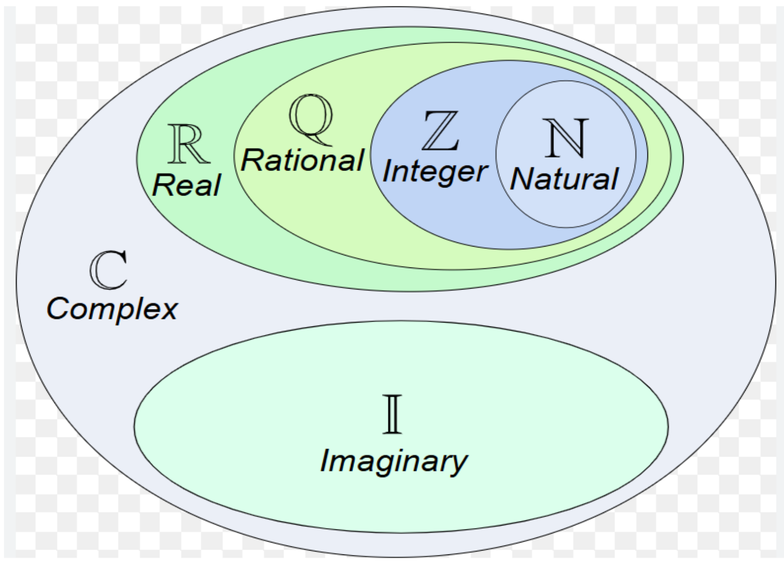
Chapter 1: Foundation

1.1 Use the Language of Algebra

Sets of Numbers:



Natural Numbers (ℕ): The counting numbers {1, 2, 3, ...} are commonly called natural numbers.

Whole numbers: Natural (counting) numbers including 0.

Integers (ℤ): Positive and negative counting numbers, as well as zero.

Rational Numbers (ℚ): Numbers that can be expressed as a ratio of an integer to a non-zero integer. Rational numbers are ending and/or non-ending but repeating decimals.

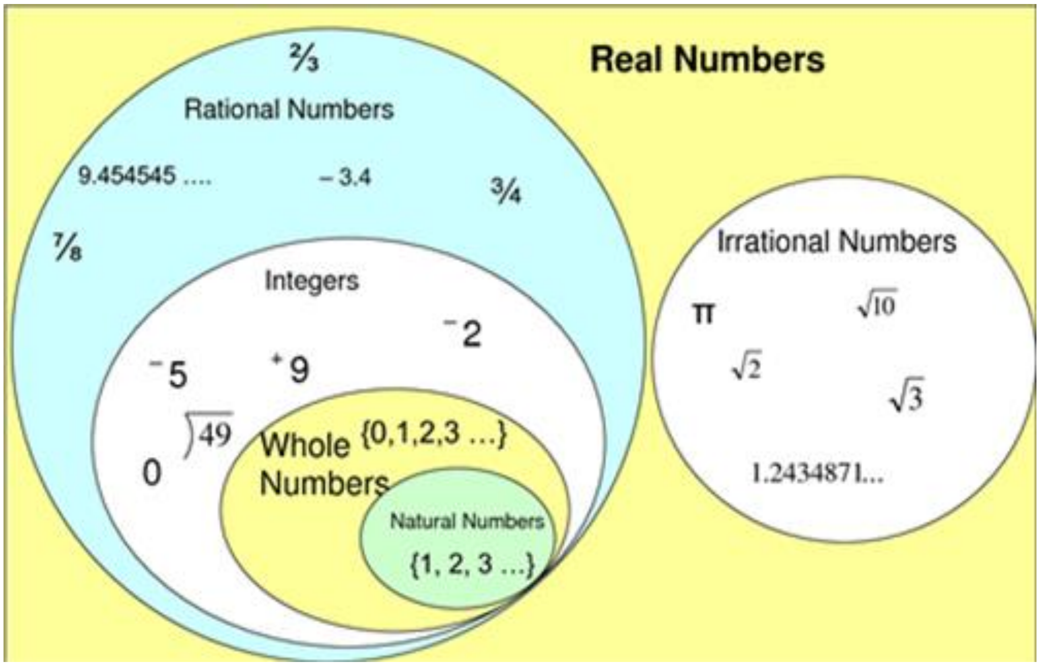
Real Numbers (ℝ): Numbers that correspond to points along a line. They can be positive, negative, or zero.

