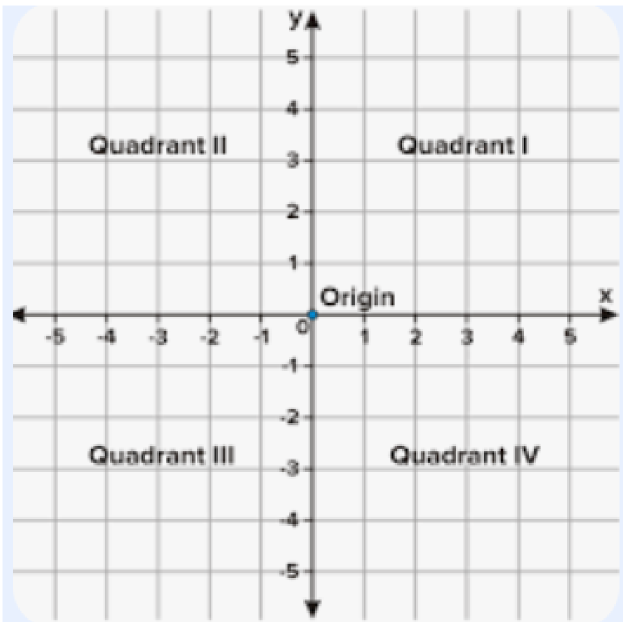
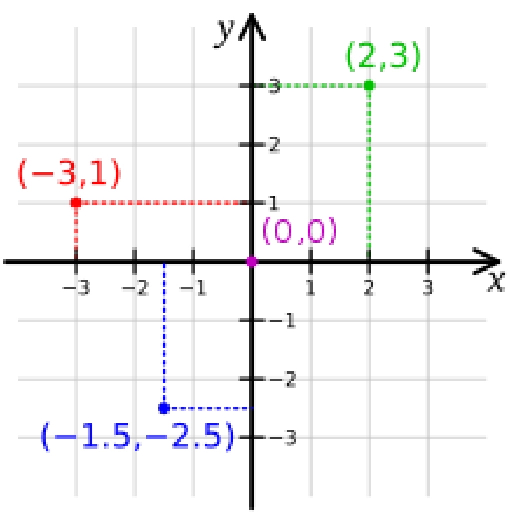
Chapter 3: Graphs and Functions

3.1 Graph Linear Equations in Two Variables

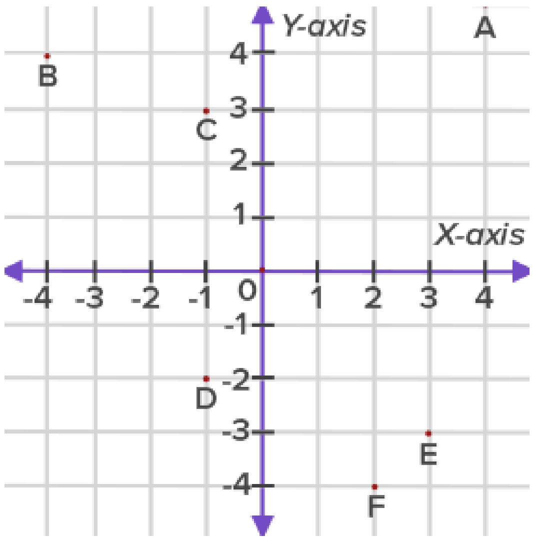
Coordinate Plane:



Points are given as ordered pairs (x,y).

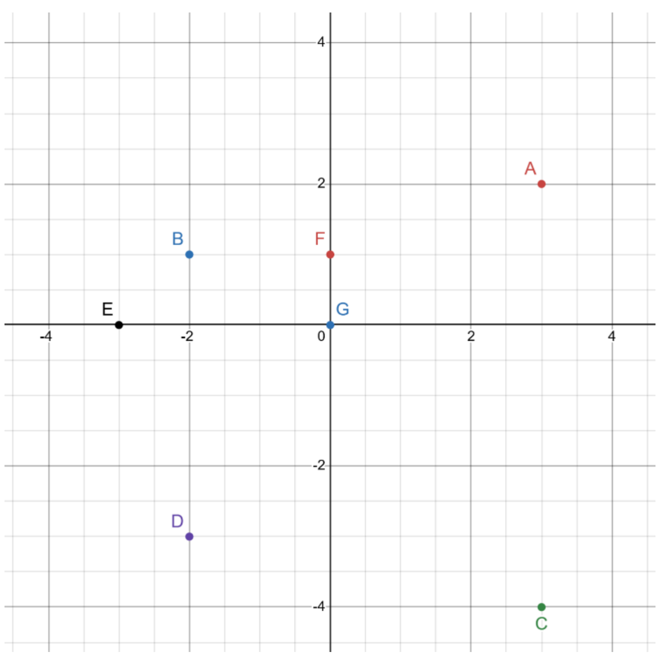


Example 3.1.1: Determine coordinates of each point.



