Chapter 6: Factoring

6.1 Greatest Common Factor and Factoring by Grouping

Greatest common factor (GCF) is the largest positive integer by which both integers can be divided. GCF of numbers is always **equal or less** than any given number.

Example 6.1.1: Find GCF of:

a) 6, 9, 12

   GCF = 3

b) 5, 15, 20

   GCF = 5

c) 15, 24, 27

   GCF = 3

d) a^2, a^4, a^7

   GCF = a^2

e) $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$

   GCF = $6x^2y$

Factoring using GCF:

- Determine GCF.
- Factor out GCF and divide each term by GCF.
- Write it as a product of GCF · (what you got in previous step).

Example 6.1.2: Factor using GCF:

a) $4x^2 - 20x + 16 = 4(x^2 - 5x + 4)$

b) $25x^4 - 15x^3 + 20x^2 = 5x^2(5x^2 - 3x + 4)$

c) $3x^3y^2z + 5x^4y^3z^5 - 4xy^4 = xy^2(3x^2z + 5x^3yz^5 - 4y^2)$

d) $21x^3 + 14x^2 + 7x = 7x(3x^2 + 2x + 1)$

Note: *When factoring, always see if there is any GCF, and start by factoring it out.*

Factoring by grouping:

- Group terms that have GCF.
- Determine GCF for each group.
- Factor out GCF for each group.
- Determine GCF for the result, if exists, and factor it out as well.

Example 6.1.3: Factor by grouping:

a) $10ab + 15b + 4a + 6 = 5b(2a + 3) + 2(2a + 3) = (2a + 3)(5b + 2)$

b) $6x^2 + 9xy - 14x - 21y = 3x(2x + 3y) - 7(2x + 3y) = (2x + 3y)(3x - 7)$

c) $5xy - 8x - 10y + 16 = x(5y - 8) - 2(5y - 8) = (5y - 8)(x - 2)$

d) $12ab - 14a - 6b + 7 = 12ab - 6b - 14a + 7 = 6b(2a - 1) - 7(2a - 1) = (2a - 1)(6b - 7)$

e) $6x^3 - 15x^2 + 2x - 5 = 6x^3 + 2x - 15x^2 - 5 = 2x(3x^2 + 1) - 5(3x^2 + 1) = (3x^2 + 1)(2x - 5)$

6.2 Factor Trinomials

Trinomials in the form $\mathbf{ax^2 + bx + c}$, where $\mathbf{a = 1}$, and b and c are some numbers, can be factored as a product of two binomials. Product of inner terms plus product of outer terms must equal to bx term.

Factoring trinomials, a = 1:

- Put trinomial in $ax^2 + bx + c$ form.
- Identify values of **a**, **b**, and **c**
- Write two pairs of binomial factors and place "x" or given variable as first terms.
  (x)(x)
- Determine factors of c – term that add up to b – term.
- Put those values as second terms in those pairs of binomial factors.

Example 6.2.1: Factor:

a) $x^2 + 9x + 18 = (x + 3)(x + 6)$

   $b = 9$ and $c = 18$

b) $x^2 + 2x - 24 = (x - 4)(x + 6)$

   $b = 2$ and $c = -24$

c) $x^2 - 4x + 3 = (x - 3)(x - 1)$

   $b = -4$ and $c = 3$

d) $x^2 - 8x - 20 = (x + 2)(x - 10)$

   $b = -8$ and $c = -20$

e) $a^2 - 9ab + 14b^2 = (a - 2b)(a - 7b)$

$b = -9b$ and $c = 14b^2$

f) $x^2 + 5x - 6 = (x - 1)(x + 6)$

$b = 5$ and $c = -6$

g) $x^2 - 7x - 18 = (x - 9)(x + 2)$

$b = -7$ and $c = -18$

h) $m^2 - mn - 30n^2 = (m - 6n)(m + 5n)$

$b = -n$ and $c = -30n^2$

i) $x^2 + 2x + 6$ is a Prime trinomial

$b = 2$ and $c = 6$

j) $3x^2 - 24x + 45 = 3(x^2 - 8x + 15) = 3(x - 3)(x - 5)$

GCF = 3, then $b = -8$ and $c = 15$

Note: *If there is any GCF, first factor it out, then continue factoring if possible.*

Factoring trinomials, $a \neq 1$:

Trinomials in the form $\mathbf{ax^2 + bx + c}$, where $\mathbf{a \neq 1}$, and b and c are some numbers, can be factored as a product of GCF, if exists, and a product of two binomials. Product of inner terms plus product of outer terms must equal to bx term.

