ are ax_1 and ax_2. This can be proven by applying the quadratic formula for both quadratic trinomials.
- Regarding transformations, converting from $ax^2 + bx + c$ to $x^2 + bx + ac$ involves a horizontal stretch by a factor of a, followed by a vertical stretch by a factor of a. This is achieved through a change of variable (substituting x/a for x) and then multiplying the resulting polynomial by a.

A visualization of the last property via an example is shown below.

Figure 1 illustrates how the polynomial $6x^2 - 11x + 3$, after the two transformations mentioned above, becomes $x^2 - 11x + 18$. The dynamic geometry software Desmos is used to provide visual representation of the procedure.

Notice that the roots of $6x^2 - 11x + 3$ are 1/3 and 3/2. After the first transformation, these roots are multiplied by 6, resulting in the roots of $6(x/6)^2 - 11(x/6) + 18$, which are $6(1/3) = 2$ and $6(3/2) = 9$. These roots remain unchanged in the last transformation, as it only affects the y-values.

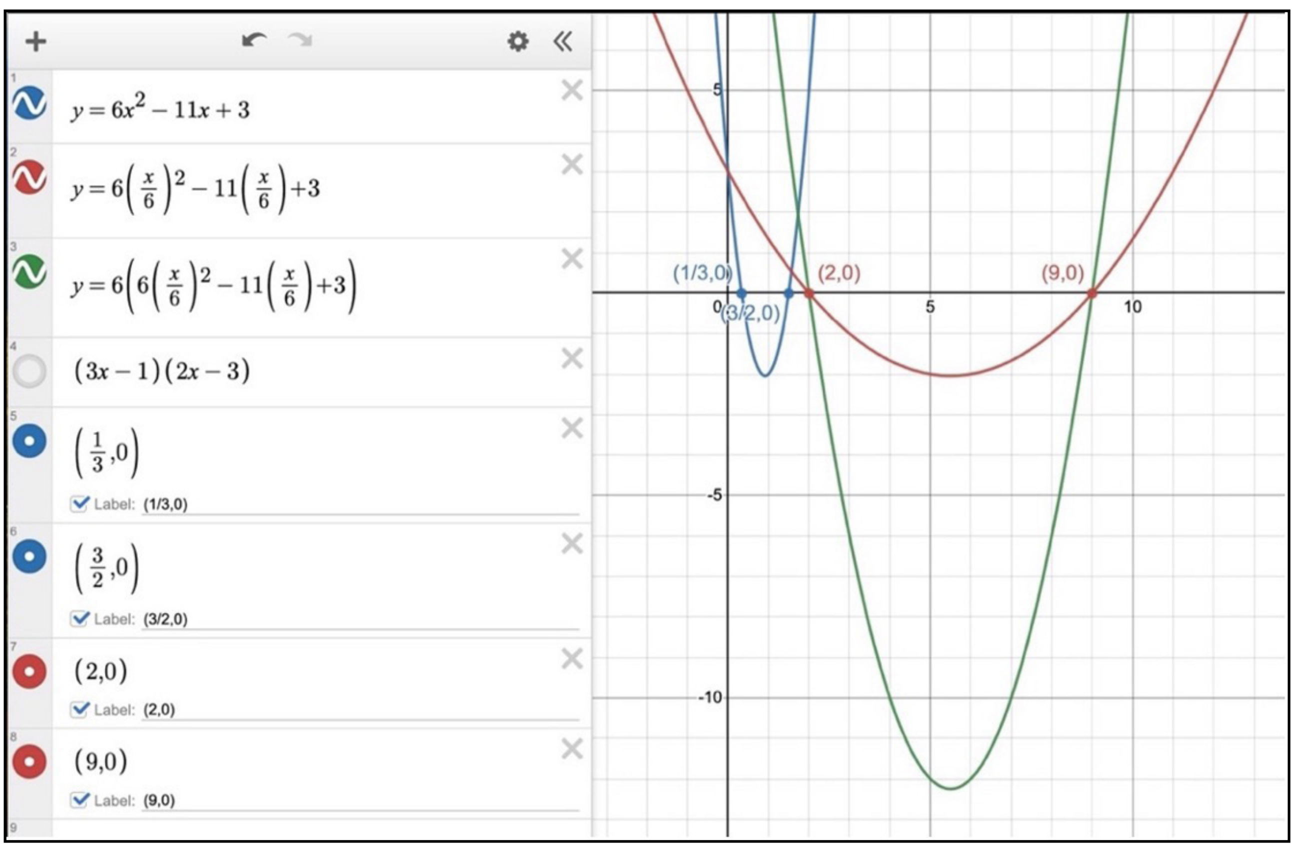

Figure 1: transformation of $6x^2 - 11x + 3$ into $x^2 - 11x + 18$ (Graphed via DESMOS)

The properties mentioned above suggest that the factorization of a quadratic trinomial $ax^2 + bx + c$ and the factorization of its associated monic quadratic trinomial $x^2 + bx + ac$ must be related; in other words, if we want to factor $ax^2 + bx + c$, we can factor $x^2 + bx + ac$ and then transform it into the given quadratic trinomial in factored form, all these remarks together with the proof of why the Slide and Divide Method works in (0), served as a motivation for developing a modified version of the Slide and Divide Method.

There are many algorithms for factoring quadratic trinomials, some of which use formulas involving radicals and derivatives from calculus, as seen in Joarder (2021). However, we will develop a method that is a variation of the Slide and Divide Method, suitable for college algebra students. For this purpose, we assume that students have mastered factoring quadratic trinomials of the form $x^2 + bx + c$ using the reverse FOIL method.