A (4, 5)   B(– 4, 4)   C(– 1, 3)   D(– 1, – 2)   E(3, – 3)   F(2, – 4)

Example 3.1.2: Graph the points A(3, 2), B (– 2, 1), C (3, – 4), D(– 2, – 3), E (– 3, 0), F (0, 2), G(0, 0).



Example 3.1.3: Check whether the given point is on the line y = 3x + 1

a) (1, 7)

   NO, 7 ≠ 3(1) + 1

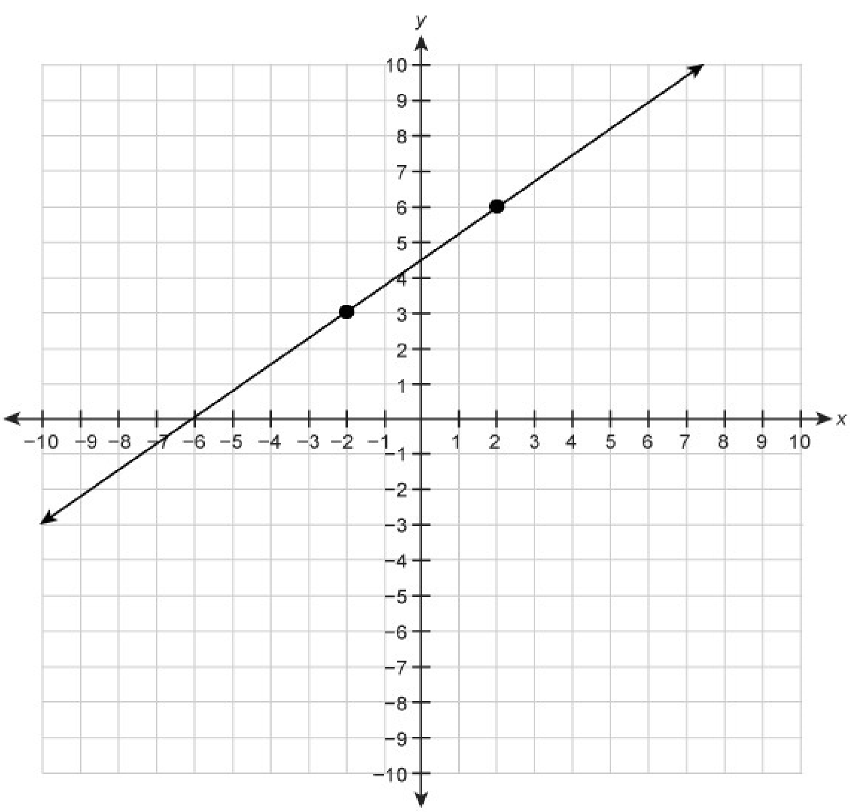
b) (0, 1)

   YES, 1 = 3(0) + 1

c) (– 2, – 5)

   YES, – 5 = 3(– 2) + 1

Lines contain infinitely many points. At least two points determine a line.



Graphing a Line:

You need at least **two points** to graph a line. The following methods can be used to find those two points.

Table of values:

- Choose any two x or y values and substitute into the equation to find the value of the other variable.
- Rewrite both as ordered pairs (x, y).
- Plot each point on a coordinate plane and draw a line through them.

X and Y intercepts:

- Substitute y = 0 into the equation to find x – intercept.
- Substitute x = 0 into the equation to find y – intercept.
- Rewrite as an ordered pair (x, 0) and (0, y).
- Plot each point on a coordinate plane and draw a line through them.

Y – intercept and slope:

- Determine values of slope and y – intercept.
- Plot y – intercept on a coordinate plane.
- Use slope to get second point. From y-intercept move up/down numerator units of the slope, then move right/left denominator units of the slope.
- Draw line through them.

Example 3.1.4: Graph using any method.