Trial and Error Method:

- Put trinomial in $ax^2 + bx + c$ form.
- Identify values of **a**, **b**, and **c**
- Write two pairs of binomial factors and place "x" or given variable as first terms.
- Think about factors of ax^2 term and place as first terms.
  (factors of ax^2)(factors of ax^2)
- Then by trial and error choose factors of c – term.
  (factors of ax^2 + factors of c)(factors of ax^2 + factors of c)
- Product of inner terms plus product of outer terms must equal to bx term.

AC – Method (Factoring by Grouping):

- Put trinomial in $ax^2 + bx + c$ form.
- Identify values of **a**, **b**, and **c**
- Determine a product of **ac**
- Think about factors of **ac**, then represent **bx** – term as sum of factors of a product of **ac**.
- Continue factoring by using factoring by grouping.

Example 6.2.2: Factor:

a) $3x^2 + 11x + 6 = (3x + 2)(x + 3)$

$a = 3$ $c = 6$ $ac = 18$

By AC – method:

$3x^2 + 2x + 9x + 6 = x(3x + 2) + 3(3x + 2) = (3x + 2)(x + 3)$

b) $8x^2 - 2x - 15 = (4x + 5)(2x - 3)$

$a = 8$ $c = -15$ $ac = -120$

$8x^2 - 12x + 10x - 15 = 4x(2x - 3) + 5(2x - 3) = (2x - 3)(4x + 5)$

c) $10x^2 - 27x + 5 = 10x^2 - 2x - 25x + 5 = 2x(5x - 1) - 5(5x - 1) = (5x - 1)(2x - 5)$

$a = 10$ $c = 5$ $ac = 50$

d) $4x^2 - xy - 5y^2 = 4x^2 - 5xy + 4xy - 5y^2 = x(4x - 5y) + y(4x - 5y) = (4x - 5y)(x + y)$

$a = 4$ $c = -5y^2$ $ac = -20y^2$

e) $18x^3 + 33x^2 - 30x = 3x(6x^2 + 11x - 10) = 3x(6x^2 - 4x + 15x - 10) =$

$3x(2x(3x - 2) + 5(3x - 2)) = 3x(3x - 2)(2x + 5)$

$a = 6$ $c = -10$ $ac = -60$

f) $3x^2 + 2x - 7$ is a Prime trinomial; not factorable

6.3 Factoring Special Products

Some Special Product Formulas:

The Sum of Squares: $a^2 + b^2$ = Prime

The Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$

Perfect Squares: $a^2 + 2ab + b^2 = (a + b)^2$ OR $a^2 - 2ab + b^2 = (a - b)^2$

Example 6.3.1: Factor:

a) $x^2 - 16 = (x + 4)(x - 4)$

b) $9a^2 - 25b^2 = (3a + 5b)(3a - 5b)$

c) $x^2 + 36$ is a Prime binomial; not factorable

d) $x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$

e) $9z^8 - 24z^4x^5 + 16x^{10} = (3z^4 - 4x^5)^2$

6.5 Polynomial Equations

A **polynomial equation** is an equation of the form:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$$

Ex.