Because our proof in (0) was constructive, we can now introduce a logical algorithm for factoring a quadratic trinomial, called the Modified Slide and Divide Method.

The Modified Slide and Divide Method

Recall that our goal is to present an algorithm with a clear logical flow, enabling students to follow each step with full understanding.

Algorithm

Factor $ax^2 + bx + c$ where a, b, and c are integers, GCD(a, b, c) = 1, and $a \neq 1$.

Step 1: Multiply $ax^2 + bx + c$ by a on the left and divide it by a on the right.

$$ax^2 + bx + c = a(ax^2 + bx + c) \div a$$

Clarification: Multiplying a polynomial on the left and right by the same number does not change its value.

Step 2: Distribute the multiplication on the left.
$ax^2 + bx + c = (a^2x^2 + abx + ac) \div a$

Step 3: Write a^2x^2 as $(ax)^2$, apply the commutative and associative properties to obtain:

$$ax^2 + bx + c = ((ax)^2 + b(ax) + ac) \div a$$

Step 4: Letting $z = ax$ and factoring $(ax)^2 + b(ax) + ac$ by finding two numbers $m = dg$ and $n = ef$ such that $m + n = b$ and $mn = ac$, to obtain:

$$ax^2 + bx + c = (ax + m)(ax + n) \div a$$

$$ax^2 + bx + c = (ax + dg)(ax + ef) \div a$$

Step 5: Factor out the Greatest Common Divisor (GCD) of a and dg, which will be **d**, since $a = df$ and f is relatively prime to g, as explained in (1) and (3).

Factor out the Greatest Common Divisor (GCD) of a and ef, which will be f, since $a = df$ and d is relatively prime to e as it was explained in (1) and (3).

So, we obtain:

$$ax^2 + bx + c = d(fx + g)f(dx + e) \div a$$

Step 6: Since $a = df$ as shown in (1), then after simplifying, the final full factorization will be:

$$ax^2 + bx + c = (fx + g)f(dx + e)$$

Examples of the application of The Modified Slide and Divide Method.

