Chapter 8: Roots and Radicals

8.1 Simplify Expressions with Roots

Perfect Squares	Perfect Cubes	Perfect 4^{th} degree	Perfect 5^{th} degree	Etc.
$1^2 = \mathbf{1}$	$1^3 = \mathbf{1}$	$1^4 = \mathbf{1}$	$1^5 = \mathbf{1}$	
$2^2 = \mathbf{4}$	$2^3 = \mathbf{8}$	$2^4 = \mathbf{16}$	$2^5 = \mathbf{32}$	
$3^2 = \mathbf{9}$	$3^3 = \mathbf{27}$	$3^4 = \mathbf{81}$	$3^5 = \mathbf{243}$	
$4^2 = \mathbf{16}$	$4^3 = \mathbf{64}$	$4^4 = \mathbf{254}$	$4^5 = \mathbf{1024}$	
$5^2 = \mathbf{25}$	$5^3 = \mathbf{125}$	$5^4 = \mathbf{625}$	$5^5 = \mathbf{3125}$	
Etc.				

Roots:

Square root

b is a square root of **a** if $\mathbf{b^2 = a}$

Ex.

5 is a square root of 25 since $5^2 = 25$

n – th root

b is an n-th root of **a** if $\mathbf{b^n = a}$

Ex.

3 is cube root of 27 since $3^3 = 27$

2 is forth root of 16 since $2^4 = 16$

$\sqrt[n]{a}$:

- If n > 1 is an **even** integer and a > 0, then $\sqrt[n]{\boldsymbol{a}}$ is the principal (positive) n-th root of **a**.

  Ex. $\sqrt[4]{16}$ = 2

- If n > 1 is an **odd** integer, then $\sqrt[n]{\boldsymbol{a}}$ is the n-th root of **a**.

  Ex. $\sqrt[3]{-8}$ = – 2

- If n > 1 is an integer, then $\sqrt[n]{\mathbf{0}}$ **= 0.**

  Ex. $\sqrt[7]{0}$ = 0

Example 8.1.1: Simplify.

a) $\sqrt{25}$ = 5

b) $\sqrt{\frac{4}{9}} = \frac{2}{3}$

c) $\sqrt{0.04}$ = 0.2

d) $\sqrt{25a^2}$ = 5a

e) $\sqrt{n^6}$ = n^3

f) $\sqrt[3]{8} = 2$

g) $\sqrt[4]{81m^{12}}$ = $3m^3$

h) $\sqrt{-144}$ Not real

i) $-\sqrt{100}$ = – 10

j) $\sqrt[3]{-8}$ = – 2

k) $\sqrt[4]{81}$ = 3

l) $\sqrt[6]{-64}$ Not real

Note: *When we are having a negative radicand under the even root it is not real. However, it can be expressed as a complex number (in terms of i).*

$\sqrt[n]{a^n}$:

- If **n** is a positive **odd** integer, then $\sqrt[n]{a^n}$ **= a**.
- If **n** is a positive **even** integer, then $\sqrt[n]{a^n}$ **=** |**a**|.

Example 8.1.2: Simplify expressions.

a) $\sqrt{x^2}$ = x

b) $\sqrt{(x-1)^2}$ = x – 1

c) $\sqrt[4]{(-3)^4}$ = 3

d) $\sqrt[5]{(-7)^5}$ = – 7

8.2 Simplify Radical Expressions

Parts of Radicals:

$$\mathbf{a\sqrt[n]{x^m}}$$

where:

a – coefficient

n – index

x – radicand

m – exponent

Note: *If there is no index it implies it is 2 (square root).*

Simplest form of a radical:

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand does not contain a fraction.
3. There are no radicals in the denominator of a fraction.

Product rule of radicals

Let **a** and **b** represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are <u>both</u> real. Then

$$\sqrt[n]{\mathbf{a \cdot b}} = \sqrt[n]{\mathbf{a}} \cdot \sqrt[n]{\mathbf{b}}$$

Quiotient rule of radicals

Let **a** and **b** represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are <u>both</u> real. Then

$$\sqrt[n]{\frac{\mathbf{a}}{\mathbf{b}}} = \frac{\sqrt[n]{\mathbf{a}}}{\sqrt[n]{\mathbf{b}}} \qquad \mathbf{b \neq 0}$$

Example 8.2.1: Simplify.

