Chapter 5: Polynomials and Polynomial Functions

5.1 Add and Subtract Polynomials

Polynomial means one or more terms.

$\mathbf{a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_2 x^2 + a_1 x + a_0}$

Some Polynomials:

Polynomial	Degree	Name by Degree	Number of Terms	Name by Number of Terms
3	0	Constant	1	Monomial
$2x + 7$	1	Linear	2	Binomial
$3x^2 + 7x - 1$	2	Quadratic	3	Trinomial
$-4x^3$	3	Cubic	1	Monomial
$8x^4 - 3x^3 + 4x^2$	5	Fourth	3	Trinomial

Example 5.1.1: Consider $-2x^3 + 3x^2 - 4x + 7$.

a) What is its degree?

   3^{rd} degree

b) What is its leading coefficient?

   -2

c) How many terms does it have?

   Four

d) What is the coefficient of x^2?

   3

e) What is a variable part of 7?

   x^0

Example 5.1.2: Evaluate.

a) $2x^2 - 4x + 6$ when $x = -4$

$2(-4)^2 - 4(-4) + 6 = 2(16) + 16 + 6 = 32 + 16 + 6 = 48 + 6 = 54$

b) $-x^2 + 2x + 6$ when $x = 3$

$-3^2 + 2(3) + 6 = -9 + 6 + 6 = -3 + 6 = 3$

Addition and Subtraction of Polynomials:

Only like terms can be added/subtracted. Add/subtract coefficients and keep the variable part as it is.

Like terms are terms that have exactly the same variable part.

Example 5.1.3: Combine by adding/subtracting.

a) $(4x^3 - 2x + 8) + (3x^3 - 9x^2 - 11) = 4x^3 - 2x + 8 + 3x^3 - 9x^2 - 11 = 7x^3 - 9x^2 - 2x - 3$

b) $(5x^2 - 2x + 7) - (3x^2 + 6x - 4) = 5x^2 - 2x + 7 - 3x^2 - 6x + 4 = 2x^2 - 8x + 11$

c) $(2x^2 - 4x + 3) + (5x^2 - 6x + 1) - (x^2 - 9x + 8) = 2x^2 - 4x + 3 + 5x^2 - 6x + 1 - x^2 + 9x - 8 =$

$6x^2 - x - 4$

5.2 Properties of Exponents

b^n = b b b ....   n-times

b is called the base, **n** is called exponent.

Rules of Exponents	**Examples**
$a^m a^n = a^{m+n}$	$x^2 x^7 = x^9$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{y^3}{y} = y^2$
$(a^m)^n$	$(z^2)^3 = z^6$
$(ab)^m$	$(3x)^2 = 9x^2$
$(\frac{a}{b})^n = \frac{a^n}{b^n}$	$(\frac{y}{2})^3 = \frac{y^3}{8}$

Example 5.2.1: Simplify:

a) $3^2 \cdot 3^6 \cdot 3 = 3^9$

b) $2x^3y^5z5xy^2z^3 = 10x^4y^7z^4$

c) $\frac{7^{13}}{7^3} = 7^8$

d) $(x^3yz^2)^4 = x^{12}y^4z^8$

e) $7a^3(2a^4)^3 = 7a^3(8a^{12}) = 56a^{15}$

f) $\frac{3m^8n^{12}}{(n^2n^3)^3} = 3m^2n^3$

Exponents Can Be:

Positive	$\mathbf{b^n = b\ b\ b\ ....}$	n-times
Zero	$\mathbf{b^0 = 1}$	
Negative	$\mathbf{b^{-n} = \frac{1}{b^n}}$	

Note:

$$\frac{1}{a^{-m}} = a^m$$

$$\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}$$

Example 5.2.2: Simplify, write your final answer using positive exponents.

a) $\frac{a^3}{a^3} = a^0 = 1$

b) $(3x^2)^0 = 1$

c) $\frac{b^3}{b^5} = b^{-2} = \frac{1}{b^2}$

d) $\frac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2} = \frac{a^3cde^4}{2b^2f^2}$

e) $\frac{(3ab^3)^{-2}ab^{-3}}{2a^{-4}b^0} = \frac{aa^4}{2(3ab^3)^2b^3} = \frac{a^5}{18b^9}$

5.3 Multiply Polynomials

To multiply polynomials, simply distribute by multiplying each term from the first polynomial with each term of the second polynomial. Multiply number with number and the same variable with the same variable (exponents will be added).