- **Linear** $\quad$ **ax + b = 0**
- **Quadratic** $\quad$ **$ax^2 + bx + c = 0$**
- **Cubic** $\quad$ **$ax^3 + bx^2 + cx + d = 0$**

Zero Product Rule:

If ab = 0 then a = 0 and/or b = 0

Example 6.5.1: Solve using zero-product rule.

a) (x + 2)(x – 1) = 0

Set each factor = 0

x + 2 = 0 $\qquad$ x – 1 = 0

x = – 2 $\qquad$ x = 1

Solution: {– 2, 1}

b) (2x − 3)(5x + 1)(x – 1) = 0

2x – 3 = 0 $\qquad$ 5x + 1 = 0 $\qquad$ x – 1 = 0

2x = 3 $\qquad$ 5x = – 1 $\qquad$ x = 1

$x = \frac{3}{2} = 1.5$ $\qquad$ $x = -\frac{1}{5} = -0.2$

Solution: {– 0.2, 1, 1.5}

Solving Polynomial Equations by Factoring:

- Put equation in standard form, if needed, open parenthesis, combine like terms.
- Set it equal to zero (ex. $ax^2 + bx + c = 0$).
- Factor side not equal to zero.
- Set each factor equal to zero (by zero product property).
- Solve each one-two step linear equations.
- Check your results.

Example 6.5.2: Solve:

a) $x^2 + 5x - 6 = 0$

$(x + 6)(x - 1) = 0$

$x + 6 = 0 \qquad x - 1 = 0$

$x = -6 \qquad x = 1$

b) $x^2 = 8x - 15$

$x^2 - 8x + 15 = 0$

$(x - 3)(x - 5) = 0$

$x - 3 = 0 \qquad x - 5 = 0$

$x = 3 \qquad x = 5$

c) $4x^2 + x - 3 = 0$

$AC = -12$

$4x^2 + 4x - 3x - 3 = 0$

$4x(x + 1) - 3(x + 1) = 0$

$(x + 1)(4x - 3) = 0$

$x + 1 = 0 \qquad 4x - 3 = 0$

$x = -1 \qquad 4x = 3$

$x = \frac{3}{4}$

d) $(x-7)(x+3)=-9$

$x^2+3x-7x-21=-9$
$x^2-4x-21+9=0$
$x^2-4x-12=0$
$(x-6)(x+2)=0$
$x-6=0 \quad x+2=0$
$x=6 \quad x=-2$

e) $4x^2=8x$

$4x^2-8x=0$ GCF = 4x
$4x(x-2)=0$
$4x=0 \quad x-2=0$
$x=0 \quad x=2$

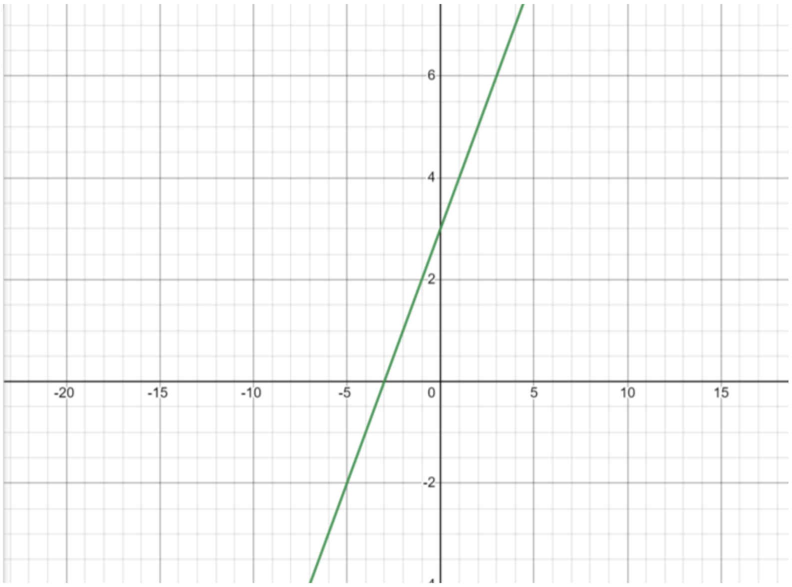
f) $2x^3-14x^2+24x=0$

GCF = 2x
$2x(x^2-7x+12)=0$
$2x(x-4)(x-3)=0$ Factored completely
$2x=0 \quad x-4=0 \quad x-3=0$
$x=0 \quad x=4 \quad x=3$

What it means graphically:

