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| M2205zambrano | Time Value of Money: Value of Cash Flow with interest. | $PV = \int_0^T f(t) e^{-rt} dt$ $FV = \int_0^T f(t) e^{+rt} dt$ |
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Suppose you won \$2,000. Would you rather collect \$2,000 today or \$2,000 five years from now? WHY?

1. Interest lost?
2. Inflation risk?
3. Collection risk?

Present Value—Recall the present value of a fixed one-time payment at future time T is: $PV = FV \cdot e^{-rT}$.

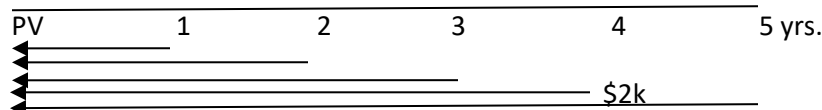
What is the present value (PV) of a \$2,000 payment to be collected five years in the future (FV)? Assume $r = 8\%$.

(Terminology: 8% is the *discount rate*, since it is the rate at which a future payment is discounted to its present value.)

Present Value of a Cash flow:

What is the *present value* of a cash flow rate of \$2000 per year, continued for five years, earning 8% interest?

The thinking: “If we assume just five payments (one each year, starting at yr 1), we compute the present value of each future payment separately, and then add them all up to get the total PV ...” A time line is used in managerial finance to diagram *discounted cash flows*:



$$PV = 2000e^{-.08(1)} + 2000e^{-.08(2)} + \dots + 2000e^{-.08(5)} = \$7916.71. \text{ (or } \underline{\$8576.07} \text{ if 1}^{st} \text{ payment starting now, at } t=0)$$

Notice that this is exactly a Riemann sum--we are adding up the PVs of all payments ($\Delta t = 1 \text{ yr}$). However, cash flows in many applications, such as revenue or sales, can be approximated as a continuous flow rate, or stream.

The definite integral can realistically approximate the present value of this same cash flow rate ($f(t) = \$2000/\text{yr}$):

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| $PV = \sum_{i=1}^5 2000e^{-.08(i)} (1\text{yr}) \xrightarrow{\text{continuous cash flow}} PV = \int_0^5 2000 e^{-.08(t)} (dt);$ | $PV = \int_0^T f(t)e^{-rt} dt.$ |
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$$\int_0^5 f(t) e^{-.08(t)} dt = \int_0^5 2000 e^{-.08(t)} dt = 2000 \cdot \left. \frac{1}{-.08} e^{-.08t} \right|_0^5 = -\frac{2000}{.08} (e^{-.4} - e^0) = \frac{200}{.08} (1 - e^{-.4}) = \$8242.00$$

Future Value of a Cash Flow: Recall, the future account value of fixed present amount PV is: $FV = PV \cdot e^{rT}$.

But now suppose that you make regular deposits at the continuous rate of $f(t) = \$2000$ per year into an account that earns 8% return, compounded continuously. What is the total accumulated future value of your account after 5 years? (The account balance grows due to TWO factors: regular continuing cash deposits AND to exponential interest growth).

Since $FV = PV \cdot e^{rT}$, and substituting for PV of the cash flow $PV = \int_0^T f(t) e^{-rt} dt$, we get $FV = e^{rT} \int_0^T f(t) e^{-rt} dt$.

So, $FV = e^{rT} \int_0^T f(t) e^{-rt} dt = e^{.08(5)} \cdot (\$8242) = \$12295.62$. So the future value of the cash flow is = \$12295.62.

OR, alternately, integrate cash flow WITH exponential growth: $FV = \int_0^T f(t) e^{rt} dt = \int_0^5 2000 e^{.08(t)} (dt) = \12295.62 .

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| Formulas for Value of cash flow, assuming cash flow rate $f(t) = R$, (constant rate) | $PV = \int_0^T f(t) e^{-rt} dt = \frac{R}{r} (1 - e^{-rT}).$ | $FV = e^{rT} \int_0^T f(t) e^{-rt} dt = \frac{R}{r} (e^{rT} - 1).$ |
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Present Value / Future Value of a Cash Flow (extending T years in the future):

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| <p>Formulas with cash flow, assuming cash flow rate $f(t) = R$ (constant rate)</p> | $PV = \int_0^T f(t)e^{-rt} dt = \frac{R}{r}(1 - e^{-rT}).$ | $FV = e^{rT} \int_0^T f(t)e^{-rt} dt = \frac{R}{r}(e^{rT} - 1),$ <p>OR, $FV = \int_0^T f(t)e^{rt} dt.$</p> |
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Exercises:

1. What is the present value of a uniform cash flow rate of \$1500 per year over a 10 year period, earning 5% interest compounded continuously? Ans. \$11,804.08
 Why is the present value less than the total of payments \$1500 (10 years)= \$15,000 ?

2. What is the future value of the same uniform cash flow rate of \$1500 per year in problem #1? Ans. \$19,461.64
 Why is the future value more than the total of payments \$1500 (10 years)= \$15,000 ?

3. Jenny makes regular payroll deposits of \$150 from each paycheck (biweekly—26 paychecks per year) to deposit into her 401k savings account. The average growth rate for the 401k mutual fund over the last 15 years is 7% annually, according to their prospectus.
 - a. How much total money in payroll deposit payments does Jenny contribute over the 10 year period? Ans. \$39,000
 - b. What is the present value of her cash flow (her payroll deposits) over the 10 year period? Ans. \$28,047.39
 - c. Compute the future value of Jenny's 401k account in 10 years, assuming continuous compounding of her cash flow at 7% annual growth rate. Ans. \$ 56,480.51
 - d. Compute the future value of Jenny's 401k account in 20 years, assuming she continues making regular contributions at the rate of \$3900 per year? Ans. #!\$\$?!?*\$!\$\$!

(P.S. Roth IRAs allow you to enjoy the entire future value of your account tax free. The IRS is onto this—they know they lose tax dollars from large future values of accounts. Hence, surprise-surprise...the IRS limits direct contributions to Roth IRA savings accounts to a maximum of about \$5-6000 per year. But Jenny is still in compliance with this regulation.)

Are YOU saving?