Example 5.3.1: Multiply

a) $(4x^3y^4z)(2x^2y^6z^3) = 8x^5y^{10}z^4$

b) $4x^3 (5x^2 - 2x + 5) = 20x^5 - 8x^4 + 20x^3$

c) $(4x + 7y)(3x - 2y) = (4x)(3x) - (4x)(2y) + (7y)(3x) - (7y)(2y) = 12x^2 - 8xy + 21xy - 14y^2 =$

$12x^2 + 13xy - 14y^2$

d) $(2x - 5)(4x^2 - 7x + 3) = 8x^3 - 14x^2 + 6x - 20x^2 + 35x - 15 = 8x^3 - 34x^2 + 41x - 15$

e) $3(2x - 4)(x + 5) = (6x - 12)(x + 5) = 6x^2 + 30x - 12x - 60 = 6x^2 + 18x - 60$

Special Products of Polynomials:

Example 5.3.2: Multiply.

a) $(x - 5)^2 = (x - 5)(x - 5) = x^2 - 5x - 5x + 25 = x^2 - 10x + 25$

b) $(a + 3)(a - 3) = a^2 - 3a + 3a - 9 = a^2 - 9$

Binomial Expansion Formula $\quad$ $\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$

$\mathbf{(a - b)^2 = a^2 - 2ab + b^2}$

Difference of Two Perfect Squares Formula $\quad$ $\mathbf{(a + b)(a - b) = a^2 - b^2}$

Note: a *represents the* ***first*** *term, and* ***b*** *represents the* ***second*** *term.*

Example 5.3.3: Multiply.

a) $(x + 4)^2 = x^2 + 2(x)(4) + 4^2 = x^2 + 8x + 16$

b) $(y - 2)(y + 2) = y^2 - 2^2 = y^2 - 4$

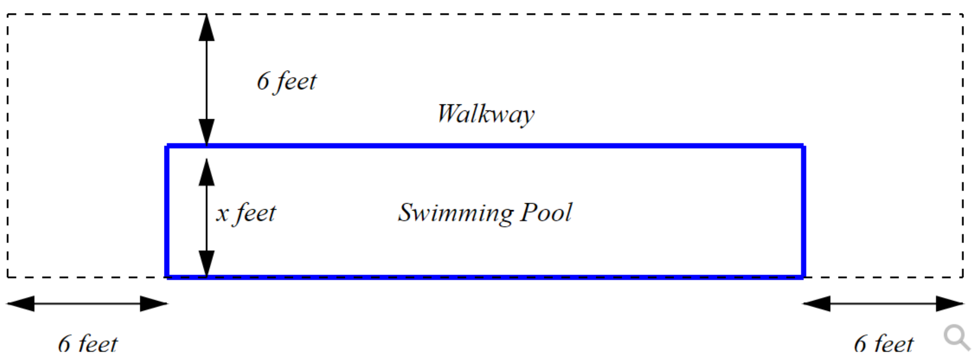
c) $(3x + 7)(3x - 7) = (3x)^2 - 7^2 = 9x^2 - 49$

d) $(2x - 6y)(2x + 6y) = (2x)^2 - (6y)^2 = 4x^2 - 36y^2$

e) $(2x + 5)^2 = (2x)^2 + 2(2x)(5) + 5^2 = 4x^2 + 20x + 25$

f) $(3x - 7y)^2 = (3x)^2 - 2(3x)(7y) + (7y)^2 = 9x^2 - 42xy + 49y^2$

Example 5.3.4: The width of a swimming pool is nine feet less than eight times its length. A six-foot-wide walkway surrounds the pool, as shown below (not drawn to scale) where *x* represents the length of the pool.



a) Write the expression giving the total area of the pool AND walkway in a factored form.