Irrational Numbers (ℝ/ℚ): Real numbers that are not rational. Irrational numbers are non-ending and non-repeating decimals.

Imaginary Numbers: Numbers that equal the product of a real number and the imaginary unit i, where $i^2 = -1$. The number 0 is both real and imaginary.

Complex Numbers (ℂ): Includes real numbers, imaginary numbers, and sums and differences of real and imaginary numbers.

Some Examples of Real Numbers:



Whole Numbers and Their Forms:

A **digit** is one of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

The position of each digit in a number tells its value, or **place value**.

Place-Value Chart														
Trillions			Billions			Millions			Thousands			Ones		
											1	4	5	6
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

The **standard form** of a number refers to a type of notation in which digits are separated into groups of three by commas. These groups of three digits are known as **periods**.

The **word names form** is when a number is written out in words.

The **expanded form** is when the number is written as a sum of the value of each digit.

Example 1.1.1

a) Identify the place value of 7 in 1987253.

   Thousands, ones.

b) Write 739464528464 in standard form.

   739,464,528,464

c) Write the word name for 376493.

   Three hundred seventy-six thousand, four hundred ninety-three

d) Write 58473 in expanded form.

   50,000 + 8,000 + 400 + 70 + 3

e) A company had a new office building constructed. The final cost was seventy-four million, three hundred sixty-two dollars. Write this number in standard form.

   74,000,362

Rounding Whole Numbers:

Step 1: Locate the place value digit of the number that is being rounded.

Step 2: If the digit to the right of the place value is:

- 0 through 4, **do not change** the place value digit

- 5 through 9, **add 1** to the place value digit

Step 3: Replace all digits to the right of the digit being rounded with zeros.

Example 1.1.2:

a) Round 38276 to the nearest tens

   38280

b) Round 28,324,647 to the nearest hundred-thousands

   28,300,000

c) Round 79808 to the nearest thousand

   80,000

Comparing Whole Numbers:

Inequality symbols are:

Less than **<**

Greater than **>**

Less than or equal **≤**

Greater than or equal **≥**

Not equal **≠**

Use **>** and **<** to compare numbers.

Example 1.1.3:

Compare numbers by placing > or < in between.

a) 7 > 2
b) 37 < 73
c) 456 < 465
d) 13652 > 13452

Basic Arithmetic Operations:

Addition **+**

Subtraction **–**

Multiplication **●**, **x**

Division **÷, /**

Example 1.1.4:

Perform an indicated operation with whole numbers, without using a calculator.

a) Add 54, 67, 89, and 34 — 244

b) Add 4678, 345, 76458 — 81,481

c) Subtract 74 from 118 — 44

d) Subtract 8976 from 100205 — 91,229

e) Multiply 31 with 72 — 2,232

f) Multiply 371 with 58 — 21,518

g) Divide 145 by 5 — 29

h) Divide 9860 by 47 — 209 R 37

Some Basic Definitions:

A **prime number** is a counting number greater than 1 that has only factors of 1 and itself.

Ex. 2, 3, 5, 7, etc.

A **composite number** is a counting number greater than 1 that has factors of 1 and itself and at least one more factor.

Ex. 6, 8, 9, 10, etc.

A **multiple** is a number obtained by multiplying a number by an integer.

Ex. 10 is a multiple of 5. Note 5 and 2 and factors of 10.

The **least common multiple (LCM)** is the smallest common multiple of given numbers.

A **variable** represents a value that may vary, represented by a letter.

A **constant** represents a value that stays the same.

An **expression** is a combination of a number(s) and/or variable(s) using basic arithmetic operations.

Ex. $2x + 3$

An **equation** is formed when two expressions are equal.

Ex. $2x + 3 = 5$

Grouping Symbols:

Parentheses ( )
Brackets [ ]
Braces { }
Absolute Value | |

Translating Sentences into Algebraic Equations:

A **variable** is a symbol (usually a letter) that stands for a value that may vary.