a) $\sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$

b) $\sqrt{x^9} = \sqrt{x^8}\sqrt{x^1} = x^4\sqrt{x}$

c) $\sqrt{\frac{a^7}{a^3}} = \sqrt{a^4} = a^2$

d) $\sqrt[3]{\frac{3}{81}} = \frac{1}{3}$

e) $-5\sqrt{18x^4y^6z^{10}} = -15x^2y^6z^{10}\sqrt{2}$

f) $\sqrt[4]{w^7z^{11}} = wz^2\sqrt[4]{(wz)^3}$

g) $\frac{15\sqrt[3]{108}}{20\sqrt[3]{2}} = \frac{9}{4}\sqrt[3]{2}$

k) $\frac{\sqrt{50}}{\sqrt{5}} = \sqrt{\frac{50}{5}} = \sqrt{10}$

8.3 Simplify Rational Exponents

$\mathbf{a^{1/n}}$:

Let **a** be a real number and let **n** be an integer such that n > 1. If $\sqrt[n]{\mathbf{a}}$ is a real number, then

$$\mathbf{a^{1/n}} = \sqrt[n]{\mathbf{a}}$$

Example 8.3.1: Convert to radical form, and simplify, if possible.

a) $9^{1/2} = \sqrt{9} = 3$

b) $-100^{1/2} = -\sqrt{100} = -10$

c) $(-8)^{1/3} = \sqrt[3]{-8} = -2$

$\mathbf{a^{m/n}}$:

Let **a** be a real number and let **m** and **n** be positive integers such that **m** and **n** share no common factors and n > 1. If $\sqrt[n]{\mathbf{a}}$ is a real number, then

$$\mathbf{a^{m/n}} = (\sqrt[n]{\mathbf{a}}\,)^m = \sqrt[n]{\mathbf{a^m}}$$

Example 8.3.2: **Example 2:** Convert to radical form and simplify.

a) $8^{2/3} = \sqrt[3]{8^2} = 4$

b) $\left(\frac{1}{5}\right)^{-3/2} = \sqrt{\left(\frac{1}{25}\right)^3}$

c) $4^{-3/2} = \sqrt{4^{-3}} = \sqrt{\frac{1}{64}} = \frac{1}{8}$

d) $(-81)^{3/4}$ Not real

e) $3y^{1/4} = 3\sqrt[4]{y}$

f) $(5x^2)^{1/3} = \sqrt[3]{5x^2}$

Example 8.3.3: Convert each expression to an equivalent expression by using rational exponents.

a) $\sqrt{7a} = (7a)^{1/2}$

b) $\sqrt[4]{b^3} = b^{3/4}$

c) $5\sqrt{c} = 5c^{1/2}$

Example 8.3.4: Use properties of exponents to simplify.

a) $y^{2/5} y^{3/5} = y^1 = y$

b) $\frac{6a^{-1/2}}{a^{3/2}} = \frac{6}{a^2}$

c) $(\frac{s^{1/2}t^{1/3}}{w^{3/4}})^4 = \frac{s^2t^{4/3}}{w^3}$

8.4 Add, Subtract, and Multiply Radical Expressions

Adding and Subtracting Radicals:

Only like radicals (with the same index and same radicand) can be **added/subtracted**. Keep radical part add/subtract coefficients.

Example 8.4.1: Simplify.

a) $5\sqrt{11} + 3\sqrt{11} - 2\sqrt{11} = (5 + 3 - 2)\sqrt{11} = 6\sqrt{11}$

b) $5\sqrt{45} + 6\sqrt{18} - 2\sqrt{98} + \sqrt{20} = 5\sqrt{9}\sqrt{5} + 6\sqrt{9}\sqrt{2} - 2\sqrt{49}\sqrt{2} + \sqrt{4}\sqrt{5} =$

$15\sqrt{5} + 18\sqrt{2} - 14\sqrt{2} + 2\sqrt{5} = 17\sqrt{5} + 4\sqrt{2}$

c) $7\sqrt[5]{6} + 4\sqrt[5]{3} - 9\sqrt[5]{3} + \sqrt[5]{6} = 8\sqrt[5]{6} - 5\sqrt[5]{3}$

d) $4\sqrt[3]{54} - 9\sqrt[3]{16} + 5\sqrt[3]{9} = 4\sqrt[3]{27}\sqrt[3]{2} - 9\sqrt[3]{8}\sqrt[3]{2} + 5\sqrt[3]{9} = 4(3)\sqrt[3]{2} - 9(2)\sqrt[3]{2} + 5\sqrt[3]{9} =$

