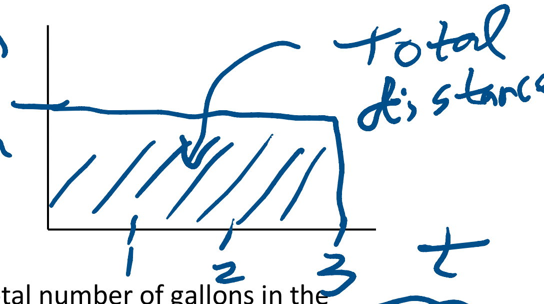
What is the significance of computing the area under a function f(x)??

Let $f(x)$ be a function representing a $rate\ of\ change$ at x.

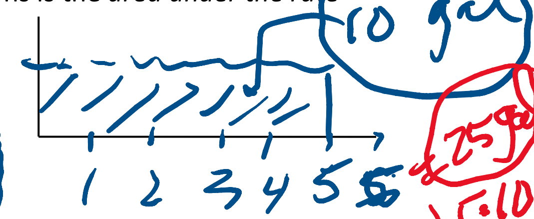
Draw a graph of $f(t)\ vs.\ t$ for the following examples:



Ex. If $f(t) = 30\ mph,\ what\ does\ area\ under\ f(t)$

$between\ t = 0\ to\ t = 3\ hours\ represent?$ ____________

Ex.1 If an empty tank is filled at the rate $f(t) = 2\ gal/min,$ what is the total number of gallons in the tank after 5 minutes? Note that this total accumulated quantity of 10 gallons is the *area under the rate function* $f(t)$ *from t=0 to t=5.*



Hence, the *area* under any *rate function* $f(x)$ between x=a and x=b represents the ________________

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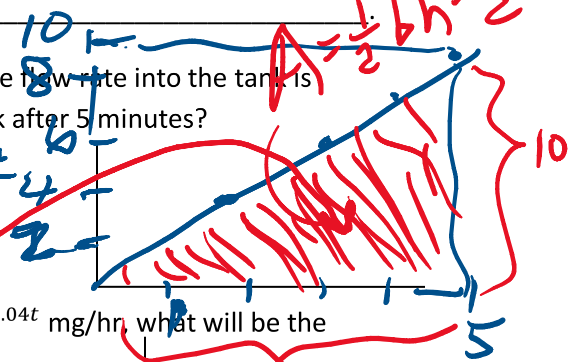
Ex.2 If an empty tank is filling at the rate given by $f(t) = 2t$ gal/min (the flow rate into the tank is continuously increasing!), what is the total number of gallons in the tank after 5 minutes?



Ex.3 If a cell culture is growing at the rate given by $f(t) = P'(t) = .10e^{.04t}$ mg/hr, what will be the total mass present after 5 hours given that $P(0) = 2.5$ mg.?

Ex. 4 If a producer has a marginal cost function given by $f(x) = C'(x) = 2x + \frac{5}{x}$ dollars/unit, and if total cost to produce 2 units is $10, what will be the total cost to produce 5 more units?

Ex.5 Insulin is absorbed by laboratory mice at a rate modeled by $f(t) = .01e^{-.21t}$ units/hr, what is the total amount of insulin absorbed by a 5 hour exposure if no insulin is present initially?

In each of the above examples, the total quantity measured that accumulates over a specified time period will be equal to the *area under the rate function* $f(x)$ given for that quantity!

Riemann sums –on a separate sheet of paper, approximate the *area* under the function $f(x) = x^2$ from x=1 to x=3 by drawing rectangles of width $\Delta x = 0.5$ under $f(x)$. There will be n=4 rectangles in this case. Draw it! Compute the height of each rectangle using the right endpoint of each interval.

1. The area desired will be the Sum of the areas of the rectangles, $S = f(x_1)\Delta x + f(x_2)\Delta x + \cdots$

2. Now, repeat the calculation using a smaller subinterval length of $\Delta x = 0.25$. (There will be n=8 rectangles)

The Definite Integral of a function f(x) on the interval [a, b] is defined to be the limit of the Riemann sum as the width of the rectangles $\Delta x \to 0$.

The Definite Integral is $\int_a^b f(x)dx = \lim_{\Delta x \to 0} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$.

P.S. Why this notation?

$$\lim_{\Delta x \to 0} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

$= \lim_{\Delta x \to 0} [\sum f(x_i)] \cdot \Delta x$, using summation notation; hence,

$$= \int_a^b f(x)dx.$$

Thus the definite integral of a function $\int_a^b f(x)dx = the\ AREA\ under\ f(x)\ from\ x = a\ to\ x = b$.

The Fundamental Theorem of Calculus

Computing a Riemann sum is tedious, even for simple functions. However, the fundamental theorem of calculus allows us to calculate most definite integrals quickly using antiderivatives!

$$\int_a^b f(x)dx = F(b) - F(a), where\ F\ is\ an\ antiderivative\ of\ f(x).$$

Ex. For $f(x) = x^2$, an antiderivative is $F(x) = \frac{x^3}{3} + C$. Hence, the definite integral of f(x) on [1,2] is

$\int_1^2 x^2 dx = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$.

Thus, the area under the graph of $f(x) = x^2$ between x=1 and x=2 is $\frac{7}{3}$. [$Note$: C will always cancel out, so we ignore it whenever we compute a definite integral.]

Exercises