Example 1: Factor $8x^2 + 10x - 3$

Step 1: Multiply $8x^2 + 10x - 3$ by 8 on the left and divide by 8 on the right.

$$8x^2 + 10x - 3 = 8(8x^2 + 10x - 3) \div 8$$

Step 2: Distribute the multiplication on the left.

$$8x^2 + 10x - 3 = (8^2x^2 + 8(10x) - 8(3)) \div 8$$

Step 3: Write 8^2x^2 as $(8x)^2$, apply the commutative, and the associative properties to obtain:

$$8x^2 + 10x - 3 = ((8x)^2 + 10(8x) - 24) \div 8$$

Step 4: Letting $z = 8x$ and factoring $(8x)^2 + 10(8x) - 24$ by finding two numbers m and n such that $m + n = 10$ and $mn = -24$ we obtain:

$$8x^2 + 10x - 3 = (8x - 2)(8x + 12) \div 8$$

Step 5: Factoring out the Greatest Common Divisor (GCD) from each binomials gives:

$$8x^2 + 10x - 3 = 2(4x - 1)4(2x + 3) \div 8$$

Step 6: Simplifying, we obtain the factorization:

$$8x^2 + 10x - 3 = (4x - 1)(2x + 3)$$

Example 2: Factor $3x^2 + 19x + 20$

Step 1: Multiply $3x^2 + 19x + 20$ by 3 on the left and divide by 3 on the right.

$$3x^2 + 19x + 20 = 3(3x^2 + 19x + 20) \div 3$$

Step 2: Distribute the multiplication on the left.

$$3x^2 + 19x + 20 = (3^2x^2 + 3(19x) + 3(20)) \div 3$$

Step 3: Write 3^2x^2 as $(3x)^2$, apply the commutative and the associative properties to obtain:

$$3x^2 + 19x + 20 = ((3x)^2 + 19(3x) + 60) \div 3$$

Step 4: Letting $z = 3x$ and factoring $(3x)^2 + 19(3x) + 60$ by finding two numbers m and n such that $m + n = 19$ and $mn = 60$ we obtain:

$$3x^2 + 19x + 20 = (3x + 4)(3x + 15) \div 3$$

Step 5: Factoring out the Greatest Common Divisor (GCD) from each binomial gives:

$$3x^2 + 19x + 20 = (3x + 4)3(x + 5) \div 3$$

Step 6: Simplifying, we obtain the factorization:

$$3x^2 + 19x + 20 = (3x + 4)(x + 5)$$

CONCLUSION

The Modified Slide and Divide Method to factor quadratic trinomials of the form $ax^2 + bx + c$ where a, b, and c are integers and $a \neq 1$, fundamentally addresses the lack of logical foundation in the Slide and Divide Method, providing mathematics Instructors an alternative method for factoring quadratic trinomials. This method can be taught as a sequence of logical steps that students can follow, based on formal mathematical reasoning and fundamental properties such as the commutative, associative, and distributive properties. It also uses the principle that multiplying and dividing a polynomial on the left and right by the same nonzero number does not change the polynomial's value. Additionally, our method does not require knowledge of factoring by grouping; instead, it relies only on the factorization of monic polynomials of the form $x^2 + bx + c$, a skill commonly mastered by undergraduate students in college algebra courses.

From an educational perspective, our method introduces the basic concept of a change of variable ($z = ax$, where a is an integer) early on. This illustrates how a one-to-one linear function can treat ax as a new variable in one-to-one correspondence with x, rather than simply as a constant times a variable.

While the Modified Slide and Divide Method may not be as appealing as the original Slide and Divide Method (it has two additional steps), its logical structure compensates for this. Additionally, it highlights the benefits of the Slide and Divide Method, even in its original form. For example, we proved in (0) that it can be used to quickly determine whether a quadratic trinomial (with $a \neq 1$) is factorable by checking if two integers m and n can be found such that $m + n = b$ and $mn = ac$. Most importantly, our Modified Slide and Divide Method provides a formal justification for instructors to teach the Slide and Divide Method openly, rather than implicitly. Using simple mathematics based on our Modified Slide and Divide method, instructors can explain to students the benefits and limitations of the original Slide and Divide Method and present it as an alternative to quickly factor quadratic trinomials.

The Modified Slide and Divide Method for factoring quadratic trinomials serves as both an alternative and a validation tool for the widely used and effective original Slide and Divide Method. It enhances understanding by emphasizing the logical foundations of mathematics and will be particularly appreciated by students who are critical thinkers, not just those who follow procedures mechanically. Lastly, hopefully it will make instructors who teach the original Slide and Divide Method feel that they are not magicians but mathematicians, who truly care about the reasons behind the algorithms that they present in class.

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