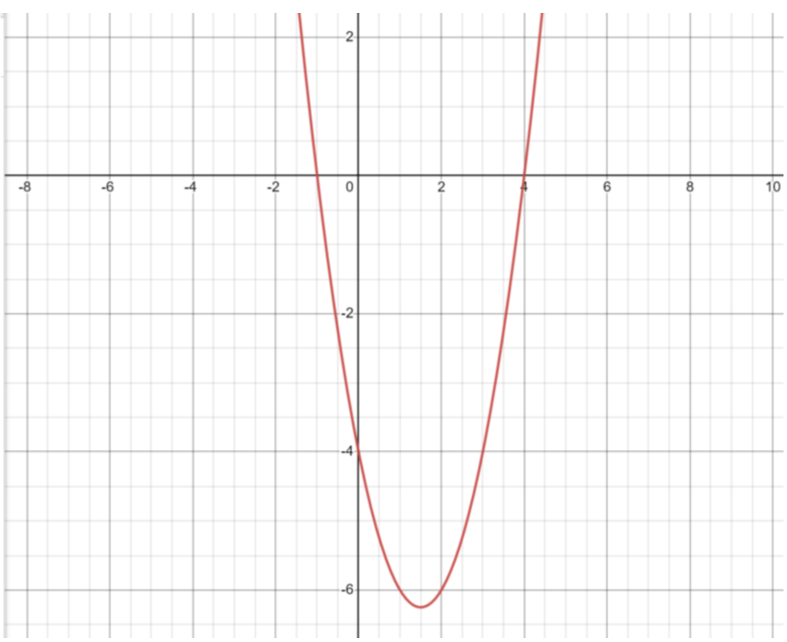
To solve a polynomial equation means to find its roots (zeros), that may or may not be x – intercepts. Set y = 0 or f(x) = 0

f(x) = x + 3



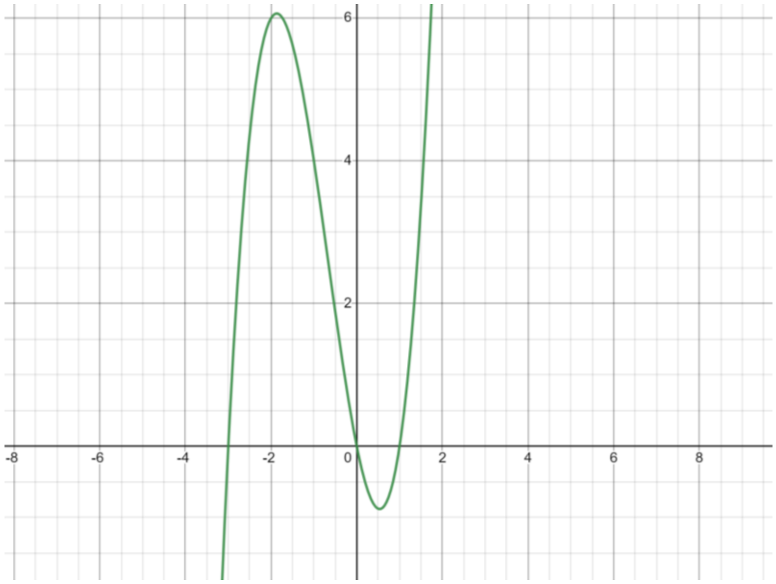
x + 3 = 0 at x = – 3, which is x – intercept for the graph.

f(x) = x^2 – 3x – 4



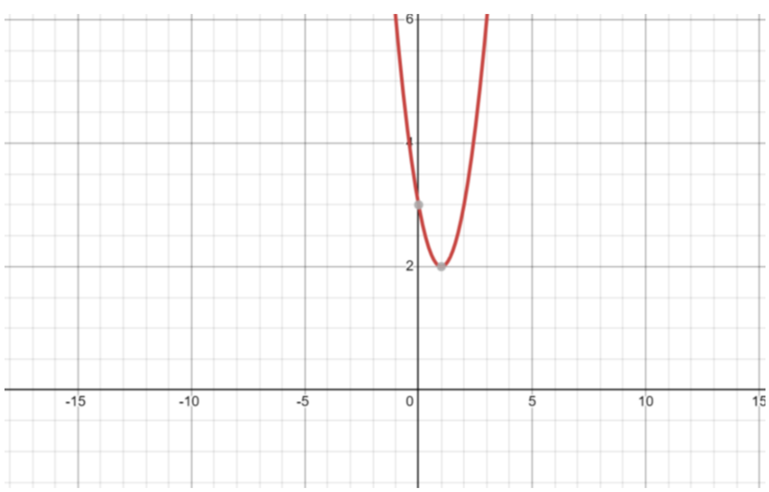
x^2 – 3x – 4 = 0 at x = – 1 and x = 4, which are x – intercepts for this graph.