a) $y = 2x - 3$

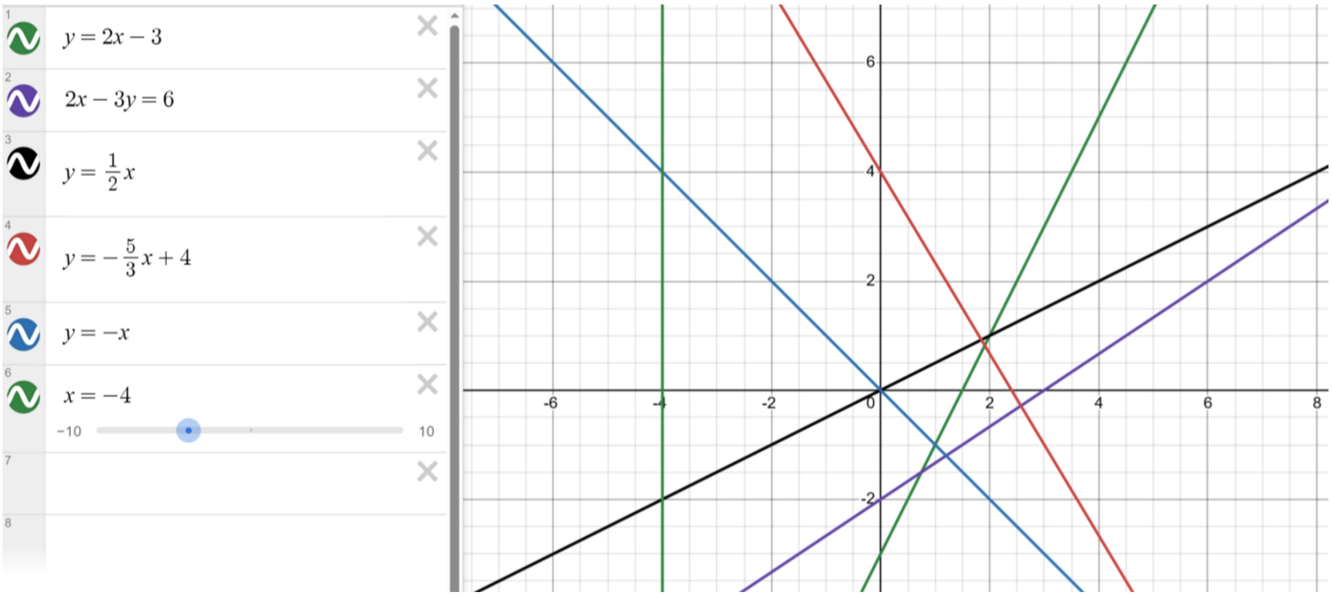
b) $2x - 3y = 6$

c) $y = \frac{1}{2}x$

d) $y = \frac{-5}{3}x + 4$

e) $y = x$

f) $x = -4$



Example 3.1.5: Suppose a barrel contains 30 ounces of salt dissolved into 100 gallons of water. The concentration of a solution is the amount of substance (salt) divided by the volume of liquid (water). Suppose that due to evaporation, the barrel loses 5 gallons of water (and only water) each day (while the salt remains behind).

a) What is the concentration of given solution?

$$\frac{30}{100}$$

b) How many gallons of water will there be in the barrel after *t* days? (Enter an expression in terms of *t*)

100 – 5t

c) What will the concentration of salt be in the water after *t* days? (Enter an expression in terms of *t*)

$$\frac{30}{100-5t}$$

Example 3.1.6: Suppose a warehouse begins with 1400 boxes in it. Workers begin removing 20 boxes per hour. Let *y* be the number of boxes in the warehouse after *x* hours.

a) Write linear equation illustrating the number of boxes present at the warehouse after x hours.

$$y = -20x + 1400$$

b) Find x and y – intercepts and write a meaning in the context of this problem.

x – intercept is (70, 0), after 70 hours there will be no boxes left at the warehouse.

y – intercept is (0, 1400), at the beginning there were 1400 boxes at the warehouse.

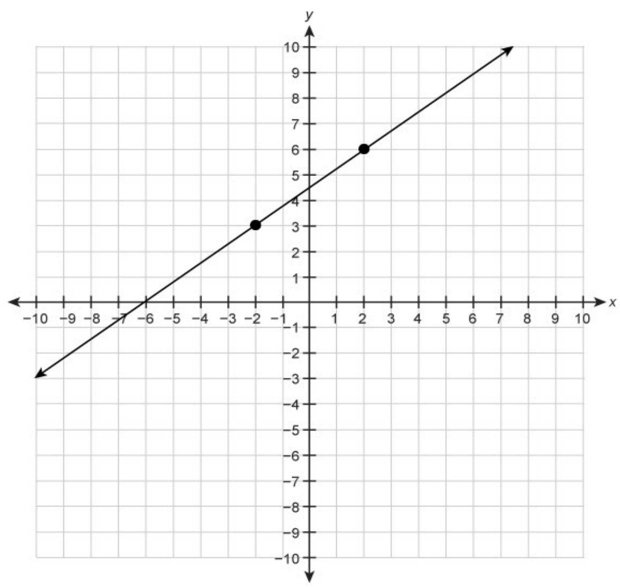
3.2 Slope of a Line

Slope of a line indicate its steepness. Denoted by **m**.

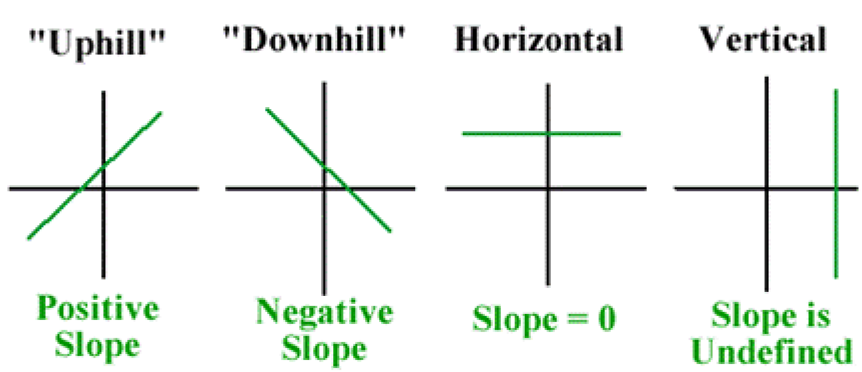
Slope = $\frac{\text{vertical change}}{\text{horizontal change}} = \frac{rise}{run}$

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are any two points on a line.