$12\sqrt[3]{2} - 18\sqrt[3]{2} + 5\sqrt[3]{9} = -6\sqrt[3]{2} + 5\sqrt[3]{9}$

Multiplication of Radicals:

Product Rule of Radicals: $\mathbf{a\sqrt[n]{b} \cdot c\sqrt[n]{d} = ac\sqrt[n]{bd}}$

Example 8.4.2: Multiply and simplify, if possible.

a) $-5\sqrt{14} \cdot 4\sqrt{6} = -20\sqrt{84} = -20\sqrt{4}\sqrt{21} = -20(2)\sqrt{21} = -40\sqrt{21}$

b) $2\sqrt[3]{18}\, 6\sqrt[3]{15} = 12\sqrt[3]{(18)(15)} = 12\sqrt[3]{270} = 12\sqrt[3]{27}\sqrt[3]{10} = 12(3)\sqrt[3]{10} = 36\sqrt[3]{10}$

c) $7\sqrt{6}(3\sqrt{10} - 5\sqrt{15}) = 21\sqrt{60} - 35\sqrt{90} = 21\sqrt{4}\sqrt{15} - 35\sqrt{9}\sqrt{10} = 42\sqrt{15} - 105\sqrt{10}$

d) $(\sqrt{5} - 2\sqrt{3})(4\sqrt{10} + 6\sqrt{6}) = 4\sqrt{50} + 6\sqrt{30} - 8\sqrt{30} - 12\sqrt{18} = 4\sqrt{25}\sqrt{2} - 2\sqrt{30} - 12\sqrt{9}\sqrt{2}$

$= 4(5)\sqrt{2} - 2\sqrt{30} - 12(3)\sqrt{2} = 20\sqrt{2} - 2\sqrt{30} - 36\sqrt{2} = -16\sqrt{2} - 2\sqrt{30}$

Example 8.4.3: Multiply special products, and simplify, if possible.

a) $(\sqrt{d} + 3)^2 = (\sqrt{d})^2 + 2(\sqrt{d})(3) + 3^2 = d + 6\sqrt{d} + 9$

b) $(5\sqrt{y} - \sqrt{2})^2 = (5\sqrt{y})^2 - 5(\sqrt{y})(\sqrt{2}) + (\sqrt{2})^2 = 25y + 5\sqrt{2y} + 2$

c) $(\sqrt{3} + 2)(\sqrt{3} - 2) = (\sqrt{3})^2 - 2^2 = 3 - 4 = -1$

8.5 Divide Radical Expressions

Division of Radicals:

Quotient Rule of Radicals: $\frac{\mathbf{a}\sqrt[\mathbf{n}]{\mathbf{b}}}{\mathbf{c}\sqrt[\mathbf{n}]{\mathbf{d}}} = \frac{\mathbf{a}}{\mathbf{c}}\sqrt[\mathbf{n}]{\frac{\mathbf{b}}{\mathbf{d}}}$

Example 8.5.1: Divide

a) $\frac{15\sqrt[3]{108}}{20\sqrt[3]{2}} = \frac{15}{20}\sqrt[3]{\frac{108}{2}} = \frac{3}{4}\sqrt[3]{54} = \frac{9\sqrt[3]{2}}{4}$

b) $\sqrt[4]{\frac{7n^4}{16}} = \frac{\sqrt[4]{7n^4}}{\sqrt[4]{16}} = \frac{n\sqrt[4]{7}}{2}$

Rationalization of Denominator:

Ex.

$\frac{6}{\sqrt{2}} =$

$\frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

Note: *The numerator and denomintor could also be multiplied by $\sqrt{8}$, $\sqrt{18}$, $\sqrt{50}$, etc. The idea is to multiply by a radical of the same index that will turn 2 (radicand) to a perfect square.*

Ex.