Mathematical Expression. When we combine numbers and variables in a valid way, using operations such as addition, subtraction, multiplication, division, exponentiation, and other operations and functions as yet unlearned, the resulting combination of mathematical symbols is called a mathematical expression.

Can be:

<u>Arithmetic</u>, purely numeric.

<u>Algebraic</u>, contains a variable.

Word Problems Translation Table:

English	Math
Equals, is, was, will be, has, costs, becomes, is the same as, etc.	=
Times, of, multiplied by, product of, twice, double, triple, etc.	•
Divided by, per, each, quotient of, out of, ratio, etc.	÷
Plus, more than, added to, and, sum, increased by, combined, etc.	+
Minus, less than, fewer, difference between, decreased by, subtracted from, etc.	–

Example 1.1.5: Translate the following phrases into mathematical expressions:

a) "12 larger than x"

   $x + 12$

b) "11 less than y"

   $y - 11$

c) "r decreased by 9"

r – 9

d) "11 times x"

11x

e) "quotient of y and 4"

y/4

f) "twice a"

2a

Example 1.1.6: Let W represent the width of the rectangle. The length of a rectangle is 4 feet longer than its width. Express the length of the rectangle in terms of its width W.

Length of a rectangle = w + 4

Example 1.1.7: A plumber has a pipe of unknown length x. He cuts it into 4 equal pieces. Find the length of each piece in terms of the unknown length x.

Length of each piece = x/4

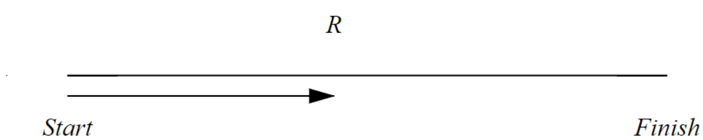
Example 1.1.8: Let the first number equal x. The second number is 3 more than twice the first number. Express the second number in terms of the first number x.

Second number = 2x + 3

Example 1.1.9: Suppose there are 800 boxes in a warehouse, and a worker removes seven boxes per hour. The number of boxes remaining after t hours can be expressed as:

800 – 7t

Example 1.1.10: A runner is on a track 500 meters long.



Suppose a runner runs at a constant speed of 3 meters per second.

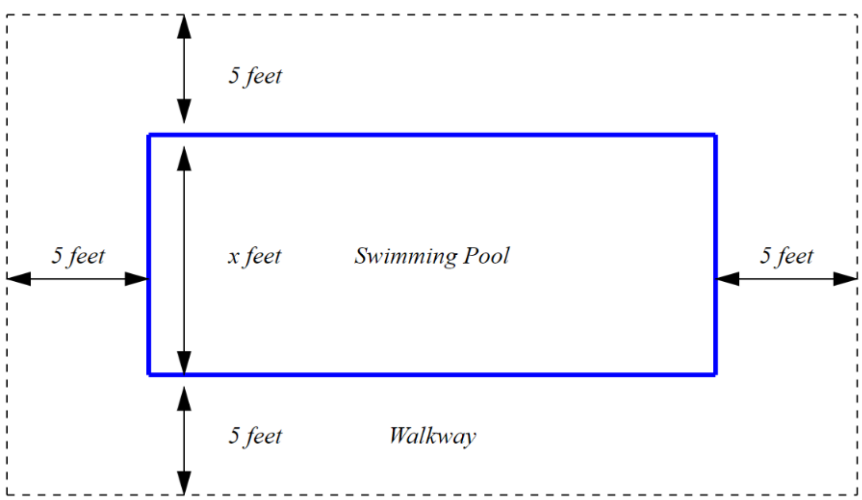
a) An expression giving the distance from the starting line will be:
(Enter an expression in terms of *t*, where *t* is time in seconds).

3t

b) An expression giving the distance to the finish line will be:
(Enter an expression in terms of *t*, where *t* is time in seconds).