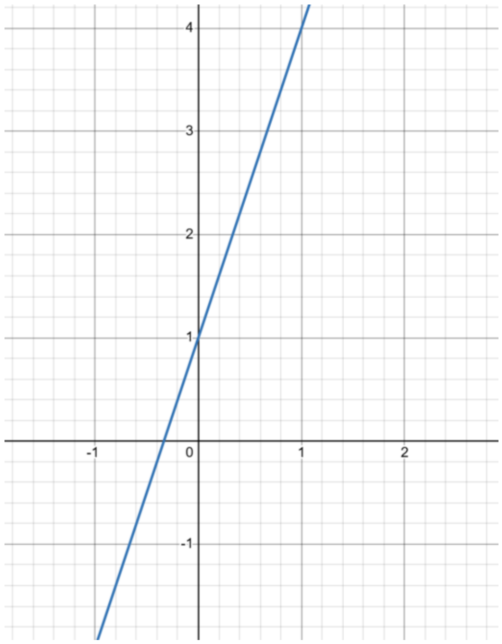


Slope Types:



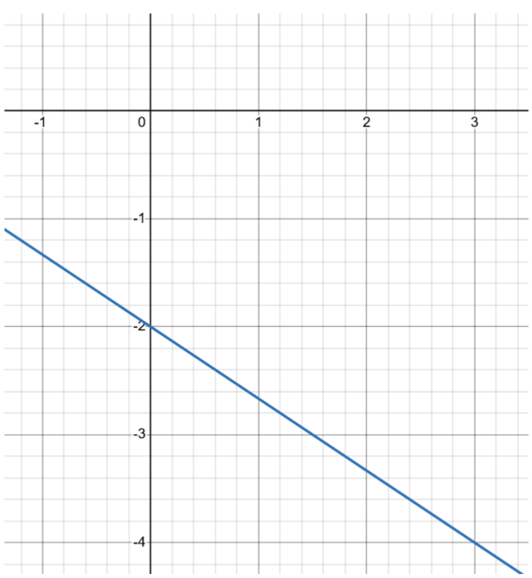
Example 3.2.1: Determine the slope of the line.

a)



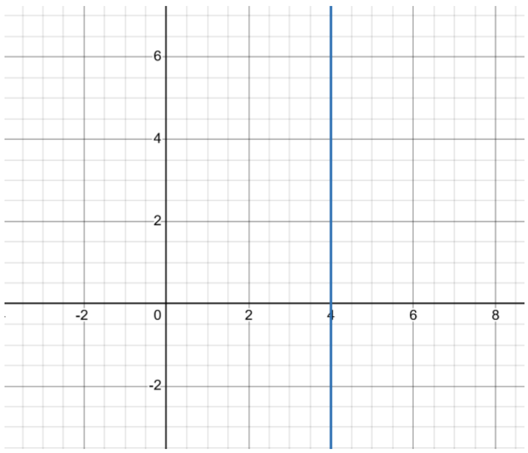
Slope = 3

b)



Slope = $-\frac{2}{3}$

c)



Slope is undefined.

Example 3.2.2: Find the slope between given points:

a) (− 4, 3) and (2, − 9)

$$m = \frac{-9-3}{2-(-4)} = \frac{-12}{6} = -2$$

b) (4, 6) and (2, − 1)

$$m = \frac{-7}{-2} = \frac{7}{2}$$

c) (− 4, − 1) and (− 4, − 5)

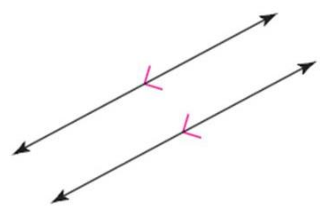
$$m = \frac{-4}{0} \Rightarrow \text{undefined}$$

d) (3, 1) and (− 2, 1)

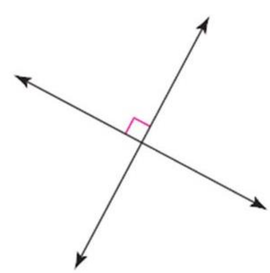
$$m = \frac{0}{-5} = 0$$

Slope of Parallel and Perpendicular line:

Parallel lines are lines in the same plane that do not intersect and are same distance apart from each other.



Perpendicular lines are lines that intersect forming a right angle (90˚).



Parallel lines have same **(equal) slopes**.

Perpendicular lines have **negative reciprocal slopes**.