$\sqrt{2}\,\sqrt{8} = \sqrt{16} = 4$

$\sqrt{2}\,\sqrt{18} = \sqrt{36} = 6$

$\sqrt{2}\,\sqrt{50} = \sqrt{100} = 10$

Example 8.5.2: Rationalize the denominator.

a) $\frac{\sqrt{3}-9}{2\sqrt{6}}=\frac{\sqrt{3}-9}{2\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}=\frac{\sqrt{18}-9\sqrt{6}}{2(6)}=\frac{3\sqrt{2}-9\sqrt{6}}{12}=\frac{\sqrt{2}+3\sqrt{6}}{4}$

b) $\frac{2}{\sqrt{7}-5}=\frac{2}{\sqrt{7}-5}\cdot\frac{\sqrt{7}+5}{\sqrt{7}+5}=\frac{2\left(\sqrt{7}+5\right)}{(\sqrt{7})^2-5^2}=\frac{2(\sqrt{7}+5)}{7-25}=\frac{2\sqrt{7}+10}{-18}=-\frac{\sqrt{7}+5}{9}$

c) $\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}\cdot\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{75}-\sqrt{45}}{(\sqrt{5})^2-\left(\sqrt{3}\right)^2}=\frac{5\sqrt{3}-3\sqrt{5}}{5-3}=\frac{5\sqrt{3}+3\sqrt{5}}{2}$

Note: *Sometimes you will be asked to rationalize the numerator.*

8.6 Solve Radical Equations

Note:

If a = b, then $a^n = b^n$

To Solve a Radical Equation:

- Isolate a radical. If there is more than one radical, isolate one of them.
- Raise both sides to the power of the index (ex. Square both sides for square roots) of the equation.
- Simplify and continue solving for a given variable.
- If there was more than one radical from beginning isolate the remaining radical and continue with same steps.
- Check your solution for extraneous solutions; solutions that make negative under even root should not be included in the solution set.

Example 8.6.1: Solve the equation

a) $\sqrt{2x} = 5$

$(\sqrt{2x})^2 = (5)^2$
$2x = 25$
$x = 12.5$

b) $\sqrt{x-3} = 7$

$(\sqrt{x-3})^2 = (7)^2$
$x - 3 = 49$
$x = 52$

c) $\sqrt{x-3} = -7$

Since it is equal to a negative number, this equation has no real solution.

d) $p - \sqrt{p-2} = 4$

$p - 4 = \sqrt{p-2}$

$(p - 4)^2 = (\sqrt{p-2})^2$

$p^2 - 8p + 16 = p - 2$
$p^2 - 9p + 18 = 0$
$(p - 6)(p - 3) = 0$
$p - 6 = 0 \quad p - 3 = 0$
$p = 6 \qquad p = 3$

Check:

$p = 6$	$p = 3$
$6 - \sqrt{6-2} = 4$	$3 - \sqrt{3-2} \neq 4$
$6 - 2 = 4$	$3 - 1 \neq 4$
$4 = 4$	$2 \neq 4$

Solution: { 6 }

e) $\sqrt{5x-1} - \sqrt{x+2} = 1$

$\sqrt{5x-1} = 1 + \sqrt{x+2}$

$(\sqrt{5x-1})^2 = (1 + \sqrt{x+2})^2$

$5x - 1 = 1 + 2\sqrt{x+2} + (x + 2)$

$4x - 4 = 2\sqrt{x+2}$
$2x - 2 = \sqrt{x+2}$
$(2x - 2)^2 = (\sqrt{x+2})^2$
$4x^2 - 8x + 4 = x + 2$
$4x^2 - 9x + 2 = 0$
$(4x - 1)(x - 2) = 0$

$x = \frac{1}{4} \qquad x = 2$

After checking results. Solution: { 2 }

8.7 Use Radicals in Functions

Example 8.7.1: Determine each of the following.

a) f(2) if f(x) = $\sqrt{5x - 1}$

f(2) = 3 2 is in the domain of f(x); a point (2,3) is on the graph.

b) f(0) if f(x) = $\sqrt[3]{7x + 8}$

f(0) = 2 0 is in the domain of f(x); a point (0, 2) the y-intercept is on the graph.

c) f(– 9) if f(x) = $4\sqrt{6 + x}$

f(– 9) = $4\sqrt{-3}$ not real, because – 9 is not in the domain of f(x).

Example 8.7.2: Suppose the height of a plane is given by *f(t)* = $\sqrt{28 - 5t}$ km, *t* hours after noon. What is the height of the plane 4 hours after noon?

f(4) = $\sqrt{28 - 5(4)}$ = $\sqrt{8}$ = $2\sqrt{2}$

The height of the plane 4 hours after noon will be $2\sqrt{2}$ kilometers.

Example 8.7.3: Let *h(t)* = $\sqrt{3t + 5}$ be the altitude of a balloon in miles, *t* minutes after noon. When will the altitude of a ballon be 5 miles?