$L_{pool} = x$ $L_{with\ walkway} = x + 6$

$W_{pool} = 8x - 9$ $W_{with\ walkway} = 8x - 9 + 6 + 6 = 8x + 3$

$A = (8x + 3)(x + 6)$

b) Write that expression in expanded form.

$A = 8x^2 + 51x + 1$

5.4 Dividing Polynomials

To divide by monomial (one term) simply divide each term of the numerator by that monomial. Divide number by number and the same variable by the same variable (subtract exponents).

Example 5.4.1: Divide:

a) $\frac{9x^5+6x^4-18x^3-24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8$

b) $\frac{8x^3+24x^2-2x-12}{4x} = 2x^2 + 6x - \frac{1}{2} - \frac{3}{x}$

To divide by binomial (two terms) use long or synthetic division.

Polynomial long division is a method for dividing a polynomial by another polynomial of a lower degree. It is very similar to dividing numbers; you start by looking at first terms of dividend and divisor.

Ex.

$$
\begin{array}{rl}
 & 2x^2 - 11x + 32 \\
\hline
x + 3 \;\big) & 2x^3 - 5x^2 - x + 3 \\
2x^2(x + 3) = & \underline{2x^3 + 6x^2} \\
 & -11x^2 - x + 3 \\
-11x(x + 3) = & \underline{-11x^2 - 33x} \\
 & 32x + 3 \\
32(x + 3) = & \underline{32x + 96} \\
 & -93
\end{array}
$$

Answer: $2x^2 - 11x + 32 + \frac{-93}{x + 3}$

Polynomial synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form **(x – c)**. In synthetic division we write only the essential parts of the long division.

Ex. $(2x^3 - 5x^2 - x + 3) \div (x + 3)$

Put only coefficients and c = – 3

– 3	2	– 5	– 1	3
	↓	+	+	+
		2(– 3)	– 11(– 3)	32(– 3)
	2	– 11	32	– 93

Answer: $2x^2 - 11x + 32 + \frac{-93}{x+3}$

Example 5.4.2: Divide parts a) and b) using synthetic division and parts c) and d) using long division.

a) $\frac{x^2 + 4x + 5}{x - 4}$

Answer: $x + 8 + \frac{37}{x-4}$

b) $\frac{x^4 + 3x^3 - 5x + 1}{x + 1}$

Answer: $x^3 + 2x^2 - 2x - 3 + \frac{4}{x+1}$

Note: *Do not forget to put 0 as a coefficient of x^2 term.*

c) $\frac{4x^3 + 2x^2 + 6x + 18}{2x - 3}$

Answer: $2x^2 - 2x + 3$

d) $\frac{x^3 + x^2 + 2x + 1}{x^2 + 3x + 1}$

Answer: $x - 2 + \frac{7x + 3}{x^2 + 3x + 1}$

Note: *If there is no remainder left, then divisor is a factor of the dividend. The quotient will be a factor of the dividend as well.*

Dividing by x – c:

The remainder theorem states that the remainder when dividing f(x) by (x − c) is r = f(c).

The factor theorem states that if f(c) = 0, then g(x) = x − c is a factor of f(x).

In other words, if there is **no remainder** when dividing f(x) by x – c, then **x – c is a factor of f(x)**.

It also means that **c** is a root (also known as zero, which may or may not be the x – intercept).

Ex. Let $f(x) = x^2 + 5x - 6$ and $g(x) = x - 1$

Since r = f(c) and r = f(1) = 0 (x – 1) is a factor of $x^2 + 5x - 6$

x = 1 is a root (x – intercept.)