Examples of negative reciprocals:

2 and $-\frac{1}{2}$

$-\frac{3}{4}$ and $\frac{4}{3}$

1 and -1

Example 3.2.3: Natalie is planning to rent a car while on vacation. The equation $C = 0.31m + 15$ models the relations between the cost in dollars C per day and the number of miles m she drives in one day.

a) Determine the cost in dollars if she drives 0 miles in one day.

   $C = 0.31(0) + 15 = \$15$

b) Determine the cost in dollars if she drive 400 miles in one day.

   $C = 0.31(400) + 15 = \$139$

c) The slope of the equation tells us which of the following:

   - The cost per day will be at least $15
   - The cost increases by $0.31 per day
   - The cost per day is increasing by $15
   - The cost increases by $0.31 per mile

   The cost increases by $0.31 per mile.

d) The C – intercept of the equation tells us which of the following:

   - The cost increases by $0.31 per day
   - The cost per day will be at least $15
   - The cost increases y $0.31 per mile
   - The cost per day is increasing by $15

   The cost per day will be at least $15.

3.3 Find the Equation of a Line

Forms of Equation of a Line:

Standard form **Ax + By = C**

Standard form, **Ax + By = C**, where A, B, and C are some integers. Where $-\frac{A}{B}$ is a slope, $\frac{C}{A}$ is x – intercept and $\frac{C}{B}$ is y – intercept.

ex. 2x + 3y = 6

Slope – intercept form **y = mx + b**

Slope intercept form, y = mx + b, where **m** is the slope and **b** is the y – coordinate of the point of intersection with y – axis (0, b).

ex. $y = 2 - \frac{2}{3}x$

Point – slope form **$y - y_1 = m(x - x_1)$**

Point slope form, $y - y_1 = m(x - x_1)$, where **m** is the slope and **(x_1, y_1)** is any point on the line.

ex. $y + 2 = -\frac{2}{3}(x - 6)$

Note: *Having equation given in any form,* to *find x-intercept, set y = 0 and solve for x. To find y-intercept, set x = 0 and solve for y.*

Example 3.3.1: Find a slope, x and y intercepts from given equation.

a) y = 4x – 1

m = 4 x-int. $(\frac{1}{4}, 0)$ y-int. (0, – 1)

b) 3x – 7y = 10

m = $-\frac{3}{-7} = \frac{3}{7}$ x-int. ($\frac{10}{3}$, 0) y-int. (0, $-\frac{10}{7}$)

c) y = 9x

m = 9 x-int. (0, 0) y-int. (0, 0)

d) x = 10

slope is undefined x-int. (10, 0) NO y-int.

e) y + 3 = 7

y = 4 m = 0 NO x-int. y-int. (0, 4)

Example 3.3.2: Use given information to write the equation of the line.

a) Slope is 2 and y-intercept is 5

y = 2x + 5

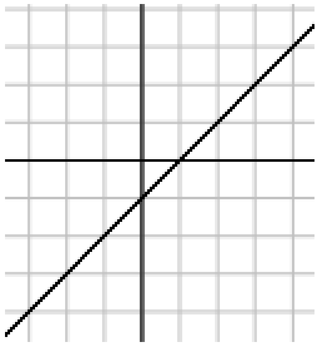
b) m = − 1, b = − 2

y = – 1x + (– 2) = – x – 2

c) Slope is $\frac{1}{4}$ and passing through a point (3, – 7)

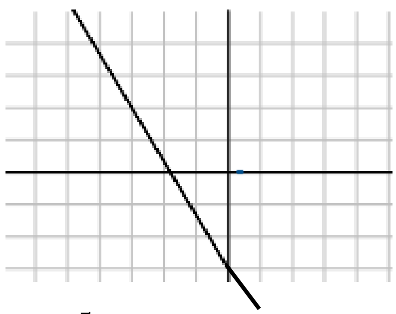
y + 7 = $\frac{1}{4}$ (x – 3)

d)



m = 1 b = – 1 y = x – 1

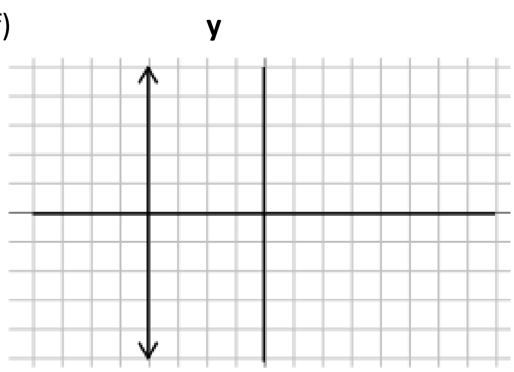
e)



$m = -\frac{5}{3}$ b = – 3

$y = -\frac{5}{3}x - 3$

f)



Slope is undefined No y-intercept

x = – 4

Example 3.3.3: Write equation of the line:

a) through the point (− 6, 2) with a slope of $-\frac{2}{3}$ in slope-intercept form.