5 = $\sqrt{3t + 5}$

$(5)^2 = (\sqrt{3t + 5})^2$

25 = 3t + 5

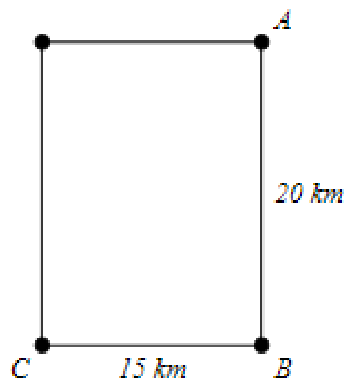
t = $\frac{20}{3}$

The altitude of the ballon will be 5 miles after $\frac{20}{3}$ seconds.

Example 8.7.4: Let *f(x)* = $\sqrt{10x - 5}$, find f(x + 5).

f(x + 5) = $\sqrt{10(x + 5) - 5}$ = $\sqrt{10x + 50 - 5}$ = $\sqrt{10x + 45}$

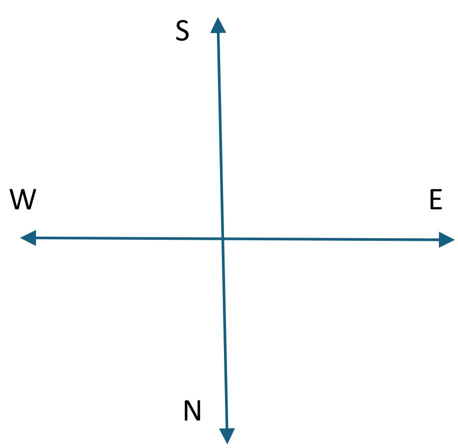
Example 8.7.5: Two runners start at different points on a rectangular field (shown below, not drawn to scale).



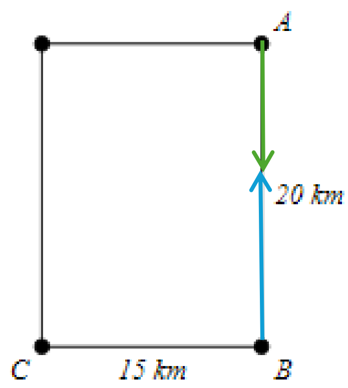
Alice starts at point A and begins running south (towards point B) at 5 km/hour, while Bob starts at point B and begins running west (towards point C) at 3 km/hour.

a) Determine how many kilometers North of point B is Alice after *t* hours.

Recall:



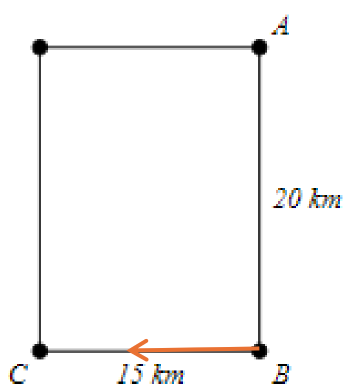
D = Vt



$D_{\text{Alice from A towards B}} = 5t$ **(Green)**

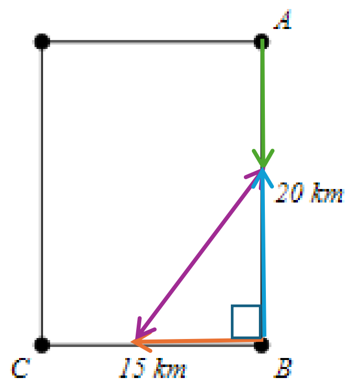
$D_{\text{Alice North of point B after t hours}} = 20 – 5t$ **(Blue)**

b) Determine how many kilometers West of point B is Bob after *t* hours.



$D_{\text{ Bob West of point B}} = 3t$ **(Orange)**

c) Use Pythagorean Theorem to find an expression for the distance between Alice and Bob after *t* hours.



Recall: ***Pythagorean Theorem***

Hypotenuse ^{2} = Leg ^{2} + Leg ^{2}

Purple ^{2} = **Orange** ^{2} + **Blue** ^{2}

An expression for the distance between Alice and Bob after *t* hours:

$$\sqrt{(3t)^2 + (20 - 5t)^2}$$

e) If *S* represents distance between them, find *S*.

$$S(t) = \sqrt{(3t)^2 + (20 - 5t)^2} = \sqrt{34t^2 - 200t + 400}$$