$f(x) = x^3 + 2x^2 - 3x$



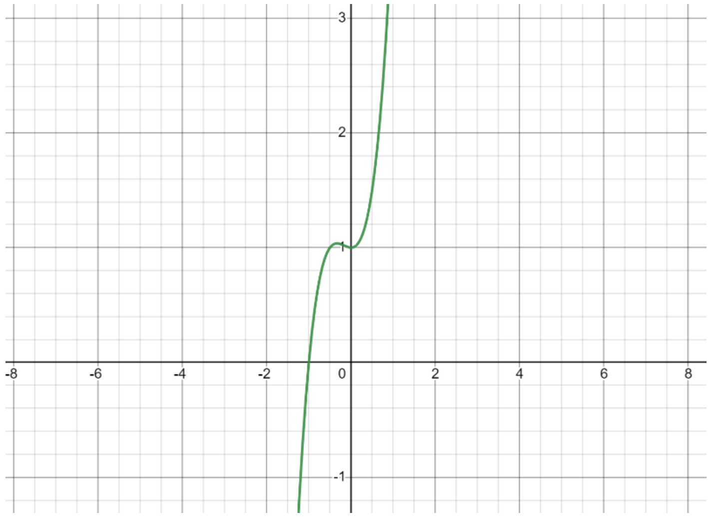
$x^3 + 2x^2 - 3x = 0$ at $x = -3$, $x = 0$, and $x = 1$, which are x – intercepts for this graph.

$f(x) = x^2 - 2x + 3$



$x^2 - 2x + 3 = 0$ at $x = 1 \pm i\sqrt{2}$, which are NOT x – intercepts, because those solution are NOT real.

$f(x) = 2x^3 + x^2 + 1$



$2x^3 + x^2 + 1 = 0$ at $x = -1$, which is x – intercept for this graph and $x = \frac{1 \pm i\sqrt{7}}{4}$, which are NOT x – intercepts, because those solutions are NOT real.

Example 6.5.3: A little league baseball player hit a pop-up foul ball. The height of the ball is given by the function $H(t) = -t^2 + 2t + 8$, where *H* is the height in inches and *t* is time in seconds.

a) From what height was the ball thrown?

   Initial value; $t = 0$ $H(0) = 8$
   The ball was thrown from a height of 8 inches above the ground.

b) How long will it take for the ball to return to the ground?

   On the ground there is no height. $H(t) = 0$

   $-t^2 + 2t + 8 = 0$
   $t^2 - 2t - 8 = 0$
   $(t - 4)(t + 2) = 0$
   $t = 4 \quad t = -2$

   Choose a positive solution, since time cannot be represented by a negative value.
   The ball will hit the ground after 4 seconds.

Example 6.5.3: The product of two consecutive odd integers is 35. Find integers.

Let *n* represent the first odd integer, then *n + 2* represents the next consecutive odd integer.

$n(n + 2) = 35$

$n^2 + 2n - 35 = 0$

$n = 5 \quad n = -7$

Integers are 5 and 7 – 7 and – 5

Example 6.5.5: The width of a rectangle is 2 feet less than its length. If *x* represents the length of the rectangle, find:

a) An expression for the width of the rectangle.

   $x - 2$

b) An expression giving the area of the rectangle.

   $x(x - 2)$ $A = LW$

c) The dimensions of the rectangle if the area is 15 square feet.

   $x(x - 2) = 15$

   $x^2 - 2x - 15 = 0$

   $x = 5 \quad x = -3$

   Length is $x = 5$ and width is $5 - 2 = 3$