What to do:

- Substitute given point (x, y) and m (slope) into y = mx + b, solve for b, then rewrite by substituting only m (slope) and b(y-int).

$$y = -\frac{2}{3}x - 2$$

OR

- Substitute given point (x, y) and m(slope) into $y - y_1 = m(x - x_1)$, then open parenthesis and solve for y, or leave it as it is.

$$y - 2 = -\frac{2}{3}(x + 6)$$

b) through the points (− 3, 4) and (1, 2).

What to do:

- Find m (slope) using the formula, then substitute m and coordinates of <u>one</u> of the given points into y = mx + b, solve for b, then rewrite by substituting only m and b.

$$y = -\frac{1}{2}x + \frac{5}{2}$$

OR

- Find m(slope) using the slope formula, then substitute m and coordinates of <u>one</u> of the given points into $y - y_1 = m(x - x_1)$.

Equations of Vertical and Horizontal Lines:

Equation of a Vertical is x = number ex. x = 5

Equation of a Horizontal is y = number ex. y = − 2

Example 3.3.4:

a) Find an equation of the vertical line passing through a point (1, 4).

$$x = 1$$

b) Find an equation of the horizontal line passing through a point (– 6, 7).

$$y = 7$$

Example 3.3.5: Find the equation of a line:

a) through (4, − 5) and parallel to 2x − 3y = 6.

First, we need to find slope of a given line. Since the line we are looking for is parallel to a given line, the slope will be the same.

$m = \frac{2}{3}$ $m \parallel = \frac{2}{3}$

$y + 5 = \frac{2}{3}(x - 4)$

b) through (6, − 9) perpendicular to $y = -\frac{3}{5}x + 4$

$m = -\frac{3}{5}$ $m \perp = \frac{5}{3}$

$y + 9 = \frac{5}{3}(x - 6)$

c) through (3, 4) perpendicular to x = − 2

Slope is undefined, because x = – 2 is a vertical line. Line perpendicular to a vertical line is a horizontal line, which means the slope is 0.

$y - 4 = 0(x - 3)$

$y - 4 = 0$

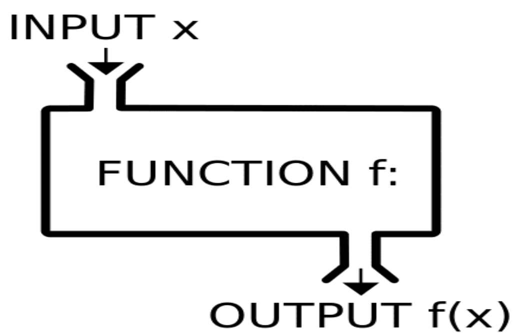
$y = 4$

3.5 Relations and Functions

A **relation** is a set of ordered pairs that define the relationships between two sets. A relation where each input has exactly one output is a function.

A **function *f*** consists of two sets, a set **D** of **inputs** called the **domain,** and a set **R** of **outputs** called the **range**, and an assignment that assigns to each input x exactly one output y.

A function ***f*** with domain D and range R is denoted by ***f: D → R***.



Two different inputs may produce **one output**, but **two different outputs** **cannot** be produced from **one input**.

In other words:

2 different X's for the same Y is a function.

$x_1 \neq x_2$ $f(x_1) = f(x_2)$

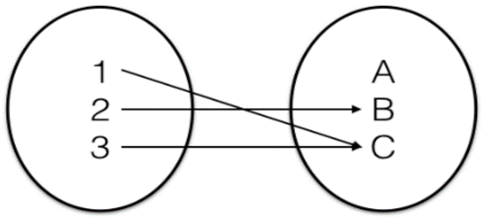
2 different Y's for the same X is not a function.

$f(x_1) \neq f(x_2)$ $x_1 = x_2$

Example 3.5.1: Determine if the following represents a function or not.

Inputs Outputs

a) Inputs Outputs



YES

b)

x	− 3	0	2	3	7	11
y	1	7	4	1	5	0

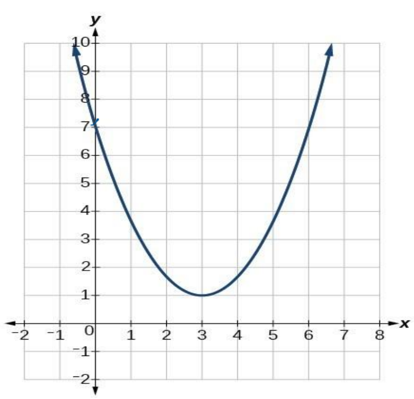
YES

c)

x	− 7	− 2	0	5	5	10
y	0	0	1	2	7	12

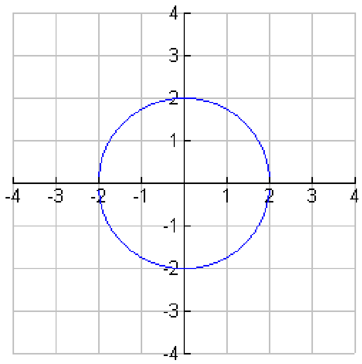
NO

d)



YES

e)



NO

Finding Function Outputs:

For **f(x) = expression in terms of x**, x is an input and f(x) (also known as y) is an output. To find an output, substitute the given input and evaluate the numerical expression.