500 – 3t

Example 1.1.11: The length of a swimming pool is five times its length. A five foot-wide walkway surrounds the pool, as shown in the figure below (NOT drawn to scale) where x represents the width of the pool.



a) The area of the pool can be expressed as:

5x (x)

b) The total area of the pool and walkway can be expressed as:

(5x + 10)(x + 10)

1.2 Integers

Zero is neither positive nor negative

Note: *Assume that there is no such thing as subtraction of integers; think it is an addition of signed numbers (positive and/or negative integers).*

Basic Arithmetic Operations:

Addition of Integers

Positive + Positive = Sum (stays positive)

Negative + Negative = Sum (stays negative)

Positive + Negative

= Difference (takes the sign of the number larger in absolute value)

Negative + Positive

Same signs; find sum *Different signs; find difference*

Note: *An absolute value is always positive. Represents the distance from the number to zero on a number line. Symbol is* |a|.

Multiplication/Division of Integers

(Positive)(Positive) = Product (stays positive)

(Positive)/(Positive) = Quotient (stays positive)

(Negative)(Negative) = Product (becomes positive)

(Negative)/(Negative) = Quotient (becomes positive)

(Positive)(Negative) = (Negative)(Positive) = Product (is negative)

(Positive)/(Negative) = (Negative)/(Positive) = Quotient (is negative)

Same signs; result is positive

Different signs; result is negative

Operations Involving Zero

Addition/Subtraction:	number + 0 = number	0 + number = number
	number – 0 = number	0 – number = – number
Multiplication/Division:	number (0) = 0	0(number) = 0
	number ÷ 0 is undefined	0 ÷ number = 0

Example 1.2.1: Simplify by using basic arithmetic operations.

a) (− 6) + 8 = 2

b) (− 4) − (− 10) = – 4 + 10 = 6

c) $-2 \div (-1) = 2$

d) $7 - 7 = 0$

e) $(-3)(-9) = 27$

f) $2 - 3 = 2 + (-3) = -1$

g) $0 \div 5 = 0$

h) $32/-8 = -4$

i) $(-5)(4) = -20$

j) $(-3) + (-1) = -4$

k) $(10)(-8) = -80$

l) $30 \div 6 = 5$

m) $-14/2 = -7$

Order of Operations:

Parenthesis (brackets)	**( ), [ ]**	
Exponents	**e^2**	
Multiplication	**x**	whichever comes first when reading from
Division	**÷**	left to right
Addition	**+**	whichever comes first when reading from
Subtraction	**–**	left to right

Example 1.2.2: Simplify using order of operations.

a) $-6 \cdot 4(-1) = -6(-4) = 24$

b) $3 + (8) \div |4| = 3 + 8 \div 4 = 3 + 2 = 5$

c) $8 \div 4 \cdot 2 = 2(2) = 4$

d) $[-9 - (2 - 5)] \div (-6) = [-9 - (-3)] \div (-6) = (-9 + 3) \div (-6) = -6 \div (-6) = 1$

e) $-6 + (-3 - 3)^2 \div 3 = -6 + (-6)^2 \div 3 = -6 + 36 \div 3 = -6 + 12 = 6$

f) $4 - 2|3^2 - 16| = 4 - 2|9 - 16| = 4 - 2|-7| = 4 - 2(7) = 4 - 14 = -10$

g) $(-1 - (-5))|3 + 2| = (4)|5| = 4(5) = 20$

h) $\frac{2 + 4|7 + 4|}{4 \cdot 2 + 5 \cdot 3} = \frac{2+4(11)}{8+15} = \frac{2+44}{23} = \frac{46}{23} = 2$

1.3 Fractions

A **fraction** (from Latin: fractus, “broken”) represents a part of a whole, or anu number of equal parts.

Fraction $\frac{\text{Numerator}}{\text{Denominator}}$ Numerator ÷ Denominator Numerator/Denominator

Proper Fraction, numerator < denominator.

Ex. $\frac{2}{5}$

Improper Fraction, numerator > denominator, can be written as a mixed number (num ÷ den).

Ex. $\frac{5}{2}$

Mixed Number, is an improper fraction represented as a whole number and fractional part.

Ex. $2\frac{1}{2}$

Equivalent fractions, Building Up and Reducing Fractions:

Equivalent fractions are fractions that are equal in value.

Ex. $\frac{1}{2} = \frac{5}{10} = \frac{41}{82} = \frac{234}{468}$ or $\frac{81}{63} = \frac{9}{7} = \frac{18}{14} = \frac{90}{70}$

To build up a fraction means to multiply both numerator and denominator by the same number.

Example 1.3.1: Build each of the following fractions to make them look exactly the same.

a) $\frac{2}{3} = \frac{6}{9}$ Multiply numerator and denominator by 3.

b) $\frac{11}{8} = \frac{-77}{-56}$ Multiply numerator and denominator by – 7.

c) $\frac{12}{4} = \frac{3}{1}$ Multiply numerator and denominator by ¼.

To reduce a fraction means to divide both numerator and denominator by the same number.

Example 1.3.2: Reduce each of the following fractions to their lowest terms.

a) $\frac{24}{40} = \frac{3}{5}$ Divide numerator and denominator by 8.

b) $\frac{81}{18} = \frac{9}{2}$ Divide numerator and denominator by 9.

c) $\frac{150}{250} = \frac{3}{5}$ Divide numerator and denominator by 50.

Least Common Denominator (LCD):

LCD is a least common multiple of given denominators. LCD must be divisible by each of the given denominators. LCD is always equal or greater than any of the given denominators.

To find LCD means to build up fraction(s) in such a way that all denominators are LCD.

Example 1.3.3: Find LCD for given fractions.

a) $\frac{1}{2}$ $\frac{3}{4}$

LCD = 4

b) $\frac{5}{7}$ $\frac{11}{8}$

LCD = 56

$\frac{9}{24}$ $\frac{-5}{12}$ $\frac{17}{36}$

LCD = 72

Comparing Fractions:

To compare fractions, you need to make denominators to be the same, then compare numerators.

Example 1.3.4: Compare by placing > or < between given fractions.

a) $\frac{9}{2} > \frac{3}{4}$

because $\frac{9}{2} = \frac{18}{4}$

b) $\frac{1}{7} < \frac{3}{4}$

because $\frac{4}{28} < \frac{21}{28}$

c) Place in ascending order:

$\frac{79}{24}$ $\frac{13}{4}$ $\frac{9}{24}$ $\frac{-5}{12}$ $\frac{17}{36}$

LCD = 72

Build fractions:

$$\frac{237}{72}, \quad \frac{234}{72}, \quad \frac{27}{72}, \quad \frac{-30}{72}, \quad \frac{34}{72}$$

In ascending order:

$$\frac{-30}{72}, \quad \frac{27}{72}, \quad \frac{34}{72}, \quad \frac{234}{72}, \quad \frac{237}{72}$$

Original fractions in ascending order:

$$\frac{-5}{12} \qquad \frac{9}{24} \qquad \frac{17}{36} \qquad \frac{13}{4} \qquad \frac{79}{24}$$

Basic Arithmetic Operations with Fractions:

To multiply means to multiply numerator of the first fraction with the numerator of the second fraction and denominator of the first with denominator of the second fraction. It is recommended to reduce before multiplying.

Example: $\frac{8}{21} \cdot \frac{14}{24} = \frac{1}{3} \cdot \frac{2}{3} = \frac{1 \cdot 2}{3 \cdot 3} = \frac{2}{9}$

To divide means to keep the first fraction as it is, change division symbol to multiplication symbol, and flip the second fraction (numerator with denominator). Then finish as a multiplication problem.

Example: $\frac{9}{5} \div \frac{12}{25} = \frac{9}{5} \cdot \frac{25}{12} = \frac{3 \cdot 5}{1 \cdot 4} = \frac{15}{4}$

Example 1.3.5: Multiply/Divide each of the following.

a) $(-2)(\frac{-2}{6}) = \frac{-2}{1}\left(\frac{-2}{6}\right) = \frac{4}{6} = \frac{2}{3}$

b) $\frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$

c) $\frac{1}{20} \div \frac{17}{2} = \frac{1}{20} \cdot \frac{2}{17} = \frac{1}{170}$

d) $\frac{12}{5} \cdot \frac{15}{2} \div 3 = \frac{6}{1} \cdot \frac{3}{1} \cdot \frac{1}{3} = 6$

Addition and Subtraction, <u>only fractions with the same denominator can be added/subtracted</u>.

If fractions have different denominators, build up so they have the same denominators then add/subtract numerators.