Example 3.5.2: For the given function f, calculate the outputs f(3), f(−3), and f(12).

a) f(x) = 3x + 4

f(3) = 13

f(– 3) = – 5

f(12) = 40

b)

$$f(x) = \begin{cases} -x-4 & , \quad x < 3 \\ x^2 - 7 & , \ 3 \le x \le 10 \\ \dfrac{120}{x} + 5 & , \quad x > 10 \end{cases}$$

f(3) = 3 ^{2} – 7 = 2 since x = 3 is in 3 ≤ x ≤ 10 interval

f(– 3) = – (– 3) – 4 = – 1 since x = – 3 is in x < 3 interval

f(12) = 120/12 + 5 = 15 since x = 12 is in x > 10 interval

Example 3.5.3: Let *f* be the function given by $f(x) = x^2 + 2x - 3$. Find the following function values.

a) $f(a)$

$f(a) = a^2 + 2a - 3$

b) $f(a + 1)$

$f(a + 1) = (a + 1)^2 + 2(a + 1) - 3 = a^2 + 4a$

c) $f(a) + 5$

$f(a) + 5 = a^2 + 2a - 3 + 5 = a^2 + 2a + 2$

d) $f(x + h)$

$f(x + h) = (x + h)^2 + 2(x + h) - 3 = x^2 + 2xh + h^2 + 2x + 2h - 3$

e) $f(x + h) - f(x)$

$f(x + h) - f(x) = x^2 + 2xh + h^2 + 2x + 2h - 3 - (x^2 + 2x - 3) = 2xh + h^2 + 2h$

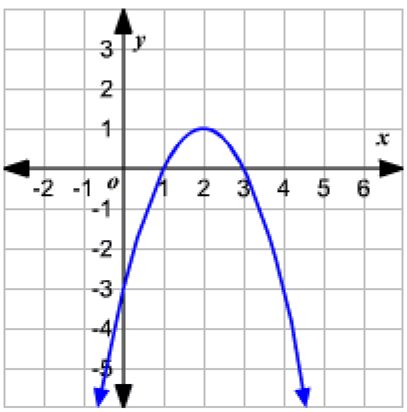
f) $\frac{f(x+h)-f(x)}{h}$

$\frac{f(x+h)-f(x)}{h} = \frac{2xh+ h^2+2h}{h} = 2x + 2 + h$

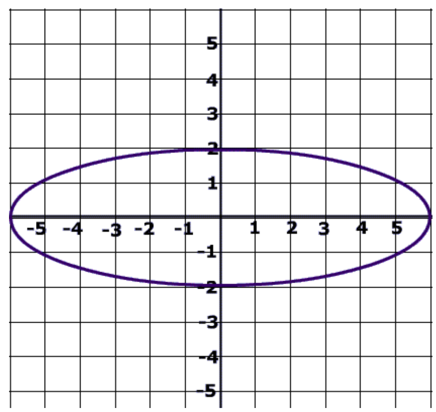
3.6 Graphs of Functions

A **Vertical Line Test** is used to determine if graph represents a function or not. In order, for a graph to be a graph of a function **__any__ vertical line** that you draw through the graph **will intersect** the graph **once** or **will not** intersect at all.

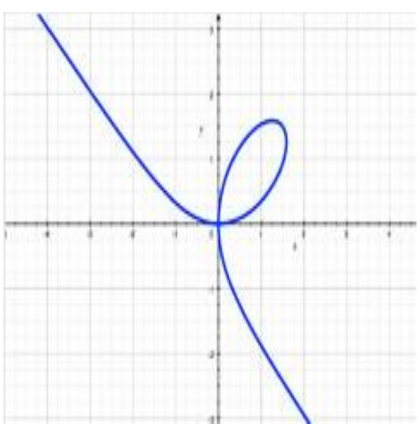
Example 3.6.1: State whether the following graphs are the graphs of functions or not? Explain.