Example: $\frac{8}{2} + \frac{13}{4} = \frac{16}{4} + \frac{13}{4} = \frac{16+13}{4} = \frac{29}{4}$ LCD = 4

$\frac{5}{4} - \frac{1}{3} = \frac{15}{12} - \frac{4}{12} = \frac{15-4}{12} = \frac{11}{12}$ LCD = 12

Example 1.3.6: Add or subtract each of the following.

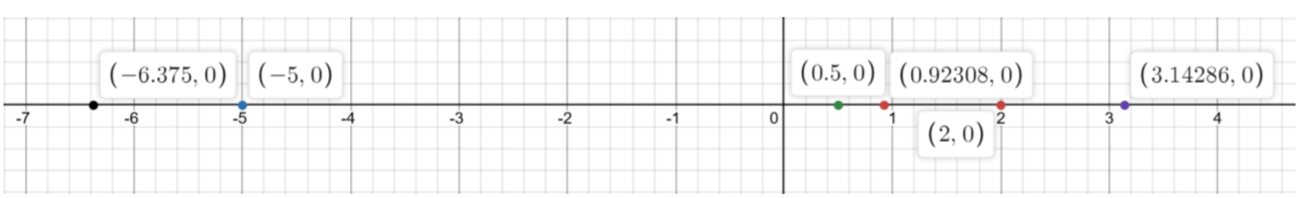
a) $\frac{14}{2} + \frac{71}{2} = \frac{14+71}{2} = \frac{85}{2}$

b) $-\frac{7}{24} + (-\frac{1}{2}) = \frac{-7+(-12)}{24} = \frac{-19}{24}$

c) $\frac{4}{5} - \frac{10}{9} = \frac{36-50}{45} = \frac{-14}{45}$

Example 1.3.7: Graph each on a number line:

a) 2 b) – 5 c) $\frac{1}{2}$ d) $3\,\frac{1}{7}$ e) – 6 f) $\frac{12}{13}$



1.5 Properties of Real Numbers

To evaluate means to substitute the value of a given variable into the expression and use order of operations to find its value.

Example 1.5.1: Evaluate each of the following.

a) $p(q + 6)$, when $p = 3$ and $q = 5$

$3(5 + 6) = 3(11) = 33$

b) $x + zx(3 - z)(x/3)$, when $x = -6$ and $z = -2$

$-6 + (-2)(-6)(3 - (-2))(-6/3) = -6 + 12(5)(-2) = -6 + 60(-2) = -6 + (-120) = -126$

c) $ab^3 - 5a + 4b^2 - 17ab$, when $a = -1$ and $b = 4$

$(-1)4^3 - 5(-1) + 4(4)^2 - 17(-1)(4) = -1(64) + 5 + 4(16) + 17(4) = -64 + 5 + 64 + 68 =$

$5 + 68 = 73$

Examples 1.5.2: Combine like terms (exactly the same variable part).

a) $5x - 2y - 8x + 7y = (5 - 8)x + (-2 + 7)y = -3x + 5y$

b) $8x^2 - 3x + 7 - 2x^2 + 4x - 3 = 6x^2 + x + 4$

Algebraic Properties:

The Commutative Property

$a + b = b + a \qquad ab = ba$

The Associative Property

$a + (b + c) = (a + b) + c$

$a\,(b\,c) = (a\,b)\,c$

The Distributive Property

$a\,(b + c) = ab + ac$

Examples 1.5.3: Distribute.

a) $7k(-k + 6) = 7k(-k) + 7k(6) = -7k^2 + 42k$

b) $-2p(9p - 1) = -18p^2 + 2p$

Examples 1.5.4: Simplify.

a) $9(b + 10) + 5b = 9b + 90 + 5b = 14b + 90$

b) $-4k^2 - 8k(8k + 1) = -4k^2 - 64k^2 - 8k = -68k^2 - 8k$

c) $(7x^2 - 3) - (5x^2 + 6x) = 7x^2 - 3 - 5x^2 - 6x = 2x^2 - 6x - 3$

d) $-3(4 + a) + 6a(9a + 10) = -12 - 3a + 54a^2 + 60a = 54a^2 + 57a - 12$