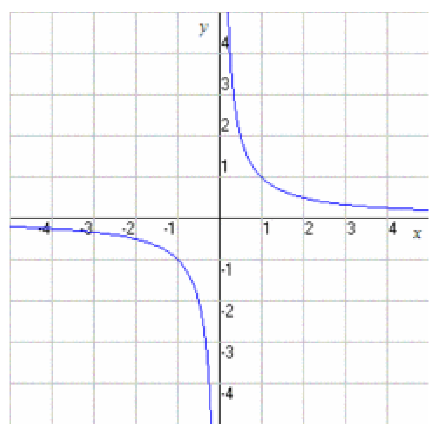
YES



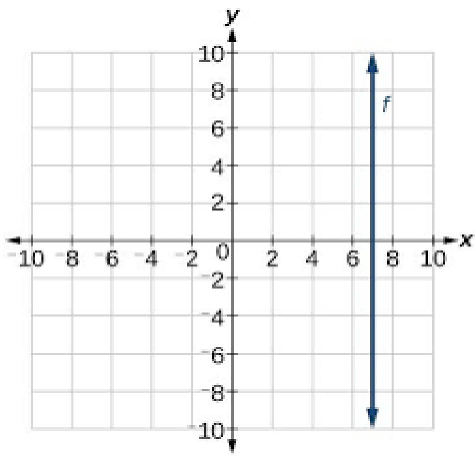
NO



NO

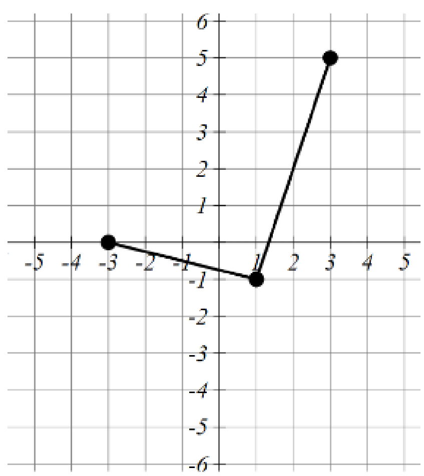


YES



NO

Example 3.6.2: The graph of a relation is shown.



a) Determine interval(s) for inputs.

   $x \in [-3, 3]$

b) Determine interval(s) for outputs.

   $y \in [-1, 5]$

c) Is the relation a function of *x*?

   YES.

Example 3.6.3: Find the domain of each of the following functions.

a) $f(x) = 4x^3 - 2x + 5$

D: All real values

b) $f(x) = |x|$

D: All real values

c) $f(x) = \frac{1 + x}{x - 7}$

$x - 7 \neq 0$
$x \neq 7$

D: All real value, $x \neq 7$

d) $f(x) = \sqrt{x + 3}$

$x + 3 \geq 0$
$x \geq -3$

D: $[-3, \infty)$

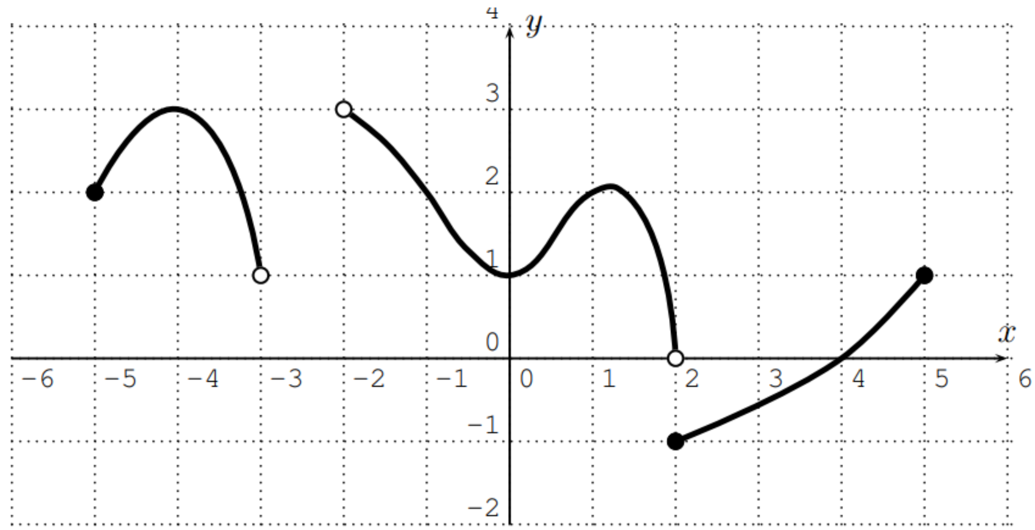
e)

$$f(x) = \begin{cases} x, & -5 < x \leq 3 \\ 3, & 3 < x \leq 7 \\ -2x + 17, & x > 7 \end{cases}$$

D: $(-5, \infty)$

Interpreting Graphs:

Consider the following graph:



Is this a graph of a function? YES.

Some functional values

f(– 4) = 3

f(x) = 2 at x = – 5, x ≈ – 3.3, x = – 1, x = 1, x ≈ 1.3

f(– 2) is undefined

f(0) = 1 (y-intercept)

f(1) = 2

f(2) = – 1

f(4) = 0 (x-intercept)

Domain and range

D: [– 5, – 3) U (– 2, 5] R: [– 1, 3]

Intervals of increase and decrease

f is increasing: (– 5, – 4) and (0, ≈1.3) and (2, 5)

f is decreasing: (– 4, – 3) and (– 2, 0) and (≈1.3, 2)